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# POSITIVE SOLUTION FOR THIRD ORDER THREE-POINT BOUNDARY VALUE PROBLEMS ON TIME SCALES

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**Abstract:** We consider the three point third order boundary value problem on time scales

$$y^{\Delta^3}(t) + f(t, y(t), y^{\Delta}(t), y^{\Delta^2}(t)) = 0, \quad t \in [t_1, \sigma^3(t_3)]\,,$$

subject to the general boundary conditions

$$\alpha_{11}y(t_1) + \alpha_{12}y^{\Delta}(t_1) + \alpha_{13}y^{\Delta^2}(t_1) = 0\,,$$
$$\alpha_{21}y(t_2) + \alpha_{22}y^{\Delta}(t_2) + \alpha_{23}y^{\Delta^2}(t_2) = 0\,,$$
$$\alpha_{31}y(\sigma^3(t_3)) + \alpha_{32}y^{\Delta}(\sigma^2(t_3)) + \alpha_{33}y^{\Delta^2}(\sigma(t_3)) = 0\,,$$

where $t_1 < t_2 < \sigma^3(t_3)$ and $\alpha_{ij}$, for $i, j = 1, 2, 3$ are real constants. We establish a criterion for the existense of at least one positive solution by utilizing Krasnosel'skii Fixed Point Theorem for operators on a cone in a Banach space.

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## 1. Introduction

The study of existence of positive solutions of boundary value problems for higher order differential equations on time scales has gained prominence and it is a rapidly growing field, since it arises in many applications. By a time scale we mean a nonempty closed subset of $\mathbb{R}$. The time scale calculus and notation for delta differentiation, integration as well as concepts for dynamic equations on time scales, we refer to the introductory book on time scales by Bohner and Peterson [6]. For advances on dynamic equations on time scales, we refer to the book by Bohner and Petrson [7]. We denote the time scale by the symbol $\mathbb{T}$. The existence of positive solutions for boundary value problems has been studied by many authors, first for ordinary differential equations, then for finite difference equations, and recently, unifying results for dynamic equations. To mention a few, we list some papers, Eloe and Henderson [8, 9, 10], Erbe and Wang [11], Ma [15] and Sun and Wen [19] for ordinary differential equations; Merdivenci Atici and Guseino [16], Anderson and Avery [4] for difference equations and Anderson [3], Kaufmann [13] and Sun [17] for time scales. Anderson and Davis [5] obtained the existence of positive solutions for the BVP

$$y'''(t) = f(t, y), \quad t_1 \le t \le t_3\,,$$
$$y(t_1) = y'(t_2) = 0, \quad \gamma y(t_3) + \delta y''(t_3) = 0\,,$$

by using Guo-Krasnosel'skii Fixed Point Theorem and Leggett-Williams Fixed Point Theorem. We generalize the results to the nonlinear differential equation on time scales

$$y^{\Delta^3}(t) + f(t, y(t), y^{\Delta}(t), y^{\Delta^2}(t)) = 0, \quad t \in [t_1, \sigma^3(t_3)]\,, \tag{1}$$

subject to the general three point boundary conditions

$$\begin{aligned} \alpha_{11}y(t_1) + \alpha_{12}y^{\Delta}(t_1) + \alpha_{13}y^{\Delta^2}(t_1) &= 0\,,\\ \alpha_{21}y(t_2) + \alpha_{22}y^{\Delta}(t_2) + \alpha_{23}y^{\Delta^2}(t_2) &= 0\,,\\ \alpha_{31}y(\sigma^3(t_3)) + \alpha_{32}y^{\Delta}(\sigma^2(t_3)) + \alpha_{33}y^{\Delta^2}(\sigma(t_3)) &= 0\,, \end{aligned} \tag{2}$$

where $t_1 < t_2 < \sigma^3(t_3)$ and $\alpha_{ij}$, for $i, j = 1, 2, 3$ are real constants. The BVPs of this form arise in the modeling of nonlinear diffusion via nonlinear sources, thermal ignition of gases, and in chemical concentrations in biological problems. In these applied settings, only positive solutions are meaningful. We make the following notation through out: Let $\beta_i = \alpha_{i1}t_i + \alpha_{i2}$ and $\gamma_i = \alpha_{i1}t_i^2 + \alpha_{i2}(t_i + \sigma(t_i)) + 2\alpha_{i3}$ for $i = 1, 2$, $\beta_3 = \alpha_{31}\sigma^3(t_3) + \alpha_{32}$ and $\gamma_3 =$

$\alpha_{31}(\sigma^3(t_3))^2+\alpha_{32}(\sigma^2(t_3)+\sigma^3(t_3))+2\alpha_{33}$. We define

$$m_{ij}=\frac{\alpha_{i1}\gamma_j-\alpha_{j1}\gamma_i}{2(\alpha_{i1}\beta_j-\alpha_{j1}\beta_i)};\quad M_{ij}=\frac{\beta_i\gamma_j-\beta_j\gamma_i}{\alpha_{i1}\beta_j-\alpha_{j1}\beta_i}.$$

Also let

$$m_1=\max\{m_{12},m_{13},m_{23}\},$$

$$m_2=\min\{m_{23}+\sqrt{m_{23}^2-M_{23}};m_{13}+\sqrt{m_{13}^2-M_{13}}\},$$

$$d=\alpha_{11}(\beta_2\gamma_3-\beta_3\gamma_2)-\beta_1(\alpha_{21}\gamma_3-\alpha_{31}\gamma_2)+\gamma_1(\alpha_{21}\beta_3-\alpha_{31}\beta_2)$$

and

$$l_i=\alpha_{i1}\sigma(s)\sigma^2(s)-(\sigma(s)+\sigma^2(s))\beta_i+\gamma_i,\quad \text{for}\quad i=1,2,3.$$

We use the following assumptions throughout the paper:

(A1) $f:[t_1,\sigma^3(t_3)]\times\mathbb{R}^{+^3}\to\mathbb{R}^+$ is continuous;

(A2) $\alpha_{11}>0,\quad \alpha_{21}>0,\quad \alpha_{31}>0$ and $\frac{\alpha_{12}}{\alpha_{11}}>\frac{\alpha_{22}}{\alpha_{21}}>\frac{\alpha_{32}}{\alpha_{31}}$;

(A3) $m_1\le t_1<t_2<t_3\le m_2;\ 2\alpha_{13}\alpha_{11}>\alpha_{12}^2,\quad 2\alpha_{23}\alpha_{21}>\alpha_{22}^2, 2\alpha_{33}\alpha_{31}>\alpha_{32}^2$;

(A4) $m_{23}^2>M_{23},\quad m_{12}^2<M_{12},\quad m_{13}^2>M_{13}$, $d>0$; and

(A5) The point $t\in[t_1,\sigma^3(t_3)]$ is not left dense and right scattered at the same time.

Define the nonnegative extended real numbers $f_0,f^0,f_\infty,f^\infty$ by

$$f_0=\lim_{(y,y^\Delta,y^{\Delta^2})\to(0^+,0^+,0^+)}\ \min_{t\in[t_1,\sigma^3(t_3)]}\frac{f(t,y,y^\Delta,y^{\Delta^2})}{y},$$

$$f^0=\lim_{(y,y^\Delta,y^{\Delta^2})\to(0^+,0^+,0^+)}\ \max_{t\in[t_1,\sigma^3(t_3)]}\frac{f(t,y,y^\Delta,y^{\Delta^2})}{y},$$

$$f_\infty=\lim_{(y,y^\Delta,y^{\Delta^2})\to(\infty,\infty,\infty)}\ \min_{t\in[t_1,\sigma^3(t_3)]}\frac{f(t,y,y^\Delta,y^{\Delta^2})}{y},$$

$$f^\infty=\lim_{(y,y^\Delta,y^{\Delta^2})\to(\infty,\infty,\infty)}\ \max_{t\in[t_1,\sigma^3(t_3)]}\frac{f(t,y,y^\Delta,y^{\Delta^2})}{y}$$

and assume that they will exist.

## 2. Green's Function and Bounds

In this section, we construct the Green's function for the homogeneous problem corresponding to the BVP (1)-(2) in six different intervals and we estimate the bounds for the Green's function.

Let $G(t,s)$ be the Green's function for the BVP $-y^{\Delta^3}(t) = 0$ satisfying (2). After computation, the Green's function $G(t,s)$ can be obtained as

$$G(t,s) = \begin{cases} \underset{t\in[t_1,t_2]}{G_1(t,s)} = \begin{cases} G_{11}(t,s), & \mathrm{t_1 \le t < s < t_2} < \sigma^3(\mathrm{t_3}), \\ G_{12}(t,s), & \mathrm{t_1} < \sigma(\mathrm{s}) < \mathrm{t \le t_2} < \sigma^3(\mathrm{t_3}), \\ G_{13}(t,s), & \mathrm{t_1 \le t < t_2 < s} < \sigma^3(\mathrm{t_3}), \end{cases} \\ \\ \underset{t\in[t_2,\sigma^3(t_3)]}{G_2(t,s)} = \begin{cases} G_{21}(t,s), & \mathrm{t_1 < t_2 \le t < s} < \sigma^3(\mathrm{t_3}), \\ G_{22}(t,s), & \mathrm{t_1 < t_2} < \sigma(\mathrm{s}) < \mathrm{t} \le \sigma^3(\mathrm{t_3}), \\ G_{23}(t,s), & \mathrm{t_1} \le \sigma(\mathrm{s}) < \mathrm{t_2 < t} < \sigma^3(\mathrm{t_3}), \end{cases} \end{cases} \tag{3}$$

where

$$\begin{aligned} G_{11}(t,s) =& \frac{1}{2d}[-(\beta_1\gamma_3 - \beta_3\gamma_1) + t(\alpha_{11}\gamma_3 - \alpha_{31}\gamma_1) - t^2(\alpha_{11}\beta_3 - \alpha_{31}\beta_1)]l_2 \\ &+ \frac{1}{2d}[(\beta_1\gamma_2 - \beta_2\gamma_1) - t(\alpha_{11}\gamma_2 - \alpha_{21}\gamma_1) + t^2(\alpha_{11}\beta_2 - \alpha_{21}\beta_1)]l_3, \\ G_{12}(t,s) =& \frac{1}{2d}[-(\beta_2\gamma_3 - \beta_3\gamma_2) + t(\alpha_{21}\gamma_3 - \alpha_{31}\gamma_2) - t^2(\alpha_{21}\beta_3 - \alpha_{31}\beta_2)]l_1, \\ G_{13}(t,s) =& \frac{1}{2d}[(\beta_1\gamma_2 - \beta_2\gamma_1) - t(\alpha_{11}\gamma_2 - \alpha_{21}\gamma_1) + t^2(\alpha_{11}\beta_2 - \alpha_{21}\beta_1)]l_3, \\ G_{21}(t,s) =& \frac{1}{2d}[(\beta_1\gamma_2 - \beta_2\gamma_1) - t(\alpha_{11}\gamma_2 - \alpha_{21}\gamma_1) + t^2(\alpha_{11}\beta_2 - \alpha_{21}\beta_1)]l_3, \\ G_{22}(t,s) =& \frac{1}{2d}[-(\beta_2\gamma_3 - \beta_3\gamma_2) + t(\alpha_{21}\gamma_3 - \alpha_{31}\gamma_2) - t^2(\alpha_{21}\beta_3 - \alpha_{31}\beta_2)]l_1, \\ &+ \frac{1}{2d}[(\beta_1\gamma_3 - \beta_3\gamma_1) - t(\alpha_{11}\gamma_3 - \alpha_{31}\gamma_1) + t^2(\alpha_{11}\beta_3 - \alpha_{31}\beta_1)]l_2, \\ G_{23}(t,s) =& \frac{1}{2d}[-(\beta_2\gamma_3 - \beta_3\gamma_2) + t(\alpha_{21}\gamma_3 - \alpha_{31}\gamma_2) - t^2(\alpha_{21}\beta_3 - \alpha_{31}\beta_2)]l_1. \end{aligned}$$

The graph in Figure 1 demonstrates the Green's function for the BVP (1)-(2) should be taken in the form of (3). Here $s \in [t_1, t_3]$.

**Theorem 2.1.** *Assume that the conditions (A2)-(A4) are satisfied. Then*

$$\gamma G(\sigma(s), s) \le G(t,s) \le G(\sigma(s), s), \quad \textit{for all} \quad (t,s) \in [t_1, \sigma^3(t_3)] \times [t_1, t_3], \tag{4}$$

*where*

$0 < \gamma$

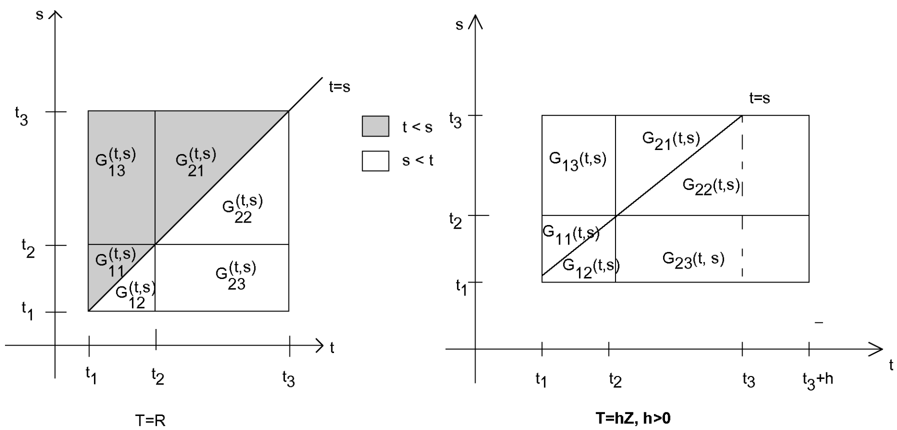


Figure 1:

$$= \min\left\{\frac{G_{12}(\sigma^3(t_3),s)}{G_{12}(t_1,s)}, \frac{G_{13}(t_1,s)}{G_{13}(\sigma^3(t_3),s)}, \frac{G_{11}(t_1,s)}{G_{11}(\sigma^3(t_3),s)}, \frac{G_{11}(\sigma^3(t_3),s)}{G_{11}(t_1,s)}\right\} < 1.$$

*Proof.* The Green's function $G(t,s)$ is given in (3) in six different cases. In each case we prove the inequality as in (4). Clearly

$$G(t,s) > 0 \quad \text{on} \quad [t_1, \sigma^3(t_3)] \times [t_1, t_3]. \tag{5}$$

*Case (i).* For $t_1 < \sigma(s) < t \le t_2 < \sigma^3(t_3)$

$$\frac{G(t,s)}{G(\sigma(s),s)} = \frac{G_{12}(t,s)}{G_{12}(\sigma(s),s)}$$

$$= \frac{[-(\beta_2\gamma_3 - \beta_3\gamma_2) + t(\alpha_{21}\gamma_3 - \alpha_{31}\gamma_2) - t^2(\alpha_{21}\beta_3 - \alpha_{31}\beta_2)]}{[-(\beta_2\gamma_3 - \beta_3\gamma_2) + \sigma(s)(\alpha_{21}\gamma_3 - \alpha_{31}\gamma_2) - (\sigma(s))^2(\alpha_{21}\beta_3 - \alpha_{31}\beta_2)]},$$

from, $(A3)$ and $(A4)$, we have $G_{12}(t,s) \le G_{12}(\sigma(s),s)$. Therefore,

$$G(t,s) \le G(\sigma(s),s), \quad \text{for all} \quad (t,s) \in [t_1, \sigma^3(t_3)] \times [t_1, t_3].$$

And also, from $(A2)$, we have

$$\frac{G(t,s)}{G(\sigma(s),s)} = \frac{G_{12}(t,s)}{G_{12}(\sigma(s),s)} \ge \frac{G_{12}(t,s)}{G_{12}(t_1,s)} \ge \frac{G_{12}(\sigma^3(t_3),s)}{G_{12}(t_1,s)}.$$

Therefore,

$$G(t,s) \ge \frac{G_{12}(\sigma^3(t_3),s)}{G_{12}(t_1,s)} G(\sigma(s),s), \quad \text{for all} \quad (t,s) \in [t_1, \sigma^3(t_3)] \times [t_1, t_3].$$

*Case (ii).* For $t_1 \leq t < t_2 < s < \sigma^3(t_3)$

$$\frac{G(t,s)}{G(\sigma(s),s)} = \frac{G_{13}(t,s)}{G_{13}(\sigma(s),s)}$$

$$= \frac{[(\beta_1\gamma_2 - \beta_2\gamma_1) - t(\alpha_{11}\gamma_2 - \alpha_{21}\gamma_1) + t^2(\alpha_{11}\beta_2 - \alpha_{21}\beta_1)]}{[(\beta_1\gamma_2 - \beta_2\gamma_1) - \sigma(s)(\alpha_{11}\gamma_2 - \alpha_{21}\gamma_1) + (\sigma(s))^2(\alpha_{11}\beta_2 - \alpha_{21}\beta_1)]},$$

from $(A3)$ and $(A4)$, we have $G_{13}(t,s) \leq G_{13}(\sigma(s),s)$. Therefore,

$$G(t,s) \leq G(\sigma(s),s) \text{ for all } (t,s) \in [t_1, \sigma^3(t_3)] \times [t_1, t_3].$$

And also, from $(A2)$, we have

$$\frac{G(t,s)}{G(\sigma(s),s)} = \frac{G_{13}(t,s)}{G_{13}(\sigma(s),s)} \geq \frac{G_{13}(t,s)}{G_{13}(\sigma^3(t_3),s)} \geq \frac{G_{13}(t_1,s)}{G_{13}(\sigma^3(t_3),s)}.$$

Therefore,

$$G(t,s) \geq \frac{G_{13}(t_1,s)}{G_{13}(\sigma^3(t_3),s)} G(\sigma(s),s), \quad \text{for all } (t,s) \in [t_1, \sigma^3(t_3)] \times [t_1, t_3].$$

*Case (iii).* For $t_1 \leq t < s < t_2 < \sigma^3(t_3)$. From $(A3)$ and Case (ii), we have $G_{11}(t,s) \leq G_{11}(\sigma(s),s)$. Therefore,

$$G(t,s) \leq G(\sigma(s),s) \text{ for all } (t,s) \in [t_1, \sigma^3(t_3)] \times [t_1, t_3].$$

And also, from $(A2)$, we have

$$\frac{G(t,s)}{G(\sigma(s),s)} \geq \min\left\{\frac{G_{11}(\sigma^3(t_3),s)}{G_{11}(t_1,s)}, \frac{G_{11}(t_1,s)}{G_{11}(\sigma^3(t_3),s)}, \frac{G_{13}(t_1,s)}{G_{13}(\sigma^3(t_3),s)}\right\}.$$

Therefore,

$$G(t,s) \geq \min\left\{\frac{G_{11}(\sigma^3(t_3),s)}{G_{11}(t_1,s)}, \frac{G_{11}(t_1,s)}{G_{11}(\sigma^3(t_3),s)}, \frac{G_{13}(t_1,s)}{G_{13}(\sigma^3(t_3),s)}\right\} G(\sigma(s),s),$$

for all $(t,s) \in [t_1, \sigma^3(t_3)] \times [t_1, t_3]$.

*Case (iv).* For $t_1 < t_2 < \sigma(s) < t \leq \sigma^3(t_3)$. From Case (i) and Case (ii), we have

$$G(t,s) \leq G(\sigma(s),s) \text{ for all } (t,s) \in [t_1, \sigma^3(t_3)] \times [t_1, t_3],$$

and

$$G(t,s) \geq \frac{G_{12}(\sigma^3(t_3),s)}{G_{12}(t_1,s)} G(\sigma(s),s), \quad \text{for all } (t,s) \in [t_1, \sigma^3(t_3)] \times [t_1, t_3].$$

*Case (v).* For $t_1 < t_2 \leq t < s < \sigma^3(t_3)$. From Case (ii), we have

$$G(t,s) \leq G(\sigma(s),s) \text{ for all } (t,s) \in [t_1, \sigma^3(t_3)] \times [t_1, t_3],$$

and

$$G(t,s) \geq \frac{G_{13}(t_1,s)}{G_{13}(\sigma^3(t_3),s)} G(\sigma(s),s), \quad \text{for all } (t,s) \in [t_1, \sigma^3(t_3)] \times [t_1, t_3].$$

*Case (vi).* For $t_1 \leq \sigma(s) < t_2 < t < \sigma^3(t_3)$. From Case (i), we have

$$G(t,s) \leq G(\sigma(s),s) \quad \text{for all} \quad (t,s) \in [t_1, \sigma^3(t_3)] \times [t_1, t_3],$$

and

$$G(t,s) \geq \frac{G_{12}(\sigma^3(t_3),s)}{G_{12}(t_1,s)} G(\sigma(s),s), \quad \text{for all} \quad (t,s) \in [t_1, \sigma^3(t_3)] \times [t_1, t_3].$$

By consolidating all the above cases, we get

$$\gamma G(\sigma(s),s) \leq G(t,s) \leq G(\sigma(s),s), \quad \text{for all} \quad (t,s) \in [t_1, \sigma^3(t_3)] \times [t_1, t_3],$$

where

$$0 < \gamma = \min\left\{ \frac{G_{12}(\sigma^3(t_3),s)}{G_{12}(t_1,s)}, \frac{G_{13}(t_1,s)}{G_{13}(\sigma^3(t_3),s)}, \frac{G_{11}(t_1,s)}{G_{11}(\sigma^3(t_3),s)}, \frac{G_{11}(\sigma^3(t_3),s)}{G_{11}(t_1,s)} \right\} < 1. \quad \square$$

## 3. Existence of Positive Solution

In this section, first we prove a lemma which is needed in our main result and establish a criteria for the existence of at least one positive solution of the BVP (1)-(2).

Let $y(t)$ be the solution of the BVP (1)- (2), and is given by

$$y(t) = \int_{t_1}^{\sigma(t_3)} G(t,s) f(s, y(s), y^{\Delta}(s), y^{\Delta^2}(s)) \Delta s, \quad \text{for all } t \in [t_1, \sigma^3(t_3)]. \qquad (6)$$

Define

$$X = \left\{ \; u \; | u \in C^3[t_1, \sigma^3(t_3)] \; \right\},$$

with norm

$$\| \, u \, \| = \max_{t \in [t_1, \sigma^3(t_3)]} | \, u(t) \, | \, .$$

Then $(X, \| \, . \, \|)$ is a Banach space. Define a set $\kappa$ by

$$\kappa = \left\{ u \in X : u(t) \geq 0 \;\; \text{on } [t_1, \sigma^3(t_3)] \;\; \text{and} \;\; \min_{t \in [t_1, \sigma^3(t_3)]} u(t) \geq \gamma \, \| \, u \, \| \right\}. \qquad (7)$$

Then it is easy to see that $\kappa$ is a positive cone in $X$.

Define the operator $T : \kappa \to X$ by

$$(Ty)(t) = \int_{t_1}^{\sigma(t_3)} G(t,s) f(s, y(s), y^{\Delta}(s), y^{\Delta^2}(s)) \Delta s, \quad \text{for all } t \in [t_1, \sigma^3(t_3)]. \qquad (8)$$

If $y \in \kappa$ is a fixed point of $T$, then $y$ satisfies (6) and hence $y$ is a positive solution of the BVP (1)-(2). We seek a fixed point of the operator $T$ in the cone $\kappa$.

**Lemma 3.1.** *The operator $T$ defined in* (8) *is a self map on* $\kappa$.

*Proof.* Let $y \in \kappa$. From (5), we have $(Ty)(t) \geq 0$, for all $t \in [t_1, \sigma^3(t_3)]$, and

$$\begin{aligned}(Ty)(t) &= \int_{t_1}^{\sigma(t_3)} G(t,s) f(s, y(s), y^{\Delta}(s), y^{\Delta^2}(s))\Delta s \\ &\leq \int_{t_1}^{\sigma(t_3)} G(\sigma(s),s) f(s, y(s), y^{\Delta}(s), y^{\Delta^2}(s))\Delta s\,,\end{aligned}$$

so that

$$\| Ty \| \leq \int_{t_1}^{\sigma(t_3)} G(\sigma(s),s) f(s, y(s), y^{\Delta}(s), y^{\Delta^2}(s))\Delta s$$

Next, if $y \in \kappa$, then by the above inequality we have

$$\begin{aligned}(Ty)(t) &= \int_{t_1}^{\sigma(t_3)} G(t,s) f(s, y(s), y^{\Delta}(s), y^{\Delta^2}(s))\Delta s \\ &\geq \gamma \int_{t_1}^{\sigma(t_3)} G(\sigma(s),s) f(s, y(s), y^{\Delta}(s), y^{\Delta^2}(s))\Delta s \geq \gamma \| Ty \|\,.\end{aligned}$$

Hence $T : \kappa \to \kappa$. Standard arguments involving the Arzela-Ascoli Theorem shows that $T$ is completely continuous. □

To establish positive solution we will employ the following fixed point theorem due to Krasnosel'skii [14].

**Theorem 3.2.** *Let $X$ be a Banach space, $K \subseteq X$ be a cone, and suppose that $\Omega_1, \Omega_2$ are open subsets of $X$ with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Suppose further that $T : K \cap (\overline{\Omega}_2 \backslash \Omega_1) \to K$ is completely continuous operator such that either*

*(i)* $\| Tu \| \leq \| u \|,\ u \in K \cap \partial\Omega_1$ *and* $\| Tu \| \geq \| u \|,\ u \in K \cap \partial\Omega_2$, *or*

*(ii)* $\| Tu \| \geq \| u \|,\ u \in K \cap \partial\Omega_1$ *and* $\| Tu \| \leq \| u \|,\ u \in K \cap \partial\Omega_2$

*holds. Then $T$ has a fixed point in $K \cap (\overline{\Omega}_2 \backslash \Omega_1)$.*

**Theorem 3.3.** *Assume that conditions (A1)-(A5) are satisfied. If $f^0 = 0$ and $f_\infty = \infty$, then the BVP* (1)-(2) *has at least one solution that lies in* $\kappa$.

*Proof.* Let $T$ be the cone preserving, completely continuous operator defined as in (8). Since $f^0 = 0$, there exist $\eta_1 > 0$ and $H_{1i} > 0$, for $i = 0, 1, 2$ so that

$$\max_{t \in [t_1, \sigma^3(t_3)]} \frac{f(t, y, y^{\Delta}, y^{\Delta^2})}{y} \leq \eta_1,$$

for $0 < y \leq H_{10}, 0 < y^{\Delta} \leq H_{11}, 0 < y^{\Delta^2} \leq H_{12}$, where $\eta_1 > 0$ satisfies

$$\eta_1 \int_{t_1}^{\sigma(t_3)} G(\sigma(s), s)\Delta s \leq 1.$$

Let $H_1 = \min\{H_{1i} : i = 0, 1, 2\}$. Thus, if $y \in \kappa$ and $\| y \| = H_1$, then we have

$$\begin{aligned}(Ty)(t) &= \int_{t_1}^{\sigma(t_3)} G(t,s) f(s, y(s), y^{\Delta}(s), y^{\Delta^2}(s))\Delta s \\ &\leq \int_{t_1}^{\sigma(t_3)} G(\sigma(s),s) f(s, y(s), y^{\Delta}(s), y^{\Delta^2}(s))\Delta s \\ &\leq \int_{t_1}^{\sigma(t_3)} G(\sigma(s),s)\eta_1 y(s)\Delta s \\ &\leq \eta_1 \int_{t_1}^{\sigma(t_3)} G(\sigma(s),s) \| y \| \Delta s \leq \| y \| .\end{aligned}$$

Therefore, $\| Ty \| \leq \| y \|$. Now if we let

$$\Omega_1 = \{y \in X : \| y \| < H_1\},$$

then

$$\| Ty \| \leq \| y \|, \quad \text{for} \quad y \in \kappa \cap \partial\Omega_1. \tag{9}$$

Further, since $f_\infty = \infty$, there exists $\eta_2 > 0$ and $\overline{H}_{2i} > 0$, for $i = 0, 1, 2$ such that

$$\min_{t \in [t_1, \sigma^3(t_3)]} \frac{f(t, y, y^{\Delta}, y^{\Delta^2})}{y} \geq \eta_2,$$

for $y \geq \overline{H}_{20}, y^{\Delta} \geq \overline{H}_{21}, y^{\Delta^2} \geq \overline{H}_{22}$ and $\eta_2 > 0$ is chosen so that

$$\eta_2 \gamma^2 \int_{t_1}^{\sigma(t_3)} G(\sigma(s), s)\Delta s \geq 1.$$

Let

$$\overline{H}_2 = \max\left\{\overline{H}_{2i} : i = 0, 1, 2\right\}.$$

Also let

$$H_2 = \max\left\{2H_1, \frac{1}{\gamma}\overline{H}_2\right\},$$

and

$$\Omega_2 = \{y \in X : \| y \| < H_2\},$$

then $y \in \kappa$ and $\| y \| = H_2$ implies

$$\min_{t \in [t_1, \sigma^3(t_3)]} y(t) \geq \gamma \| y \| \geq \overline{H}_2,$$

and so

$$
\begin{aligned}
(Ty)(t) &= \int_{t_1}^{\sigma(t_3)} G(t,s) f(s, y(s), y^{\Delta}(s), y^{\Delta^2}(s))\Delta s \\
&\geq \int_{t_1}^{\sigma(t_3)} \gamma G(\sigma(s),s) f(s, y(s), y^{\Delta}(s), y^{\Delta^2}(s))\Delta s \\
&\geq \gamma \int_{t_1}^{\sigma(t_3)} G(\sigma(s),s)\eta_2 y(s)\Delta s \geq \gamma^2\eta_2 \int_{t_1}^{\sigma(t_3)} G(\sigma(s),s) \parallel y \parallel \Delta s \geq \parallel y \parallel .
\end{aligned}
$$

Hence,

$$\parallel Ty \parallel \geq \parallel y \parallel, \quad \text{for} \quad y \in \kappa \cap \partial\Omega_2. \tag{10}$$

Therefore, by part (i) of Theorem 3.2 applied to (9) and (10), $T$ has a fixed point $y(t) \in \kappa \cap (\overline{\Omega}_2 \backslash \Omega_1)$ such that $H_1 \leq \parallel y \parallel \leq H_2$. This fixed point is a positive solution of the BVP (1)-(2). □

**Theorem 3.4.** *Assume that conditions (A1)-(A5) are satisfied. If $f_0 = \infty$ and $f^{\infty} = 0$, then the BVP* (1)-(2) *has at least one positive solution that lies in $\kappa$.*

*Proof.* Let $T$ be the cone preserving, completely continuous operator defined as in (8). Since $f_0 = \infty$, there exist $\overline{\eta}_1 > 0$ and $J_{1i} > 0$ for $i = 0,1,2$ such that

$$\min_{t \in [t_1, \sigma^3(t_3)]} \frac{f(t, y, y^{\Delta}, y^{\Delta^2})}{y} \geq \overline{\eta}_1,$$

for $0 < y \leq J_{10}, 0 < y^{\Delta} \leq J_{11}, 0 < y^{\Delta^2} \leq J_{12}$ and

$$\overline{\eta}_1 \gamma^2 \int_{t_1}^{\sigma(t_3)} G(\sigma(s),s)\Delta s \geq 1.$$

Let $J_1 = \min\{J_{1i} : i = 0,1,2\}$. Then for $y \in \kappa$ and $\parallel y \parallel = J_1$, we have

$$
\begin{aligned}
(Ty)(t) &= \int_{t_1}^{\sigma(t_3)} G(t,s) f(s, y(s), y^{\Delta}(s), y^{\Delta^2}(s))\Delta s \\
&\geq \int_{t_1}^{\sigma(t_3)} \gamma G(\sigma(s),s) f(s, y(s), y^{\Delta}(s), y^{\Delta^2}(s))\Delta s \\
&\geq \gamma \int_{t_1}^{\sigma(t_3)} G(\sigma(s),s)\overline{\eta}_1 y(s)\Delta s \geq \gamma^2\overline{\eta}_1 \int_{t_1}^{\sigma(t_3)} G(\sigma(s),s) \parallel y \parallel \Delta s \geq \parallel y \parallel .
\end{aligned}
$$

Thus, we may let

$$\Omega_1 = \{y \in X : \quad \parallel y \parallel < J_1\},$$

so that

$$\parallel Ty \parallel \geq \parallel y \parallel, \quad \text{for} \quad y \in \kappa \cap \partial\Omega_1. \tag{11}$$

Now, since $f^{\infty}=0$, there exists $\overline{\eta}_2>0$ and $\overline{J}_{2i}>0$, for $i=0,1,2$ so that

$$\max_{t\in[t_1,\sigma^3(t_3)]}\frac{f(t,y,y^{\Delta},y^{\Delta^2})}{y}\leq\overline{\eta}_2,$$

for $y\geq\overline{J}_{20},y^{\Delta}\geq\overline{J}_{21},y^{\Delta^2}\geq\overline{J}_{22}$ and $\overline{\eta}_2>0$ satisfies

$$\overline{\eta}_2\int_{t_1}^{\sigma(t_3)}G(\sigma(s),s)\Delta s\leq 1.$$

Let $\overline{J}_2=\max\left\{\overline{J}_{2i}:i=0,1,2\right\}$. There are two cases.

*Case (i).* $f$ is bounded. Suppose $L>0$ is such that $f(t,y,y^{\Delta},y^{\Delta^2})\leq L$, for all $0<y<\infty,0<y^{\Delta}<\infty,0<y^{\Delta^2}<\infty$. In this case, we may choose

$$J_2=\max\left\{2J_1,L\int_{t_1}^{\sigma(t_3)}G(\sigma(s),s)\Delta s\right\},$$

so that $y\in\kappa$ with $\| y \|=J_2$, we have

$$\begin{aligned}(Ty)(t)&=\int_{t_1}^{\sigma(t_3)}G(t,s)f(s,y(s),y^{\Delta},y^{\Delta^2})\Delta s\\&\leq\int_{t_1}^{\sigma(t_3)}G(\sigma(s),s)f(s,y(s),y^{\Delta},y^{\Delta^2})\Delta s\\&\leq L\int_{t_1}^{\sigma(t_3)}G(\sigma(s),s)\Delta s\leq J_2=\| y \|,\end{aligned}$$

and therefore

$$\| Ty \|\leq\| y \|.$$

*Case (ii).* $f$ is unbounded. Choose $J_{2i}>\max\{2J_{1i},\overline{J}_{2i}\}$ be such that $f(t,y,y^{\Delta},\ y^{\Delta^2})\leq f(t,J_{20},J_{21},J_{22})$ for $0<y^{\Delta^i}\leq J_{2i}$, $i=0,\ 1,\ 2$. Let $J_2=\max\{J_{2i}:i=0,\ 1,2\}$. Then for $y\in\kappa$ and $\| y \|=J_2$, we have

$$\begin{aligned}(Ty)(t)&=\int_{t_1}^{\sigma(t_3)}G(t,s)f(s,y(s),y^{\Delta}(s),y^{\Delta^2}(s))\Delta s\\&\leq\int_{t_1}^{\sigma(t_3)}G(\sigma(s),s)f(s,y(s),y^{\Delta}(s),y^{\Delta^2}(s))\Delta s\end{aligned}$$

$$\leq\int_{t_1}^{\sigma(t_3)}G(\sigma(s),s)f(s,J_{20},J_{21},J_{22})\Delta s\leq\int_{t_1}^{\sigma(t_3)}G(\sigma(s),s)\overline{\eta}_2J_2\Delta s\leq J_2=\| y \|.$$

Therefore, in either case we put

$$\Omega_2=\{y\in X:\| y \|<J_2\},$$

we have

$$\parallel Ty \parallel \leq \parallel y \parallel, \quad \text{for} \quad y \in \kappa \cap \partial\Omega_2. \tag{12}$$

Therefore, by the part $(ii)$ of Theorem 3.2 applied to (11) and (12), $T$ has a fixed point $y(t) \in \kappa \cap (\overline{\Omega}_2 \backslash \Omega_1)$ such that $J_1 \leq \parallel y \parallel \leq J_2$. This fixed point is a positive solution of the BVP (1)-(2). □

## 4. Examples

Now, we give examples to illustrate the main results.

**Example 1.** Consider the following boundary value problem

$$y^{\Delta^3} + y^2(e^{-y} + 3e^{-y^{\Delta}} + 4te^{-y^{\Delta^2}}) = 0, \quad \forall t \in [0, \sigma^3(1)] \cap \mathbb{T}, \tag{13}$$

where $\mathbb{T} = \{0\} \cup \{\frac{1}{2^{n+1}} : n \in \mathbb{N}\} \cup [\frac{1}{2}, \frac{3}{2}]$, subject to the boundary conditions

$$\begin{aligned} y(0) + \frac{6}{5}y^{\Delta}(0) + \frac{11}{8}y^{\Delta^2}(0) = 0\,, \\ y(\frac{1}{2}) + \frac{1}{2}y^{\Delta}(\frac{1}{2}) + y^{\Delta^2}(\frac{1}{2}) = 0\,, \\ y(\sigma^3(1)) + \frac{1}{4}y^{\Delta}(\sigma^2(1)) + \frac{1}{2}y^{\Delta^2}(\sigma(1)) = 0\,. \end{aligned} \tag{14}$$

The Green's function for the boundary value problem can be expressed in the form

$$\underset{t\in[0,\frac{1}{2}]}{G(t,s)} = \begin{cases} G_{11}(t,s), & 0 \leq \mathrm{t} < \mathrm{s} < \frac{1}{2} < \sigma^3(1)\,, \\ G_{12}(t,s), & 0 < \sigma(\mathrm{s}) < \mathrm{t} \leq \frac{1}{2} < \sigma^3(1)\,, \\ G_{13}(t,s), & 0 \leq \mathrm{t} < \frac{1}{2} < \mathrm{s} < \sigma^3(1)\,, \end{cases}$$

$$\underset{t\in[\frac{1}{2},\sigma^3(1)]}{G(t,s)} = \begin{cases} G_{21}(t,s), & 0 < \frac{1}{2} \leq \mathrm{t} < \mathrm{s} < \sigma^3(1)\,, \\ G_{22}(t,s), & 0 < \frac{1}{2} < \sigma(\mathrm{s}) < \mathrm{t} \leq \sigma^3(1)\,, \\ G_{23}(t,s), & 0 \leq \sigma(\mathrm{s}) < \frac{1}{2} < \mathrm{t} < \sigma^3(1)\,. \end{cases}$$

After computation, the functions $G_{ij}(t,s)$ for $i = 1, 2$ and $j = 1, 2, 3$ are given by

$$\begin{aligned} G_{11}(t,s) = [\frac{7}{8} - t - \frac{t^2}{10}][5\sigma(s)\sigma^2(s) - 5(\sigma(s) + \sigma^2(s)) + \frac{11}{4}] \\ + [\frac{11}{4} - t^2][2\sigma(s)\sigma^2(s) - \frac{5}{2}(\sigma(s) + \sigma^2(s)) + 5]\,, \end{aligned}$$

$$G_{12}(t,s) = G_{23}(t,s) = [\frac{15}{8} - \frac{t}{2} - \frac{t^2}{2}][5\sigma(s)\sigma^2(s) - 6(\sigma(s) + \sigma^2(s)) + \frac{55}{4}]$$

$$G_{13}(t,s) = G_{21}(t,s) = [\frac{11}{4} - t^2][2\sigma(s)\sigma^2(s) - \frac{5}{2}(\sigma(s) + \sigma^2(s)) + 5]\,,$$

$$G_{22}(t,s) = [\frac{15}{8} - \frac{t}{2} - \frac{t^2}{2}][5\sigma(s)\sigma^2(s) - 6(\sigma(s) + \sigma^2(s)) + \frac{55}{4}] + [\frac{7}{8} - t - \frac{t^2}{10}][5\sigma(s)\sigma^2(s) - 5(\sigma(s) + \sigma^2(s)) + \frac{11}{4}]\,.$$

Clearly, the coefficients of the boundary conditions in (2) satisfies (A1)-(A5) and a simple calculation show that $\gamma = \frac{1}{3}$, $f^0 = 0$ and $f_\infty = \infty$. It follows from Theorem 3.3 the BVP (13)-(14) has at least one positive solution.

**Example 2.** Consider the following boundary value problem

$$y^{\Delta^3} + \frac{(t^4 e^{-y} + t^3 e^{-y^\Delta} + e^{-y^{\Delta^2}})}{y} = 0, \quad \forall t \in [0, \sigma^3(1)] \cap \mathbb{T}\,, \tag{15}$$

where $\mathbb{T} = \{0\} \cup \{\frac{1}{2^{n+1}} : n \in \mathbb{N}\} \cup [\frac{1}{2}, \frac{3}{2}]$, subject to the boundary conditions

$$\begin{aligned} y(0) + \frac{4}{3}y^\Delta(0) + \frac{5}{4}y^{\Delta^2}(0) &= 0\,, \\ y(\frac{1}{2}) + \frac{1}{2}y^\Delta(\frac{1}{2}) + y^{\Delta^2}(\frac{1}{2}) &= 0\,, \\ y(\sigma^3(1)) + \frac{1}{4}y^\Delta(\sigma^2(1)) + \frac{1}{2}y^{\Delta^2}(\sigma(1)) &= 0\,. \end{aligned} \tag{16}$$

The Green's function for the boundary value problem can be expressed in the form

$$\underset{t\in[0,\frac{1}{2}]}{G(t,s)} = \begin{cases} G_{11}(t,s), & 0 \le \mathrm{t} < \mathrm{s} < \frac{1}{2} < \sigma^3(1)\,, \\ G_{12}(t,s), & 0 < \sigma(\mathrm{s}) < \mathrm{t} \le \frac{1}{2} < \sigma^3(1)\,, \\ G_{13}(t,s), & 0 \le \mathrm{t} < \frac{1}{2} < \mathrm{s} < \sigma^3(1)\,, \end{cases}$$

$$\underset{t\in[\frac{1}{2},\sigma^3(1)]}{G(t,s)} = \begin{cases} G_{21}(t,s), & 0 < \frac{1}{2} \le \mathrm{t} < \mathrm{s} < \sigma^3(1)\,, \\ G_{22}(t,s), & 0 < \frac{1}{2} < \sigma(\mathrm{s}) < \mathrm{t} \le \sigma^3(1)\,, \\ G_{23}(t,s), & 0 \le \sigma(\mathrm{s}) < \frac{1}{2} < \mathrm{t} < \sigma^3(1)\,. \end{cases}$$

After computation, the functions $G_{ij}(t,s)$ for $i = 1,2$ and $j = 1,2,3$ are given by

$$G_{11}(t,s) = [\frac{-5}{6} + \frac{t^2}{3}][6\sigma(s)\sigma^2(s) - 6(\sigma(s) + \sigma^2(s)) + \frac{33}{2}] + [14 - 3t - 4t^2][2\sigma(s)\sigma^2(s) - \frac{5}{2}(\sigma(s) + \sigma^2(s)) + 5]\,,$$

$$G_{12}(t,s) = G_{23}(t,s) = [\frac{15}{4} - t - t^2][6\sigma(s)\sigma^2(s) - 8(\sigma(s) + \sigma^2(s)) + 15]\,,$$

$$G_{13}(t,s) = G_{21}(t,s) = [14 - 3t - 4t^2][2\sigma(s)\sigma^2(s) - \frac{5}{2}(\sigma(s) + \sigma^2(s)) + 5]\,,$$

$$G_{22}(t,s) = [\frac{15}{4} - t - t^2][6\sigma(s)\sigma^2(s) - 8(\sigma(s) + \sigma^2(s)) + 15] + [\frac{5}{6} - \frac{t^2}{3}][6\sigma(s)\sigma^2(s) - 6(\sigma(s) + \sigma^2(s)) + \frac{33}{2}].$$

We found that $\gamma = 0.4666$, $f_0 = \infty$, and $f^\infty = 0$. It follows from Theorem 3.4 the BVP (15)-(16) has at least one positive solution.

## References

[1] R.P. Agarwal, D. O'Regan, P.J.Y. Wong, *Positive Solutions of Differential, Difference and Integral Equations*, Kluwer Academic Publishers, The Netherlands (1999).

[2] B. Alubach, S. Hilger, A unified approach to continuous and discrete dynamics in qualitative theory of differential equations (Szeged 1988), volume 53 of *Colloq. Math. Soc. Janas Bolya*, North-Holland, Amsterdam (1990), 37-56.

[3] D.R. Anderson, Eigenvalue intervals for a second-order mixed conditions problem on time scales, *Int. J. Nonlinear Diff. Eqns. T.M. A*, **74**, No. 1 and 2 (2002), 97-104.

[4] D.R. Anderson, R.I. Avery, Multiple positive solutions to a third-order discrete focal boundary value problem, *J. Comp. and Math. with Appl.*, **42** (2001), 333-340.

[5] D.R. Anderson, J.M. Davis, Mulitple solutions and eigenvalues for third-order right focal boundary value problems, *J. Math. Anal. Appl.*, **267** (2002), 135-157.

[6] M. Bohner, A.C. Peterson, *Dynamic Equations on Time Scales. An Introduction with Applications*, Birkhäuser, Boston, MA (2001).

[7] M. Bohner, A.C. Peterson, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston (2003).

[8] P.W. Eloe, J. Henderson, Positive solutions for $(n-1, 1)$ conjugate boundary value problems. *Nonlinear. Anal.*, **28** (1997), 1669-1680.

[9] P.W. Eloe, J. Henderson, Positive solutions and nonlinear $(k, n-k)$ conjugate eigenvalue problems, *Diff. Eqns. Dyn. Sys.*, **6** (1998), 309-317.

[10] P.W. Eloe, J. Henderson, Positive solutions and nonlinear multipoint conjugate eigenvalue problems, *Elec. J. Diff. Eqns.*, **1997**, (1997) No. 3, 1-11.

[11] L.H. Erbe, H. Wang, On the existence of positive solutions of ordinary differential equations, *Proc. Amer. Math. Soc.*, **120** (1994), 743-748.

[12] J. Henderson, H. Wang, Positive solutions for nonlinear eigenvalue problems, *J. Math. Anal. Appl.*, **208** (1997), 252-259.

[13] E.R. Kaufmann, Positive solutions of a three point boundary value problem on time scales, *Elec. J. Diff. Eqns.*, **82** (2003), 1-11.

[14] M.A. Krasnosel'skii, *Positive Solutions of Operator Equations*, P. Noordhoff Ltd., Groningen, The Netherlands (1964).

[15] R. Ma, Positive solutions of a nonlinear three point problem, *Elec. J. Diff. Eqns.*, **1998**, No. 34 (1998), 1-8.

[16] F. Merdivenci Atici, G.Sh. Guseinov, Positive periodic solutions for nonlinear difference equations with periodic coefficients, *J. Math. Anal. Appl.*, **232** (1999), 166-182.

[17] J.P. Sun, Existense of solution and positive solution of BVP for nonlinear third order dynamic equation, *Nonlinear Anal.*, **64**, No. 1 (2006), 165-176.

[18] Y. Sun, Positive solutions of singular third-order three-point boundary value problem, *J. Math. Anal. Appl.*, **306** (2005), 589-603.

[19] H.R. Sun, W.K. Wen, On the number of positive solutions for a nonlinear third order boundary value problem, *Int. J. Diff. Eqns.*, **1** , No. 1 (2006), 165-176.

[20] H.R. Sun, W.T. Li, Positive solutions for nonlinear three-point boundary value problems on time scales, *J. Math. Anal. Appl.*, **299** (2004), 508-524.

[21] Q. Yao, The existense and multiplicity of positive solutions for a third order three point boundary value problem, *Acta. Math. Appl. Sinica*, **19** (2003), 117-122.