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ON GENERALIZED DIFFERENCE OPERATOR OF THIRD KIND AND ITS APPLICATIONS IN NUMBER THEORY

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Abstract: In this paper, the authors extend the theory of the generalized difference operator Δ_ℓ and the second kind Δ_{ℓ_1,ℓ_2} to the generalized difference operator of the third kind $\Delta_{\ell_1,\ell_2,\ell_3}$ for the positive reals ℓ_1, ℓ_2 and ℓ_3. We also present the discrete version of Leibnitz Theorem, binomial theorem, Newton's formula with reference to $\Delta_{\ell_1,\ell_2,\ell_3}$. Also by defining its inverse, we establish a few formulae for the sum of the second partial sums of higher powers of arithmetic progression in number theory.

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1. Introduction

The theory of difference equations is based on the operator Δ defined as

$$\Delta u(n) = u(n+1) - u(n), \quad n \in \mathbb{N}\,, \tag{1}$$

where $\mathbb{N} = \{0, 1, 2, 3, \cdots, \}$. Eventhough many authors [1], [8], [9] have suggested the definition of Δ as

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$$\Delta u(n) = u(n+\ell) - u(n), \qquad n \in \mathbb{N}, \qquad \ell \in \mathbb{R} - \{0\}, \tag{2}$$

no significant progress took place on this line. But recently, when we took up the definition of Δ as given in (2) and developed the theory of difference equations in a different direction and obtained some interesting results in the application of number theory. For convinence we labelled the operator Δ defined by (2) as Δ_ℓ and by defining its inverse Δ_ℓ^{-1} many interesting results on number theory were obtained. By extending the study for sequences of complex numbers and ℓ to be real, some new qualitative properties like rotatory, expanding and shrinking, spiral and weblike were studied for the solutions of difference equations involving Δ_ℓ. The results obtained can be found in [2]-[6]. The goal of this paper is to obtain some significant results on Δ_{ℓ_1,ℓ_2}, $\Delta_{\ell_1,\ell_2,\ell_3}$ and to obtain the values of S^n, PS^n and P^2S^n where

$$S^n = j^n + (j+\ell)^n + (j+2\ell)^n + \cdots + (j+k\ell)^n,$$

$$PS^n = j^n + \overline{j^n + (j+\ell)^n} + \overline{j^n + (j+\ell)^n + (j+2\ell)^n} + \cdots + \overline{j^n + (j+\ell)^n + \cdots + (j+k\ell)^n},$$

$$P^2S^n = j^n + \left[j^n + \overline{j^n + (j+\ell)^n}\right] + \left[j^n + \overline{j^n + (j+\ell)^n} + \overline{j^n + (j+\ell)^n + (j+2\ell)^n}\right] + \cdots + \left[j^n + \overline{j^n + (j+\ell)^n} + \overline{j^n + (j+\ell)^n + (j+2\ell)^n} + \ldots + \overline{j^n + (j+\ell)^n + (j+2\ell)^n + \cdots + (j+k\ell)^n}\right].$$

S^n denotes the sum of the n-th powers of an A.P., PS^n denotes the sum of the partial sums of S^n and P^2S^n denotes sum of partial sums of PS^n.

The formula for the value of S^n, PS^n are derived in [2], [7] using Δ_ℓ, $\Delta_{\ell,\ell}$ respectively. Hence, in this paper, we derive the formulae for finding the value of P^2S^n using $\Delta_{\ell,\ell,\ell}$, Stirling numbers of the second kind and present here some results and applications on $\Delta_{\ell_1,\ell_2,\ell_3}$.

Throughout this paper, we make use of the following assumptions:

(i) $N = \{0, 1, 2, 3, ...\}$,

(ii) $N(a) = \{a, a+1, a+2, ...\}$,

(iii) r and n are positive integers and ℓ_1, ℓ_2 and ℓ_3 are positive reals,

(iv) n^* is the largest non negative integer such that $k - n^*\ell \geq 0$,

(v) $c, c_0, c_1, c_2, \ldots$ are constants,

(vi) $rC_i = \frac{r!}{(r-i)!i!}$ where $0! = 1$, $r! = 1.2.3...r$ and

(vii) $u : [0, \infty) \to \mathbb{C}$ is a complex valued function on $[0, \infty)$.

2. Basic Definitions and Examples

In this section, we present the discrete versions of Leibnitz and binomial theorems with reference to $\Delta_{\ell_1,\ell_2,\ell_3}$. Suitable examples are provided to illustrate the results.

Definition 2.1. Let $u : [0,\infty) \to \mathbb{C}$ be any complex valued function on $[0,\infty)$. We define the generalized difference operator of the third kind for $u(k)$ as

$$\Delta_{\ell_1,\ell_2,\ell_3}u(k) = u(k+\ell_1+\ell_2+\ell_3) - [u(k+\ell_1+\ell_2) + u(k+\ell_1+\ell_3) + u(k+\ell_2+\ell_3)] + [u(k+\ell_1) + u(k+\ell_2) + u(k+\ell_3)] - u(k). \quad (3)$$

Lemma 2.2. *If E^{ℓ} is the usual shift operator defined as $E^{\ell}u(k) = u(k+\ell)$, then the following are simple to derive. If $\ell_j, j = 1,2,3$ are positive reals, then:*

(i) $$\Delta_{\ell_1,\ell_2,\ell_3} = E^{\ell_1+\ell_2+\ell_3} - (E^{\ell_1+\ell_2} + E^{\ell_1+\ell_3} + E^{\ell_2+\ell_3}) + (E^{\ell_1} + E^{\ell_2} + E^{\ell_3}) - 1, \quad (4)$$

(ii) $$\Delta_{\ell_1,\ell_2,\ell_3} = \Delta_{\ell_1+\ell_2+\ell_3} - (\Delta_{\ell_1+\ell_2} + \Delta_{\ell_1+\ell_3} + \Delta_{\ell_2+\ell_3}) + (\Delta_{\ell_1} + \Delta_{\ell_2} + \Delta_{\ell_3}), \quad (5)$$

(iii) $$\Delta_{\ell_1,\ell_2,\ell_3} = \Delta_{\ell_1}\ \Delta_{\ell_2}\Delta_{\ell_3}, \quad \textit{and} \quad (6)$$

(iv) $$\Delta_{\ell_1,\ell_2,\ell_3} = \prod_{j=1}^{3}\left(\sum_{i=1}^{\ell_j} \ell_j C_i \Delta^i\right). \quad (7)$$

Definition 2.3. The second order of the generalized difference operator of the third kind is $\Delta^2_{\ell_1,\ell_2,\ell_3} = \Delta_{\ell_1,\ell_2,\ell_3}(\Delta_{\ell_1,\ell_2,\ell_3})$ and in general the n-th order of the generalized difference operator of the third kind is defined as $\Delta^n_{\ell_1,\ell_2,\ell_3} = \Delta_{\ell_1,\ell_2,\ell_3}\left(\Delta^{n-1}_{\ell_1,\ell_2,\ell_3}\right)$.

Remark 2.4. For the positive integers p and q,

$$\Delta^p_{\ell_1,\ell_2,\ell_3}\Delta^q_{\ell_1,\ell_2,\ell_3} = \Delta^q_{\ell_1,\ell_2,\ell_3}\Delta^p_{\ell_1,\ell_2,\ell_3}.$$

As a consequence of Definition 2.3, the following results can be obtained easily.

Lemma 2.5. *(i) If $P_{3p-1}(k) = c_{3p-1}k^{3p-1} + c_{3p-2}k^{3p-2} + \cdots + c_1k^1 + c_0$ is any polynomial in k of degree $(3p-1)$, then $\Delta^p_{\ell_1,\ell_2,\ell_3}P_{3p-1}(k) = 0$.*

(ii) If m and n are positive integers and ℓ is a real number, then

$$\Delta^n_{\ell,\ell,\ell} k^m = \begin{cases} m!\ \ell^m, & \text{if} \quad m = 3n; \\ 0, & \text{if} \quad m < 3n. \end{cases} \tag{8}$$

(iii) If $P_k = a_0 k^{3n} + a_1 k^{3n-1} + a_2 k^{3n-2} + \cdots + a_n$ is any polynomial in k of degree $3n$, then

$$\Delta^n_{\ell,\ell,\ell} P_k = a_0(3n!)\ell^{3n}. \tag{9}$$

(iv) For the positive integer r,

$$\Delta^r_{\ell_1,\ell_2,\ell_3} = \prod_{j=1}^{3} \Big(\sum_{i=0}^{r} (-1)^i\ rC_i E^{\ell_j(r-i)} \Big) \tag{10}$$

which is equivalent to

$$\Delta^r_{\ell_1,\ell_2,\ell_3} u(k) = \prod_{j=1}^{3} \left(\sum_{i=0}^{r} (-1)^i\ rC_i u\big(k + \ell_j(r-i)\big) \right). \tag{11}$$

(v) *If $\ell_j = \sum\limits_{i=1}^{n} \ell_{j_i}$, $j = 1, 2, 3$, then $\Delta_{\ell_1,\ell_2,\ell_3} = \prod\limits_{j=1}^{3} \Big[\prod\limits_{i=1}^{n} \Big(\Delta_{\ell_{j_i}} + 1 \Big) - 1 \Big]$.*

(vi) For the positive integer n,

$$(a)\quad \Delta_{n\ell_1,n\ell_2,n\ell_3} = E^{n(\ell_1+\ell_2+\ell_3)} - (E^{n(\ell_1+\ell_2)} + E^{n(\ell_1+\ell_3)} + E^{n(\ell_2+\ell_3)}) + (E^{n\ell_1} + E^{n\ell_2} + E^{n\ell_3}) - 1, \tag{12}$$

$$(b)\quad \Delta_{n\ell_1,n\ell_2,n\ell_3} = (1 + \Delta_{\ell_1+\ell_2+\ell_3})^n - [(1 + \Delta_{\ell_1+\ell_2})^n + (1 + \Delta_{\ell_1+\ell_3})^n + (1 + \Delta_{\ell_2+\ell_3})^n] + [(1 + \Delta_{\ell_1})^n + (1 + \Delta_{\ell_2})^n + (1 + \Delta_{\ell_3})^n] - 1, \tag{13}$$

$$(c)\quad \Delta_{n\ell_1,n\ell_2,n\ell_3} = \sum_{r=1}^{n} n\ C_r \{ \Delta^r_{\ell_1+\ell_2+\ell_3} - (\Delta^r_{\ell_1+\ell_2} + \Delta^r_{\ell_1+\ell_3} + \Delta^r_{\ell_2+\ell_3}) + (\Delta^r_{\ell_1} + \Delta^r_{\ell_2} + \Delta^r_{\ell_3}) \}, \tag{14}$$

$$(d)\quad \Delta^n_{\ell_1,\ell_2,\ell_3} = \textstyle\sum_{r=0}^{n} (-1)^r\ n\ C_r \Delta^{n-r}_{\ell_1+\ell_2+\ell_3} \left\{ \displaystyle\sum_{i=0}^{r} (-1)^r r\ C_i (\Delta_{\ell_1+\ell_2} + \Delta_{\ell_1+\ell_3} + \Delta_{\ell_2+\ell_3})^{r-i} (\Delta_{\ell_1} + \Delta_{\ell_2} + \Delta_{\ell_3})^i \right\}, \tag{15}$$

$$(e)\quad \Delta^n_{\ell_1,\ell_2,\ell_3} = \prod_{j=1}^{3} \left(\sum_{i=0}^{n-1} (-1)^i\ nC_i \Delta_{(n-i)\ell_j} \right). \tag{16}$$

The following is the discrete version of Leibnitz Theorem according to $\Delta_{\ell_1,\ell_2,\ell_3}$.

Theorem 2.6. *For the functions* $u : [0, \infty) \to \mathbb{C}$, $v : [0, \infty) \to \mathbb{C}$,

$$\Delta^n_{\ell_1,\ell_2,\ell_3}[u(k)v(k)] = \Delta^n_{\ell_1}\big(\Delta^n_{\ell_2}[u(k)\Delta^n_{\ell_3}v(k)]\big) + nC_1\Delta^n_{\ell_1}\big(\Delta^n_{\ell_2}[\Delta_{\ell_3}u(k)\Delta^{n-1}_{\ell_3}v(k+\ell_3)]\big) + \cdots + nC_n\Delta^n_{\ell_1}\big(\Delta^n_{\ell_2}[\Delta^n_{\ell_3}u(k)v(k+n\ell_3)]\big). \quad (17)$$

Proof. The proof follows from the generalized Leibnitz Theorem (Theorem 2.5 [2]) and (6). □

Lemma 2.7. *If* n *is a positive integer, then*

$$E^{n(\ell_1+\ell_2+\ell_3)} - \big(E^{n(\ell_1+\ell_2)} + E^{n(\ell_1+\ell_3)} + E^{n(\ell_2+\ell_3)}\big) + \big(E^{n\ell_1} + E^{n\ell_2} + E^{n\ell_3}\big)$$
$$= \sum_{r=1}^{n} nC_r\{\sum_{i=0}^{r-1}(-1)^i rC_i[\Delta_{(r-i)(\ell_1+\ell_2+\ell_3)} - (\Delta_{(r-i)(\ell_1+\ell_2)} + \Delta_{(r-i)(\ell_1+\ell_3)} + \Delta_{(r-i)(\ell_2+\ell_3)}) + \Delta_{(r-i)\ell_1} + \Delta_{(r-i)\ell_2} + \Delta_{(r-i)\ell_3}]\}. \quad (18)$$

Theorem 2.8. *If* n *and* p *are the positive integers, then*

$$(k+n(\ell_1+\ell_2+\ell_3))^p - [(k+n(\ell_1+\ell_2))^p + (k+n(\ell_1++\ell_3))^p + (k+n(\ell_2+\ell_3))^p] + [(k+n\ell_1)^p + (k+n\ell_2)^p + (k+n\ell_3)^p]$$
$$= \sum_{r=1}^{n} nC_r\{[(k+r(\ell_1+\ell_2+\ell_3))^p - [(k+r(\ell_1+\ell_2))^p + (k+r(\ell_1+\ell_3))^p + (k+r(\ell_2+\ell_3))^p] + [(k+r\ell_1)^p + (k+r\ell_2)^p + (k+r\ell_3)^p] - k^p] - rC_1[(k+(r-1)(\ell_1+\ell_2+\ell_3))^p - [(k+(r-1)(\ell_1+\ell_2))^p + (k+(r-1)(\ell_1+\ell_3))^p + (k+(r-1)(\ell_2+\ell_3))^p] + [(k+(r-1)\ell_1)^p + (k+(r-1)\ell_2)^p + (k+(r-1)\ell_3)^p] - k^p] + \cdots + (-1)^{r-1}rC_{r-1}[(k+\ell_1+\ell_2+\ell_3)^p - [(k+\ell_1+\ell_2)^p + (k+\ell_1+\ell_3)^p + (k+\ell_2+\ell_3)^p] + [(k+\ell_1)^p + (k+\ell_2)^p + (k+\ell_3)^p] - k^p]\}. \quad (19)$$

Proof. The proof follows by operating (18) on $u(k) = k^p$. □

Example 2.9. If θ_1, θ_2 and θ_3 are angles measured along the anticlockwise direction, then

$$\sin(k+n(\theta_1+\theta_2+\theta_3)) - [\sin(k+n(\theta_1+\theta_2)) + \sin(k+n(\theta_1+\theta_3)) + \sin(k+n(\theta_2+\theta_3))] + [\sin(k+n\theta_1) + \sin(k+n\theta_2) + \sin(k+n\theta_3)]$$
$$= \sum_{r=1}^{n} nC_r\{[\sin(k+r(\theta_1+\theta_2+\theta_3)) - [\sin(k+r(\theta_1+\theta_2)) + \sin(k+r(\theta_1+\theta_3)) + \sin(k+r(\theta_2+\theta_3))] + [\sin(k+r\theta_1) + \sin(k+r\theta_2) + \sin(k+r\theta_3)] - \sin k] - rC_1[\sin(k+(r-1)(\theta_1+\theta_2+\theta_3)) - [\sin(k+(r-1)(\theta_1+\theta_2)) + \sin(k+(r-1)(\theta_1+\theta_3)) + \sin(k+(r-1)(\theta_2+\theta_3))] + [\sin(k+(r-1)\theta_1) + \sin(k+(r-1)\theta_2) + \sin(k+(r-1)\theta_3)] - \sin k] + \cdots +$$

$$+(-1)^{r-1}rC_{r-1}[\sin(k+\theta_1+\theta_2+\theta_3)-[\sin(k+\theta_1+\theta_2)+\sin(k+\theta_1+\theta_3)$$
$$+\sin(k+\theta_2+\theta_3)]+[\sin(k+\theta_1)+\sin(k+\theta_2)+\sin(k+\theta_3)]-\sin k]\}.$$

Lemma 2.10. *Let $u:[0,\infty)\to\mathbb{C}$ is a function and x is real. Then*

$$\sum_{j=0}^{\infty}\Big\{\frac{x^{j(\ell_1+\ell_2+\ell_3)}u(j(\ell_1+\ell_2+\ell_3))}{j!(\ell_1+\ell_2+\ell_3)^j}-\Big[\frac{x^{j(\ell_1+\ell_2)}u(j(\ell_1+\ell_2))}{j!(\ell_1+\ell_2)^j}$$
$$+\frac{x^{j(\ell_1+\ell_3)}u(j(\ell_1+\ell_3))}{j!(\ell_1+\ell_3)^j}+\frac{x^{j(\ell_2+\ell_3)}u(j(\ell_2+\ell_3))}{j!(\ell_2+\ell_3)^j}\Big]+\Big[\frac{x^{j\ell_1}u(j\ell_1)}{j!\ell_1^j}+\frac{x^{j\ell_2}u(j\ell_2)}{j!\ell_2^j}$$
$$+\frac{x^{j\ell_3}u(j\ell_3)}{j!\ell_3^j}\Big]\Big\}=\Big\{e^{\frac{x^{(\ell_1+\ell_2+\ell_3)}E^{(\ell_1+\ell_2+\ell_3)}}{(\ell_1+\ell_2+\ell_3)}}-\Big[e^{\frac{x^{(\ell_1+\ell_2)}E^{(\ell_1+\ell_2)}}{(\ell_1+\ell_2)}}+e^{\frac{x^{(\ell_1+\ell_3)}E^{(\ell_1+\ell_3)}}{(\ell_1+\ell_3)}}$$
$$+e^{\frac{x^{(\ell_2+\ell_3)}E^{(\ell_2+\ell_3)}}{(\ell_2+\ell_3)}}\Big]+\Big[e^{\frac{x^{\ell_1}E^{\ell_1}}{\ell_1}}+e^{\frac{x^{\ell_2}E^{\ell_2}}{\ell_2}}+e^{\frac{x^{\ell_3}E^{\ell_3}}{\ell_3}}\Big]\Big\}u(0)$$
$$=\Big\{e^{\frac{x^{(\ell_1+\ell_2+\ell_3)}}{(\ell_1+\ell_2+\ell_3)}}\quad e^{\frac{x^{(\ell_1+\ell_2+\ell_3)}\Delta_{\ell_1+\ell_2+\ell_3}}{(\ell_1+\ell_2+\ell_3)}}-\Big[e^{\frac{x^{(\ell_1+\ell_2)}}{(\ell_1+\ell_2)}}\quad e^{\frac{x^{(\ell_1+\ell_2)}\Delta_{\ell_1+\ell_2}}{(\ell_1+\ell_2)}}$$
$$+e^{\frac{x^{(\ell_1+\ell_3)}}{(\ell_1+\ell_3)}}\quad e^{\frac{x^{(\ell_1+\ell_3)}\Delta_{\ell_1+\ell_3}}{(\ell_1+\ell_3)}}+e^{\frac{x^{(\ell_2+\ell_3)}}{(\ell_2+\ell_3)}}\quad e^{\frac{x^{(\ell_2+\ell_3)}\Delta_{\ell_2+\ell_3}}{(\ell_2+\ell_3)}}\Big]$$
$$+\Big[e^{\frac{x^{\ell_1}}{\ell_1}}\quad e^{\frac{x^{\ell_1}\Delta_{\ell_1}}{\ell_1}}+e^{\frac{x^{\ell_2}}{\ell_2}}\quad e^{\frac{x^{\ell_2}\Delta_{\ell_2}}{\ell_2}}+e^{\frac{x^{\ell_3}}{\ell_3}}\quad e^{\frac{x^{\ell_3}\Delta_{\ell_3}}{\ell_3}}\Big]\Big\}u(0).$$

Proof. The proof follows from (4), $u(k)=E^k u(0)$ and $E^\ell=1+\Delta_\ell$. □

Corollary 2.11. *If ℓ is a positive real, then*

$$\sum_{j=0}^{\infty}\Big\{\frac{x^{j(3\ell)}u(j(3\ell))}{j!(3\ell)^j}-3\frac{x^{j(2\ell)}u(j(2\ell))}{j!(2\ell)^j}+3\frac{x^{j\ell}u(j\ell)}{j!\ell^j}\Big\}$$
$$=\Big\{e^{\frac{x^{3\ell}E^{3\ell}}{3\ell}}-3e^{\frac{x^{2\ell}E^{2\ell}}{2\ell}}+3e^{\frac{x^{\ell}E^{\ell}}{\ell}}\Big\}u(0)$$
$$=\Big\{e^{\frac{x^{3\ell}}{3\ell}}\quad e^{\frac{x^{3\ell}\Delta_{3\ell}}{3\ell}}-3e^{\frac{x^{2\ell}}{2\ell}}\quad e^{\frac{x^{2\ell}\Delta_{2\ell}}{2\ell}}+3e^{\frac{x^{\ell}}{\ell}}\quad e^{\frac{x^{\ell}\Delta_{\ell}}{\ell}}\Big\}u(0).$$

3. Generalized Polynomial Factorial of the Third Kind

In this section, we establish the relation between the generalized polynomial factorial and polynomial and discrete version of Newton's formula on $\Delta_{\ell,\ell,\ell}$.

Definition 3.1. For the positive integer n, the generalized polynomial factorial in k of the third kind is defined as

$$k^{(n)}_{\ell_1,\ell_2,\ell_3} = (k+\ell_2+\ell_3)^{(n)}_{\ell_1} + (k+\ell_1+\ell_3)^{(n)}_{\ell_2} + (k+\ell_1+\ell_2)^{(n)}_{\ell_3} - \{(k+\ell_2)^{(n)}_{\ell_1} + (k+\ell_3)^{(n)}_{\ell_1} + (k+\ell_1)^{(n)}_{\ell_2} + (k+\ell_3)^{(n)}_{\ell_2} + (k+\ell_1)^{(n)}_{\ell_3} + (k+\ell_2)^{(n)}_{\ell_3}\} + k^{(n)}_{\ell_1} + k^{(n)}_{\ell_2} + k^{(n)}_{\ell_3}. \quad (20)$$

Using the Stirling numbers of the first kind s^n_r, the following can be easily obtained.

Lemma 3.2. *If n is any positive integer and any real t, then*

$$\sum_{r=1}^{n} s^n_r t^{n-r} \Delta_{\ell_1,\ell_2,\ell_3} k^r = \Delta_{\ell_1,\ell_2,\ell_3} k^{(n)}_t \quad (21)$$

and

$$\Delta^m_{\ell_1,\ell_2,\ell_3} k^{(n)}_t = \sum_{r=0}^{m} \Bigg[(-1)^r mC_r \sum_{i=1}^{n} s^n_i t^{n-i} \Big\{ \sum_{j=0}^{m} (-1)^j mC_j \Big[\sum_{p=0}^{m} (-1)^p mC_p (k+(m-r)\ell_1+(m-j)\ell_2+(m-p)\ell_3)^i\Big]\Big\}\Bigg]. \quad (22)$$

Proof. The proof follows from (6) and the relation

$$\Delta^m_\ell k^{(n)}_t = \sum_{r=0}^{m} \big[(-1)^r mC_r \sum_{i=1}^{n} s^n_i t^{n-i} (k+(m-r)\ell)^i\big]. \qquad \square$$

Lemma 3.3. *If n is a positive integer, then $\Delta_{\ell_1,\ell_2,\ell_3} k^{(n)}_t$ is*

$$\begin{cases} n\ell_1[(k+\ell_2+\ell_3)^{(n-1)}_{\ell_1} - ((k+\ell_2)^{(n-1)}_{\ell_1} + (k+\ell_3)^{(n-1)}_{\ell_1})], & \text{if } t{=}\ell_1; \\ n\ell_2[(k+\ell_1+\ell_3)^{(n-1)}_{\ell_2} - ((k+\ell_1)^{(n-1)}_{\ell_2} + (k+\ell_3)^{(n-1)}_{\ell_2})], & \text{if } t{=}\ell_2; \\ n\ell_3[(k+\ell_1+\ell_2)^{(n-1)}_{\ell_3} - ((k+\ell_1)^{(n-1)}_{\ell_3} + (k+\ell_2)^{(n-1)}_{\ell_3})], & \text{if } t{=}\ell_3. \end{cases} \quad (23)$$

Proof. The proof follows from (3), (20) and $k^{(n-1)}_t$. $\square$

Lemma 3.4. *If n is a positive integer, then*

$$\Delta_{\ell_1,\ell_2,\ell_3} k^{(n)}_{\ell_1,\ell_2,\ell_3} = n\ell_1 \Delta^2_{\ell_2,\ell_3} k^{(n-1)}_{\ell_1} + n\ell_2 \Delta^2_{\ell_1,\ell_3} k^{(n-1)}_{\ell_2} + n\ell_3 \Delta^2_{\ell_1,\ell_2} k^{(n-1)}_{\ell_3}. \quad (24)$$

Proof. The proof follows from (6) and $\Delta_\ell k_\ell^{(n)} = n\ell k_\ell^{(n-1)}$. □

Corollary 3.5. *Let n be a positive integer. Then*

$$\Delta_{\ell,\ell,\ell} k_{\ell,\ell,\ell}^{(n)} = (n\ell)_\ell^{(3)} k_{\ell,\ell,\ell}^{(n-3)}. \tag{25}$$

The following theorem is the generalized version of Newton's formula with reference to $\Delta_{\ell,\ell,\ell}$.

Theorem 3.6. *Let $f(k)$ be a polynomial in k of degree $3n$. Then $f(k)$ can be expressed as*

$$f(k) = f(0) + \frac{\Delta_{\ell,\ell,\ell} f(0)}{3!\ell^3} k_\ell^{(3)} + \frac{\Delta^2_{\ell,\ell,\ell} f(0)}{6!\ell^6} k_\ell^{(6)} + \cdots + \frac{\Delta^n_{\ell,\ell,\ell} f(0)}{(3n)!\ell^{3n}} k_\ell^{(3n)}. \tag{26}$$

Proof. Assume that

$$f(k) = a_0 + a_1 k_\ell^{(3)} + a_2 k_\ell^{(6)} + \cdots + a_n k_\ell^{(3n)}. \tag{27}$$

The coefficients are determined from the relation

$$\Delta^r_{\ell,\ell,\ell} f(0) = a_r (3r)! \ell^{3r}. \tag{28}$$

The rest of the proof follows from (27) and (28). □

Corollary 3.7. *Let $f(k)$ be a polynomial in k of degree $3n$. Then $f(k-t)$ can be expressed as*

$$f(t) + \frac{\Delta_{\ell,\ell,\ell} f(t)}{3!\ell^3} (k-t)_\ell^{(3)} + \frac{\Delta^2_{\ell,\ell,\ell} f(t)}{6!\ell^6} (k-t)_\ell^{(6)} + \cdots + \frac{\Delta^n_{\ell,\ell,\ell} f(t)}{(3n)!\ell^{3n}} (k-t)_\ell^{(3n)}. \tag{29}$$

Proof. Replacing 0 by t and k by $(k-t)$ in (26) we obtain the result as desired. □

4. Inverse of Generalized Difference Operator of the Third Kind and its Applications

In this section, we define the inverse $\Delta^{-1}_{\ell_1,\ell_2,\ell_3}$ and present some results using the inverse which will be used to find $P^2 S^n$.

Definition 4.1. The inverse of generalized difference operator of the third kind denoted by $\Delta^{-1}_{\ell_1,\ell_2,\ell_3}$ is defined as follows.

If $\Delta_{\ell_1,\ell_2,\ell_3} z(k) = y(k)$, then

$$z(k) = \Delta^{-1}_{\ell_1,\ell_2,\ell_3} y(k) + c_{2j}\left(\frac{k^{(2)}_{\ell_2}}{2\ell_2^2}\right) + c_{1j}\left(\frac{k^{(1)}_{\ell_1}}{\ell_1}\right) + c_{0j}\,, \tag{30}$$

respectively, where c_{0j}, c_{1j} and c_{2j}'s are constants which depend on $k - n^*\ell$.

Lemma 4.2. *If n is a positive integer and $k \in [n\ell, \infty)$, then*

$$\Delta^{-1}_{\ell_1,\ell_2,\ell_3}\big(k^{(n)}_{\ell_1,\ell_2,\ell_3}\big) = \frac{k^{(n+1)}_{\ell 1}}{\ell_1(n+1)} + \frac{k^{(n+1)}_{\ell 2}}{\ell_2(n+1)} + \frac{k^{(n+1)}_{\ell 3}}{\ell_3(n+1)}$$

$$+c_{2j}\left(\frac{k^{(2)}_{\ell_2}}{2\ell_2^2}\right) + c_{1j}\left(\frac{k^{(1)}_{\ell_1}}{\ell_1}\right) + c_{0j} \tag{31}$$

and

$$\Delta^{-1}_{\ell,\ell,\ell}\big(k^{(n)}_{\ell,\ell,\ell}\big) = 3\,\frac{k^{(n+1)}_{\ell}}{\ell(n+1)} + c_{2j}\left(\frac{k^{(2)}_{\ell_2}}{2\ell_2^2}\right) + c_{1j}\left(\frac{k^{(1)}_{\ell_1}}{\ell_1}\right) + c_{0j}. \tag{32}$$

Proof. The proof follows from (30) and $\Delta_\ell(k^{(n+1)}_\ell + c) = \ell(n+1)k^{(n)}_\ell$. □

Theorem 4.3. *There exits constants c_{0j}, c_{1j} and c_{2j} which depend on $k - n^*\ell$ such that*

$$\Delta^{-1}_{\ell,\ell,\ell} y(k) = \sum_{t=2}^{n^*}\sum_{s=1}^{n^*}\sum_{r=0}^{n^*} y(k - t\ell - s\ell - r\ell) + c_{2j}\Big(\frac{k^{(2)}_\ell}{2\ell^2}\Big) + c_{1j}\Big(\frac{k}{\ell}\Big) + c_{0j}. \tag{33}$$

Proof. The proof follows by the relation $\Delta_{\ell,\ell,\ell}\Big\{\sum\limits_{t=2}^{n^*}\sum\limits_{s=1}^{n^*}\sum\limits_{r=0}^{n^*} y(k - t\ell - s\ell - r\ell) + c_{2j}\left(\frac{k^{(2)}_\ell}{2\ell^2}\right) + c_{1j}\left(\frac{k}{\ell}\right) + c_{0j}\Big\} = y(k)$. □

Lemma 4.4. *If $\lambda \neq 1, k \geq 3\ell$ and P_k is any function of k, then*

$$\sum_{t=2}^{n^*}\sum_{s=1}^{n^*}\sum_{r=0}^{n^*} \lambda^{k-t\ell-s\ell-r\ell} P_{k-t\ell-s\ell-r\ell} = \frac{\lambda^k}{(\lambda^\ell-1)^3}\Big\{1 - \frac{\lambda^\ell\Delta_\ell}{(\lambda^\ell-1)}$$

$$+ \frac{\lambda^{2\ell}\Delta^2_\ell}{(\lambda^\ell-1)^2} - \cdots\Big\}^3 P_k + c_{2j}\left(\frac{k^{(2)}_\ell}{2\ell^2}\right) + c_{1j}\left(\frac{k}{\ell}\right) + c_{0j}. \tag{34}$$

Proof. Let $\Delta_{\ell,\ell,\ell}\lambda^k F_k = \lambda^k P_k$, where $P_k = (\lambda^\ell E^\ell - 1)^3 F_k$. Operating both sides by $\Delta^{-1}_{\ell,\ell,\ell}$, we obtain

$$\Delta^{-1}_{\ell,\ell,\ell}\lambda^k P_k = \lambda^k F_k + c_{2j}\left(\frac{k_\ell^{(2)}}{2\ell^2}\right) + c_{1j}\left(\frac{k}{\ell}\right) + c_{0j}$$

$$= \lambda^k(\lambda^\ell E^\ell - 1)^{-3} P_k + c_{2j}\left(\frac{k_\ell^{(2)}}{2\ell^2}\right) + c_{1j}\left(\frac{k}{\ell}\right) + c_{0j}.$$

The rest of the proof follows from (33) and the binomial theorem. □

Lemma 4.5. *The relation between* $\Delta^{-1}_{\ell,\ell,\ell}$ *and* Δ^{-1}_ℓ *is*

$$\sum_{p=0}^{\ell-1}\sum_{q=0}^{\ell-1}\sum_{r=0}^{\ell-1}\Delta^{-1}_{\ell,\ell,\ell}u(k+p+q+r) = \Delta^{-1}(\Delta^{-1}(\Delta^{-1}u(k)))$$

$$+c_{2j}\left(\frac{k_\ell^{(2)}}{2\ell^2}\right) + c_{1j}\left(\frac{k}{\ell}\right) + c_{0j}.$$

Proof. The proof follows form $\sum_{i=0}^{\ell-1}\Delta^{-1}_\ell u(k+i) = \Delta^{-1}u(k) + c$ and (33). □

The following two lemmas are easy deductions.

Lemma 4.6. *If* S_r^n*'s are the Stirling numbers of the second kind, then*

$$(k+2\ell)^n - 2(k+\ell)^n + k^n = \frac{1}{3}\sum_{r=1}^{n} S_r^n \ell^{n-r} k^{(r)}_{\ell,\ell,\ell}. \tag{35}$$

Lemma 4.7. *If* n *is a positive integer, then*

$$(i)\quad \Delta^{-1}_{\ell,\ell,\ell}k_\ell^{(n)} = \frac{k^{(n+5)}_{\ell,\ell,\ell}}{3n(n+1)\cdots(n+5)\ell^5} + c_{2j}\left(\frac{k_\ell^{(2)}}{2\ell^2}\right) + c_{1j}\left(\frac{k}{\ell}\right) + c_{0j}, \tag{36}$$

$$(ii)\quad k^n = \frac{1}{3}\sum_{r=1}^{n} S_r^n \ell^{n-r}\Delta^{-1}_{\ell,\ell,\ell}k^{(r)}_{\ell,\ell,\ell}. \tag{37}$$

Lemma 4.8. *If* ℓ *is any positive real number and* $k \in [4\ell, \infty)$*, then*

$$\sum_{t=2}^{n^*}\sum_{s=1}^{n^*}\sum_{r=0}^{n^*}(k - t\ell - s\ell - r\ell)^3 + c_{2j}\left(\frac{k_\ell^{(2)}}{2\ell^2}\right) + c_{1j}\left(\frac{k}{\ell}\right) + c_{0j}$$

$$= \frac{k_\ell^{(6)}}{120\ell^3} + \frac{k_\ell^{(5)}}{20\ell^2} + \frac{k_\ell^{(4)}}{24\ell}. \tag{38}$$

Proof. The proof follows from (33), (6) and $k^n = \sum_{r=1}^{n} S_r^n \ell^{n-r} k_\ell^{(r)}$ (see [2]). □

The following theorem is the general rule to find the value of $P^2 S^n$, where S^n is the sum of n-th powers of an arithmetic progression.

Theorem 4.9. *If S_t^n's are the Stirling numbers of second kind, then*

$$\sum_{t=2}^{n^*}\sum_{s=1}^{n^*}\sum_{r=0}^{n^*}(k-t\ell-s\ell-r\ell)^n$$

$$= \sum_{t=1}^{n} \frac{S_t^n \ell^{n-(t+3)}}{(t+1)(t+2)(t+3)} \left\{ k_\ell^{(t+3)} - \big((n+2)\ell+j\big)_\ell^{(t+3)} \right\}$$

$$+ \sum_{t=2}^{n^*}\sum_{s=1}^{n^*}\sum_{r=0}^{n^*} \big((n+2)\ell - t\ell - s\ell - r\ell + i\big)^n + \left((n+2) + \frac{(j-k)}{\ell}\right)$$

$$\left\{ \sum_{t=1}^{n} \frac{S_t^n \ell^{n-(t+3)}}{(t+1)(t+2)(t+3)} \left[\big((n+3)\ell+j\big)_\ell^{(t+3)} - \big((n+2)\ell+j\big)_\ell^{(t+3)} \right] \right.$$

$$\left. - \sum_{t=2}^{n^*}\sum_{s=1}^{n^*} \big((n+3)\ell - t\ell - s\ell + i\big)^n \right\} + \frac{1}{2\ell^2}[((n+2)\ell+j)\big(2k-((n-3)\ell+j)\big)$$

$$-k_\ell^{(2)}]\left\{ \sum_{t=1}^{n} \frac{S_t^n \ell^{n-(t+3)}}{(t+1)(t+2)(t+3)} \left[((n+4)\ell+j)_\ell^{(t+3)} - ((n+2)\ell+j)_\ell^{(t+3)} \right] \right.$$

$$+(-2)\left(\sum_{t=1}^{n} \frac{S_t^n \ell^{n-(t+3)}}{(t+1)(t+2)(t+3)} \left[\big((n+3)\ell+j\big)_\ell^{(t+3)} \right.\right.$$

$$\left.\left.\left. - \big((n+2)\ell+j\big)_\ell^{(t+3)} \right] \right) - \sum_{t=2}^{n} \big((n+3)\ell - t\ell + j\big)^n \right\} = P^2 S^n. \tag{39}$$

Proof. From (38), we obtain

$$\sum_{t=2}^{n^*}\sum_{s=1}^{n^*}\sum_{r=0}^{n^*}(k-t\ell-s\ell-r\ell)^n + c_{2j}\left(\frac{k_\ell^{(2)}}{2\ell^2}\right) + c_{1j}\left(\frac{k}{\ell}\right) + c_{0j}$$

$$= \sum_{t=1}^{n} \frac{S_t^n \ell^{n-(t+3)}}{(t+1)(t+2)(t+3)} k_\ell^{(t+3)}. \tag{40}$$

Replacing k by $(n+2)\ell+j$, we get

$$c_{0j}=\sum_{t=1}^{n}\frac{S_t^n\ell^{n-(t+3)}}{(t+1)(t+2)(t+3)}((n+2)\ell+i)_\ell^{(t+3)}-\sum_{s=2}^{n^*}\sum_{r=0}^{n^*}((n+2)\ell$$
$$-s\ell-r\ell+j)^n-c_{2j}\left(\frac{((n+2)\ell+j)_\ell^{(2)}}{2\ell^2}\right)-\frac{c_{1j}}{\ell}((n+2)\ell+j). \qquad (41)$$

Substituting (41) in (40) and replacing k by $(n+3)\ell+j$, we obtain

$$c_{1j}=\sum_{t=1}^{n}\frac{S_t^n\ell^{n-(t+3)}}{(t+1)(t+2)(t+3)}[((n+2)\ell+j)_\ell^{(t+3)}-((n+1)\ell+j)_\ell^{(t+2)}]$$
$$+\frac{c_{2j}}{2\ell^2}[((n+2)\ell+j)_\ell^{(2)}-((n+3)\ell+j)_\ell^{(2)}]-\sum_{s=2}^{n^*}((n+3)\ell-t\ell-s\ell+j)^n. \quad (42)$$

Substituting (42) and (41) in (40) and replacing k by $((n+4)\ell+j)$, we obtain

$$c_{2j}=\sum_{t=1}^{n}\frac{S_t^n\ell^{n-(t+3)}}{(t+1)(t+2)(t+3)}[((n+4)\ell+j)_\ell^{(t+3)}-((n+2)\ell+i)_\ell^{(t+2)}]$$
$$+\sum_{t=2}^{n^*}\sum_{s=1}^{n^*}\sum_{r=0}^{n^*}((n+2)\ell-t\ell-s\ell-r\ell+j)^n$$
$$-2\Big\{\sum_{t=1}^{n}\frac{S_t^n\ell^{n-(t+3)}}{(t+1)(t+2)(t+3)}[((n+3)\ell+j)_\ell^{(t+3)}-((n+2)\ell+i)_\ell^{(t+2)}]$$
$$-\sum_{t=2}^{n^*}\sum_{s=1}^{n^*}((n+3)\ell-t\ell-s\ell+j)^n\Big\}-\sum_{t=2}^{n^*}\sum_{s=1}^{n^*}\sum_{r=0}^{n^*}((n+4)\ell-t\ell-s\ell-r\ell+j)^n. \qquad (43)$$

The proof follows from (40), (41), (42) and (43). □

Example 4.10. For the A.P. $3,10,17,\cdots,101$, we shall find S^5, PS^5 and P^2S^5 where $S^5=3^5+10^5+17^5+\cdots+101^5$, the sum of fifth powers of the A.P., $PS^5=3^5+\overline{3^5+10^5}+\overline{3^5+10^5+17^5}+\cdots+\overline{3^5+10^5+17^5+\cdots+101^5}$, the sum of all partial sums from S^5 and $P^2S^5=3^5+(3^5+\overline{3^5+10^5})+(3^5+\overline{3^5+10^5}+\overline{3^5+10^5+17^5})+\cdots+(3^5+\overline{3^5+10^5}+\cdots+\overline{3^5+10^5+17^5+\cdots+101^5})$, the sum of all partial sums from PS^5.

Solution. Taking $n=5, j=3, \ell=7$ and $k=122$ in Theorem 4.9, we obtain $P^2S^5=$ 201550764800. Similarly we can find the value of PS^n and S^n (refer [2], [7]).

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