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EXACT MULTIPLICITY OF SOLUTIONS FOR *P*-LAPLACE TYPE DIFFERENTIAL EQUATIONS

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Abstract: In this paper we study multiple solutions for boundary value problem of p-Laplace type equation

$$\begin{cases} -(\varphi(u'))' = \lambda\psi(u), & -T < x < T; \\ u(-T) = 0, & u(T) = 0. \end{cases}$$

There are lots of works devoted to such boundary value problem with odd function φ and almost no known result for the equation with non-odd function φ. In this paper we study a particular non-odd case, e.g. $\varphi(s) = \alpha s^{p-1}, s \geq 0, \varphi(s) = -\beta|s|^{q-1}, s \leq 0$. We prove several exact multiplicity results for the solutions to the above equation.

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1. Introduction

We in this paper consider boundary value problem

$$\begin{cases} -(\varphi(u'))' = \lambda\psi(u), & -T < x < T\,, \\ u(-T) = 0, & u(T) = 0\,, \end{cases} \tag{1.1}$$

where φ, ψ are continuous, φ is increasing and also satisfies condition $\varphi(s)s > 0, \forall s \neq 0$. Many studies on multiple solutions are devoted to semilinear [2, 3,

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9, 12], e.g. $\varphi(s) = s$, or $p-$Laplace type quasilinear equations [1, 8, 11, 13, 15], e.g., $\varphi(s) = |s|^{p-2}s$ and there are few results on general φ and specially non known result on equations with non-odd function φ. In this paper, we take up the case with non-odd function φ, which is very natural with consideration that there are many variational problems where the associated functional $E(u) = \int f(x, u, u')$ is not even in (u, u'). The big difference here is that the non-oddness of φ clearly breaks down the symmetry of positive solutions and can even force the maximum point, which is always at the middle of the interval for semilinear equations [10], vary from solution to solution. Moreover the property of solutions appearing in pairs $(u(x), -u(x))$ or $(u(x), u(-x))$ to equations with odd φ, ψ is lost in general for non-odd φ, and thereby nodal solutions cannot be simply obtained via odd-extension of positive solutions [7, 14] and bifurcation of single nodal solution would be possible (see the figure). For simplicity, we shall focus on a prototype equation, namely, assuming, $\varphi(s) = \alpha s_+^{p-1} - \beta s_-^{q-1}, \psi(s) = a s_+^{r-1} - b s_-^{s-1}$, where $\alpha, \beta, a, b > 0, p, q, r, s > 1$ are constants, $s_+ = \max\{s, 0\}, s_- = \max\{-s, 0\}$. The main results in this paper are existences of exact two, one or non solutions, i.e., Theorem 1 in Section 3 and Theorem 3 in Section 4 for positive and negative solutions, and Theorem 2 in Section 3 and Theorem 4 in Section 4 for nodal solutions. The main tool in our proof is the time-mapping method [11], which we shall review it in next section

2. The Time Mapping

Since the equation in (1.1) is integrable and it admits an first integral

$$\Phi(u') + \lambda\Psi(u) = C\,, \tag{2.1}$$

where $\Psi(u) = \int_0^u \psi(t)\,dt, \Phi(s) = \int_0^{\varphi(s)} \varphi^{-1}(t)\,dt$ is strictly decreasing respectively increasing on $(-\infty, 0]$ and $[0, +\infty)$ therefore there exist inverse functions Φ_-^{-1}, Φ_+^{-1} on the above intervals whence the equation (2.1) can be solved explicitly, e.g.

$$u'(x) = \begin{cases} \Phi_-^{-1}(C - \lambda\Psi(u)), & \text{if } u'(x) \le 0\,, \\ \Phi_+^{-1}(C - \lambda\Psi(u)), & \text{if } u'(x) \ge 0\,. \end{cases} \tag{2.2}$$

If $\varphi(s) = \alpha s_+^{p-1} - \beta s_-^{q-1}, \alpha, \beta > 0, p, q > 1$ are constants, then

$$\Phi(s) = \frac{\alpha}{p'} s_+^p + \frac{\beta}{q'} s_-^q\,,$$

where $\frac{1}{p} + \frac{1}{p'} = 1, \frac{1}{q} + \frac{1}{q'} = 1$. It follows that $\Phi_+^{-1}(v) = \sqrt[p]{\frac{vp'}{\alpha}}, \Phi_-^{-1}(v) = -\sqrt[q]{\frac{vq'}{\beta}}$

and

$$u'(x) = \begin{cases} \sqrt[p]{\frac{p'}{\alpha}}\sqrt[p]{C - \lambda\Psi(u)}, & \text{if } u'(x) \geq 0\,, \\ -\sqrt[q]{\frac{q'}{\beta}}\sqrt[q]{C - \lambda\Psi(u)}, & \text{if } u'(x) \leq 0\,. \end{cases} \tag{2.3}$$

For a positive solution $u(x)$ with a maximum u_0 at the point x_0, i.e., $u_0 = u(x_0)$ then $C = \lambda\Psi(u_0)$, and $u(x)$ is given by

$$\begin{cases} \int_0^{u(x)} \frac{ds}{\sqrt[p]{\Psi(u_0)-\Psi(s)}} = \sqrt[p]{\frac{\lambda p'}{\alpha}}(T+x), & \text{if } x \in (-T, x_0)\,, \\ \int_0^{u(x)} \frac{ds}{\sqrt[q]{\Psi(u_0)-\Psi(s)}} = \sqrt[q]{\frac{\lambda q'}{\beta}}(T-x), & \text{if } x \in (x_0, T). \end{cases} \tag{2.4}$$

By choosing $x = x_0$ we deduce that (λ, u_0, x_0) should satisfy

$$\begin{cases} \int_0^{u_0} \frac{ds}{\sqrt[p]{\Psi(u_0)-\Psi(s)}} = \sqrt[p]{\frac{\lambda p'}{\alpha}}(T+x_0), \\ m(\lambda, u_0) =: \sqrt[p]{\frac{\alpha}{\lambda p'}}\int_0^{u_0} \frac{ds}{\sqrt[p]{\Psi(u_0)-\Psi(s)}} + \sqrt[q]{\frac{\beta}{\lambda q'}}\int_0^{u_0} \frac{ds}{\sqrt[q]{\Psi(u_0)-\Psi(s)}} = 2T. \end{cases} \tag{2.5}$$

The time mapping m above will be the main tool in our proof for existence of solutions [11].

3. Eigenvalue Type Problem

We consider the nonlinear eigenvalue type boundary value problem

$$\begin{cases} -(\varphi(u'))' = \lambda\varphi(u), & -T < x < T\,, \\ u(-T) = 0, \quad u(T) = 0\,, \end{cases} \tag{3.1}$$

where $\varphi(s) = \alpha s_+^{p-1} - \beta s_-^{q-1}, \alpha, \beta > 0$ constants. This type of equations have been studied in [1] with a general but odd homomorphism φ, and in [7] for a special case $p = q$. In [7], it has been observed that switches of frequency and amplitude occur at the max- and minimum, or nodes of solution $u(x)$, which is a new phenomenon.

First we consider a positive solution $u(x)$ and let x_0 be the maximum point of $u(x)$ and let $u_0 = u(x_0)$ then $u'(x) > 0, -T < x < x_0, u'(x_0) = 0, u'(x) < 0, x_0 < x < T$, and the constant C in (2.1) will be $C = \lambda\Psi(u_0) = \lambda\frac{\alpha}{p}u_0^p$. It follows from (2.4) that on the interval $(-T, x_0), u(x)$ satisfies the equation, since $p'/p = p' - 1$

$$u'(x) = \sqrt[p]{\lambda(p'-1)(u_0^p - u^p(x))}$$

and thus is implicitly given by

$$\int_0^{u(x)} \frac{ds}{\sqrt[p]{u_0^p - s^p}} = \sqrt[p]{\lambda(p'-1)}(T+x). \tag{3.2}$$

Choose $x = x_0$, we obtain

$$\int_0^{u_0} \frac{du}{\sqrt[p]{u_0^p - u^p}} = \sqrt[p]{\lambda(p'-1)}(T+x_0)$$

$$\Longrightarrow \frac{\pi}{p\sin(\frac{\pi}{p})} = \sqrt[p]{\lambda(p'-1)}(T+x_0). \tag{3.3}$$

Passing to the next interval (x_0, T) we have

$$-u'(x) = \sqrt[q]{\lambda\frac{\alpha q'}{\beta p}}\sqrt[q]{u_0^p - u^p(x)}$$

and the solution $u(x)$ is determined by

$$\int_0^{u(x)} \frac{ds}{\sqrt[q]{u_0^p - s^p}} = \sqrt[q]{\lambda\frac{\alpha q'}{\beta p}}(T-x). \tag{3.4}$$

Letting $x = x_0$ yields

$$\int_0^{u_0} \frac{ds}{\sqrt[q]{u_0^p - s^p}} = \sqrt[q]{\lambda\frac{\alpha q'}{\beta p}}(T-x)\,.$$

Using change of variables, we deduce that

$$\int_0^{u_0} \frac{ds}{\sqrt[q]{u_0^p - s^p}} = \int_0^1 \frac{u_0 dt}{\sqrt[q]{u_0^p(1-t^p)}}$$

$$= \frac{1}{p}u_0^{1-\frac{p}{q}}\int_0^1 (1-\theta)^{-\frac{1}{q}}\theta^{\frac{1}{p}-1}\,d\theta = \frac{1}{p}u_0^{1-\frac{p}{q}}B(\frac{1}{p},\frac{1}{q'})\,,$$

where $B(x, y)$ is the beta function, and thus u_0, x_0 are related in following way

$$\frac{1}{p}u_0^{\frac{q-p}{q}}B(\frac{1}{p},\frac{1}{q'}) = \sqrt[q]{\lambda\frac{\alpha q'}{\beta p}}(T-x_0)\,. \tag{3.5}$$

Combining relations in (3.3) and (3.5), we get that λ, u_0, x_0 satisfy

$$x_0 = -T + \frac{c(p)}{\sqrt[p]{\lambda}}, \tag{3.6}$$

$$L_+(\lambda, u_0) =: \frac{c(p)}{\sqrt[p]{\lambda}} + \frac{b(p,q)}{\sqrt[q]{\lambda}}\sqrt[q]{\frac{\beta}{\alpha}}u_0^{\frac{q-p}{q}} = 2T\,, \tag{3.7}$$

where

$$c(z) = \frac{\pi \sqrt[z]{z-1}}{z \sin(\frac{\pi}{z})}, \quad b(p,q) = \frac{1}{p} B(\frac{1}{p}, \frac{1}{q'}) \sqrt[q]{\frac{p}{q'}}.$$

It is worth to notice here that the maximum point x_0 depends on λ (see (3.6)), which is in contrast to the well known symmetry result for semilinear elliptic equations, where the maximum occurs always at the origin [11].

In analogy for a negative solution $u(x)$ of (3.1) let $v_0 = \max|u(x)| = |u(x_0)|$, then we have

$$x_0 = -T + \frac{c(q)}{\sqrt[q]{\lambda}}, \tag{3.6'}$$

$$L_-(\lambda, v_0) =: \frac{c(q)}{\sqrt[q]{\lambda}} + \frac{b(q,p)}{\sqrt[p]{\lambda}} \sqrt[p]{\frac{\alpha}{\beta}} v_0^{\frac{p-q}{p}} = 2T\,. \tag{3.7'}$$

If $p \neq q$, then both functions L_+, L_- are monotone function with respect to the variable u_0 respective v_0, and thus for any given $\lambda > 0$ the solution u_0 of equation (3.7) respective v_0 of equation (3.7') will be unique, whenever they exist. But, it is clearly that equation (3.7) respective equation (3.7') is solvable, if and only if $\lambda > \lambda(p) =: (c(p)/(2T))^p$ respective $\lambda > \lambda(q) =: (c(q)/(2T))^q$.

Let $\Lambda_0 = \max\{\lambda(p), \lambda(q)\}, \lambda_0 = \min\{\lambda(p), \lambda(q)\}$, whereafter we have shown

Theorem 1. *For any given $1 < p, q, p \neq q$, then the following results hold*

1) (3.1) has a non-trivial solution if and only if $\lambda > \lambda_0$.

2) (3.1) has precise one non-trivial solution if and only if $\lambda \in (\lambda_0, \Lambda_0]$.

3) (3.1) has exact one positive (negative) solution if and only if $\lambda > \lambda(p)$ ($\lambda > \lambda(q)$).

4) (3.1) has precise one positive and one negative solutions if and only if $\lambda > \Lambda_0$.

Next we consider nodal solutions, which are also called sign changing solutions in many literature. We say that x_0 is a nodal point (or simply node) of $u(x)$, if $x_0 \in (-T, T)$ and $u(x_0) = 0$. Since neither φ nor ϕ is assumed to be odd, so it is necessary to distinguish solutions which have different signs near the starting point $x = -T$, we say that the solution u is a initially positive (negative), if $u(x)$ is positive (negative) on $(-T, -T + \delta)$ for some $\delta > 0$. Moreover the interval on which a solution is positive or negative is as well important for our investigation, so we call u a positive wave on (x_1, x_2), if $u(x) > 0, x_1 < x < x_2, u(x_1) = u(x_2) = 0$ and call $x_2 - x_1$ the associated wavelength; and define a negative wave in a similar way. Therefore, any nodal solution has at least one positive wave and one negative wave. If a nodal solution has at least two positive waves, then it follows from (2.1) that those positive waves have same height under the condition $\Psi(s)$ being strictly increasing on $(0,+\infty)$. Thus by (3.7) they even have the same wavelength; and similar results hold for negative waves, if $\Psi(s)$ is strictly decreasing. For our choice of function ψ, Ψ, those conditions are fulfilled, thus positive and negative waves have this property.

We begin with a solution having only one nodal point at T_0. First we consider an initially positive solution $u(x)$ with the node at T_0, i.e., $u(x)>0$ on $(-T,T_0), u(x)<0$ on (T_0,T).

Let $u_0=\max u(x), v_0=\max(-u(x))$ then we infer from (2.1) that

$$\frac{\beta}{q}v_0^q=\frac{\alpha}{p}u_0^p \tag{3.8}$$

furthermore deduce as in single wave cases that the positive wavelength is

$$L_+(u)=\frac{c(p)}{\sqrt[p]{\lambda}}+\frac{b(p,q)}{\sqrt[q]{\lambda}}\sqrt[q]{\frac{\beta}{\alpha}}u_0^{\frac{q-p}{q}}=T+T_0$$

and the negative wavelength is

$$L_-(u)=\frac{c(q)}{\sqrt[q]{\lambda}}+\frac{b(q,p)}{\sqrt[p]{\lambda}}\sqrt[p]{\frac{\alpha}{\beta}}v_0^{\frac{p-q}{p}}=T-T_0$$

but the sum of the positive and negative wavelengthes must be the whole interval, i.e., $L_+(u)+L_-(u)=2T$ and consequently λ, u_0, v_0 also satisfy

$$\frac{c(p)}{\sqrt[p]{\lambda}}+\frac{b(p,q)}{\sqrt[q]{\lambda}}\sqrt[q]{\frac{\beta}{\alpha}}u_0^{\frac{q-p}{q}}+\frac{c(q)}{\sqrt[q]{\lambda}}+\frac{b(q,p)}{\sqrt[p]{\lambda}}\sqrt[p]{\frac{\alpha}{\beta}}v_0^{\frac{p-q}{p}}=2T \tag{3.9}$$

and the node is determined by

$$T_0=\frac{c(p)}{\sqrt[p]{\lambda}}+\frac{b(p,q)}{\sqrt[q]{\lambda}}\sqrt[q]{\frac{\beta}{\alpha}}u_0^{\frac{q-p}{q}}-T\,. \tag{3.10}$$

If $u(x)$ is an initially negative nodal solution with also only one nodal point T_0, *i.e.*, $u(x)<0, x<T_0; u(x)>0, x>T_0$, then the negative and positive wave lengthes satisfy

$$L_-(u)=T+T_0, \quad L_+(u)=T-T_0$$

and the node is $T_0=L_-(u)-T$. However by adding the above two equalities we arrive at the same relation (3.9).

It follows from (3.10) that for any given $\lambda>0$, nodal solutions from same class do not share the nodes anymore, and nodes shall shift from one solution to another, which is different compared with quasilinear equations with odd nonlinearity.

The relations between u_0 respectively T_0 and λ can be written explicitly in view of (3.8)

$$\frac{c(p)}{\sqrt[p]{\lambda}} + \frac{b(p,q)}{\sqrt[q]{\lambda}}\sqrt[q]{\frac{\beta}{\alpha}}u_0^{\frac{q-p}{q}} + \frac{c(q)}{\sqrt[q]{\lambda}} + \frac{b(q,p)}{\sqrt[p]{\lambda}}\sqrt[q]{\frac{\alpha}{\beta}}\left(\frac{q}{p}\right)^{(p-q)/(pq)} u_0^{\frac{p-q}{q}} = 2T\,, \quad (3.11)$$

$$\left(T + T_0 - \frac{c(p)}{\sqrt[p]{\lambda}}\right)\left(T - T_0 - \frac{c(q)}{\sqrt[q]{\lambda}}\right) = \frac{b(p,q)b(q,p)}{\sqrt[q]{\lambda}\sqrt[p]{\lambda}}\left(\frac{\alpha q}{\beta p}\right)^{(q-p)/q}.$$

Since $\frac{q-p}{q} = -\frac{p-q}{q}$ we can solve u_0 in (3.11) and obtain that

$$u_0^{\frac{q-p}{q}} = \frac{2T - \frac{c(p)}{\sqrt[p]{\lambda}} - \frac{c(q)}{\sqrt[q]{\lambda}} \pm \sqrt{(2T - \frac{c(p)}{\sqrt[p]{\lambda}} - \frac{c(q)}{\sqrt[q]{\lambda}})^2 - 4\frac{b(p,q)}{\sqrt[q]{\lambda}}\frac{b(q,p)}{\sqrt[p]{\lambda}}\left(\frac{q}{p}\right)^{\frac{p-q}{pq}}}}{2\frac{b(p,q)}{\sqrt[q]{\lambda}}}$$

$$:= \Delta_{\pm}(\lambda)\,, \quad (3.12)$$

which are real solutions if and only if

$$f(\lambda) =: 2\sqrt{\frac{b(p,q)b(q,p)}{\sqrt[p]{\lambda}\sqrt[q]{\lambda}}\left(\frac{q}{p}\right)^{\frac{p-q}{pq}}} + \frac{c(p)}{\sqrt[p]{\lambda}} + \frac{c(q)}{\sqrt[q]{\lambda}} \le 2T\,. \quad (3.13)$$

For the nodes, it follows from (3.10) and (3.12) that

$$T_{0\pm} = -T + \frac{c(p)}{\sqrt[p]{\lambda}} + \frac{b(p,q)}{\sqrt[q]{\lambda}}\sqrt[q]{\frac{\beta}{\alpha}}\Delta_{\pm}\,. \quad (3.14)$$

Observe that the function $f(\lambda)$ defined in (3.13) is a strictly decreasing function and satisfies

$$f(\lambda) \to +\infty, \text{ as } 0 < \lambda \to 0; \quad f(\lambda) \to 0, \text{ as } \lambda \to +\infty\,,$$

thus there is a unique $\Lambda_1 > 0$ depending only on p, q, α, β such that $f(\Lambda_1) = 2T$. Summing up this part analysis we can conclude that for $\lambda < \Lambda_1$ there is no nodal solution to (3.1); if $\lambda > \Lambda_1$ there are two initially positive respectively negative solutions having one node; and has precise one initially positive respective negative solutions, when $\lambda = \Lambda_1$.

If $u(x)$ is a solution with $(2k+1)$ nodes and then there are $k+1$ positive waves and $k+1$ negative waves. Since positive (negative) waves share the same wavelength, we see that λ, u_0, v_0 satisfy a relation $(k+1)(L_+(u) + L_-(u)) = 2T$ i.e.,

$$(k+1)\left(\frac{c(p)}{\sqrt[p]{\lambda}} + \frac{b(p,q)}{\sqrt[q]{\lambda}}\sqrt[q]{\frac{\beta}{\alpha}}u_0^{\frac{q-p}{q}} + \frac{c(q)}{\sqrt[q]{\lambda}} + \frac{b(q,p)}{\sqrt[p]{\lambda}}\sqrt[p]{\frac{\alpha}{\beta}}v_0^{\frac{p-q}{p}}\right) = 2T \quad (3.15)$$

and the equation (3.15) is solvable in (u_0, v_0) if and only if

$$2\sqrt{\frac{b(p,q)b(q,p)}{\sqrt[p]{\lambda}\sqrt[q]{\lambda}}\left(\frac{q}{p}\right)^{\frac{p-q}{pq}}} + \frac{c(p)}{\sqrt[p]{\lambda}} + \frac{c(q)}{\sqrt[q]{\lambda}} \le \frac{2T}{k+1}\,. \tag{3.16}$$

If $u(x)$ has $2k$ nodes with $(k+1)$ positive waves and k negative waves, then it must be initially positive and moreover $(k+1)L_+(u) + kL_-(u) = 2T$, that is

$$(k+1)\left(\frac{c(p)}{\sqrt[p]{\lambda}} + \frac{b(p,q)}{\sqrt[q]{\lambda}}\sqrt[q]{\frac{\beta}{\alpha}}u_0^{\frac{q-p}{q}}\right) + k\left(\frac{c(q)}{\sqrt[q]{\lambda}} + \frac{b(q,p)}{\sqrt[p]{\lambda}}\sqrt[p]{\frac{\alpha}{\beta}}v_0^{\frac{p-q}{p}}\right) = 2T \tag{3.17}$$

which in view of (3.8) is solvable in (u_0, v_0) only if

$$2\sqrt{\frac{k(k+1)b(p,q)b(q,p)}{\sqrt[p]{\lambda}\sqrt[q]{\lambda}}\left(\frac{q}{p}\right)^{\frac{p-q}{pq}}} + \frac{(k+1)c(p)}{\sqrt[p]{\lambda}} + \frac{kc(q)}{\sqrt[q]{\lambda}} \le 2T\,. \tag{3.18}$$

If $u(x)$ is a solution with $(k+1)$ negative waves and k positive waves, then u must be initially negative and $kL_+(u) + (k+1)L_-(u) = 2T$, i.e.,

$$k\left(\frac{c(p)}{\sqrt[p]{\lambda}} + \frac{b(p,q)}{\sqrt[q]{\lambda}}\sqrt[q]{\frac{\beta}{\alpha}}u_0^{\frac{q-p}{q}}\right) + (k+1)\left(\frac{c(q)}{\sqrt[q]{\lambda}} + \frac{b(q,p)}{\sqrt[p]{\lambda}}\sqrt[p]{\frac{\alpha}{\beta}}v_0^{\frac{p-q}{p}}\right) = 2T \tag{3.17$'$}$$

and which has positive solution u_0, v_0 if

$$2\sqrt{\frac{k(k+1)b(p,q)b(q,p)}{\sqrt[p]{\lambda}\sqrt[q]{\lambda}}\left(\frac{q}{p}\right)^{\frac{p-q}{pq}}} + \frac{kc(p)}{\sqrt[p]{\lambda}} + \frac{(k+1)c(q)}{\sqrt[q]{\lambda}} \le 2T\,. \tag{3.18$'$}$$

To summarize the above analysis, we have proven

Theorem 2. *For any given $1 < p, q, p \neq q, \alpha, \beta > 0$ and integer $k \geq 0$ then there are constants $\Lambda_k, \Lambda_k^+, \Lambda_k^-$, depending only on p, q, α, β such that:*

1) (3.1) has two : one : zero initially positive and negative solutions with $(2k+1)$ nodes, if $\lambda > \Lambda_k : \lambda = \Lambda_k : \lambda < \Lambda_k$.

2) (3.1) has two : one : zero initially positive solutions with $2k$ nodes, if $\lambda > \Lambda_k^+ : \lambda = \Lambda_k^+ : \lambda < \Lambda_k^+$.

2') (3.1) has two : one : zero initially negative solutions with $2k$ nodes, if $\lambda > \Lambda_k^- : \lambda = \Lambda_k^- : \lambda < \Lambda_k^-$.

4. The Combined Convex – Concave Effect

In the study of semilinear boundary value problem

$$\begin{cases} -\Delta u = \lambda f(u), & \text{in } \Omega\,, \\ u = 0\,, & \text{on } \partial\Omega\,, \end{cases}$$

we know that a proper combination convex- and concavity in nonlinearity f, typically, $f = au^p + bu^q, a, b > 0, q < 1 < p$, leads to existence of two positive solutions [2, 4, 5]. Here we shall consider a variant of such phenomenon

$$\begin{cases} -(\varphi(u'))' = \lambda\psi(u), & -T < x < T\,, \\ u(-T) = 0, \quad u(T) = 0\,, \end{cases} \tag{4.1}$$

where $\varphi(s) = \alpha s_+^{p-1} - \beta s_-^{q-1}, \alpha, \beta > 0, p, q > 1, \alpha, \beta, \lambda > 0$ constants. If ψ has a property that $\varphi \circ \psi^{-1}(s)$ will be concave on $(-\infty, 0]$ and convex on $[0, \infty)$, for example $\psi(s) = |s|^{r-2}s, q < r < p$, then it is plausible that the combined convex- and concave effect could occur even in this case. In the sequel, we shall confirm that it indeed happens.

Our first result in this section is again about positive solutions

Theorem 3. *For any given $\alpha, \beta, a, b > 0, q > p > 1, \psi(s) = as_+^{r-1} - bs_-^{s-1}$ then*

1) (4.1) has a unique positive solution for all $\lambda > 0$, whenever r does not belong to the interval $[p, q]$.

1') (4.1) has a unique negative solution for all $\lambda > 0$, whenever s does not belong to the interval $[p, q]$.

2) if $r \in (p, q)$ there is a constant Λ_0^+ depending only on $p, q, r, a, b, \alpha, \beta, T$, (4.1) has two: one: zero positive solutions, in case $\lambda > \Lambda_0^+ : \lambda = \Lambda_0^+ : \lambda < \Lambda_0^+$.

2') if $s \in (p, q)$ there is a constant Λ_0^- depending only on $p, q, s, a, b, \alpha, \beta, T$, (4.1) has two: one: zero negative solutions, in case $\lambda > \Lambda_0^- : \lambda = \Lambda_0^- : \lambda < \Lambda_0^-$.

Proof. Let $u(x)$ be a positive solution of (4.1) then it solves

$$\begin{cases} -(\varphi(u'))' = \lambda a u^{r-1}, & -T < x < T\,, \\ u(-T) = 0, \quad u(T) = 0\,. \end{cases}$$

Let $u_0 = u(x_0) = \max u(x)$ as before, then it follows from (2.4) and (2.5) that $u(x)$ solves

$$u'(x) = \begin{cases} \sqrt[p]{\frac{\lambda a}{\alpha}\frac{p'}{r}(u_0^r - u^r(x))}, & -T \le x \le x_0\,, \\ -\sqrt[q]{\frac{\lambda a}{\beta}\frac{q'}{r}(u_0^r - u^r(x))}, & x_0 \le x \le T\,, \end{cases}$$

and (λ, u_0, x_0) satisfies

$$x_0 = C_1 \sqrt[p]{\frac{\alpha}{\lambda a}} u_0^{\frac{p-r}{p}} - T, \tag{4.2}$$

$$m_+(u_0) =: C_1 \sqrt[p]{\frac{\alpha}{\lambda a}} u_0^{\frac{p-r}{p}} + C_2 \sqrt[q]{\frac{\beta}{\lambda a}} u_0^{\frac{q-r}{q}} = 2T\,, \tag{4.3}$$

where

$$C_1 = \frac{1}{r}B(\frac{1}{p},\frac{1}{r})\sqrt[p]{\frac{r}{p'}}, \quad C_2 = \frac{1}{r}B(\frac{1}{q},\frac{1}{r})\sqrt[q]{\frac{r}{q'}}.$$

In analogy, we derive if $u(x)$ is a negative solution of

$$\begin{cases} -(\varphi(u'))' = \lambda b|u|^{s-1}, & -T < x < T\,, \\ u(-T) = 0, \quad u(T) = 0\,, \end{cases}$$

$v_0 = -u(x_0) = \max|u(x)|$, then

$$x_0 = D_1\sqrt[q]{\frac{\beta}{\lambda b}}v_0^{\frac{q-s}{q}} - T, \tag{4.2'}$$

$$m_-(v_0) =: D_1\sqrt[q]{\frac{\beta}{\lambda b}}v_0^{\frac{q-s}{q}} + D_2\sqrt[p]{\frac{\alpha}{\lambda b}}v_0^{\frac{p-s}{p}} = 2T\,, \tag{4.3'}$$

where

$$D_1 = \frac{1}{s}B(\frac{1}{q},\frac{1}{s})\sqrt[q]{\frac{s}{q'}}, \quad D_2 = \frac{1}{s}B(\frac{1}{p},\frac{1}{s})\sqrt[p]{\frac{s}{p'}}.$$

Without loss of generality, we shall in the sequel assume that $p < q$. It follows from (4.3) and (4.3') that m_+, m_- will be monotone functions, whenever r, s does not belong to the interval (p, q) and equation (4.3) respectively (4.3') is uniquely solvable when r, s do not belong to $[p, q]$.

If $r \in (p, q)$ then we see clearly that

$$m_+(u_0) \to +\infty, \qquad \text{as either } u_0 \to 0+, \qquad \text{or } u_0 \to +\infty\,,$$

and m_+ has a unique minimum point $\tilde{u}_0$ and is decreasing on $(0, \tilde{u}_0)$ and is increasing on $(\tilde{u}_0, +\infty)$, furthermore and get after some calculations the minimum of $m_+(u_0)$

$$M_+ = \frac{1}{\sqrt[r]{\lambda a}} \cdot C_1^{\frac{(q-r)p}{(q-p)r}} \cdot C_2^{\frac{(r-p)q}{(q-p)r}} \cdot \alpha^{\frac{q-r}{(q-p)r}} \cdot \beta^{\frac{p-r}{(p-q)r}} \cdot \frac{(q-p)r}{(q-r)p} \cdot \left(\frac{(r-p)q}{(q-r)p}\right)^{\frac{(p-r)q}{(q-p)r}}$$

which is achieved at

$$\tilde{u}_0 =: \frac{1}{\sqrt[r]{\lambda a}} \cdot \alpha^{\frac{q}{(q-p)r}} \cdot \beta^{\frac{p}{(p-q)r}} \cdot \left(\frac{(r-p)qC_1}{(q-r)pC_2}\right)^{\frac{pq}{(q-p)r}}.$$

Obviously, (4.3) has two solutions; one solution; non solution, precisely when inequalities $M_+ < 2T; M_+ = 2T; M_+ > 2T$ are satisfied. A simplification shows that $M_+ \le 2T$ is equivalent to $\lambda \ge \Lambda_0^+$ where

$$\Lambda_0^+ =: C_1^{\frac{(q-r)p}{q-p}} \cdot C_2^{\frac{(r-p)q}{q-p}} \cdot \alpha^{\frac{q-r}{q-p}} \cdot \beta^{\frac{r-p}{q-p}} \cdot (a(2TC_+)^r)^{-1}$$

$$C_+ = \frac{p}{r} \cdot \frac{q-r}{q-p}\left(\frac{r-p}{q-r} \cdot \frac{q}{p}\right)^{\frac{q}{r}\cdot\frac{r-p}{q-p}}.$$

In similar way, we can show that if $r \in (p,q)$ then the unique minimum M_- of function $m_-(v_0)$ will be achieved at

$$\tilde{v}_0 = \frac{1}{\sqrt[s]{\lambda b}} \cdot \beta^{\frac{p}{(p-q)s}} \cdot \alpha^{\frac{q}{(q-p)s}} \cdot \left(\frac{(s-q)pD_1}{(p-s)qD_2} \right)^{\frac{pq}{(p-q)s}}$$

and the minimum is

$$M_- = \frac{1}{\sqrt[s]{\lambda b}} \cdot D_1^{\frac{(p-s)q}{(p-q)s}} \cdot D_2^{\frac{(s-q)p}{(p-q)s}} \cdot \alpha^{\frac{p-s}{(p-q)s}} \cdot \beta^{\frac{q-s}{(q-p)s}} \cdot \frac{(p-q)s}{(p-s)q} \cdot \left(\frac{(s-q)p}{(p-s)q} \right)^{\frac{(q-s)p}{(p-q)s}} .$$

Consequently, equation (4.3'), similarly as (4.3), has two solutions; one solution; non solution, whenever λ satisfies inequalities $\lambda > \Lambda_0^-; \lambda = \Lambda_0^-; \lambda < \Lambda_0^-$, where

$$\Lambda_0^- =: D_1^{\frac{(p-s)p}{p-q}} \cdot D_2^{\frac{(s-q)p}{p-q}} \cdot \alpha^{\frac{p-s}{p-q}} \cdot \beta^{\frac{s-q}{p-q}} \cdot (b(2TC_-)^s)^{-1} ,$$

where

$$C_- = \frac{q}{s} \cdot \frac{p-s}{p-q} \left(\frac{s-q}{p-s} \cdot \frac{p}{q} \right)^{\frac{p}{s} \cdot \frac{s-q}{p-q}}$$

and the proof is complete. □

Theorem 4. *For any given $\alpha, \beta, a, b > 0, p, q, r, s > 1, p \neq q$ and $\psi(s) = as_+^{r-1} - bs_-^{s-1}$ then the following statements hold*

I) if either $\max\{r,s\} < \min\{p,q\}$ or $\min\{r,s\} > \max\{p,q\}$, the boundary value problem (4.1) has for all $\lambda > 0$ a unique initially positive solution and a unique initially negative solution with a fixed number of nodes. Henceforth, for any $\lambda > 0$ there are infinitely many solutions to (4.1).

II) if $\max\{r,s\} > \min\{p,q\}, \min\{r,s\} < \max\{p,q\}$, there exist positive, increasing, un-bounded sequences $\{\Lambda_k\}_0^\infty, \{\Lambda^+(k)\}_0^\infty, \{\Lambda^-(k)\}_0^\infty$, which depends only on $p,q,r,s,\alpha,\beta,a,b,T$ and have a property

$$\Lambda_k < \Lambda_k^\pm < \Lambda_{k+1}, \quad \forall k \geq 0, \quad \text{and} \quad \Lambda_k = Ck^\delta(1 + \circ(1)), \quad \text{as} \quad k \to +\infty,$$

where $\delta = \min\{\max\{p,q\}, \max\{r,s\}\}$, such that

1) (4.1) has two: one : zero initially positive and negative solutions with $(2k+1)$ nodes, if $\lambda > \Lambda_k : \lambda = \Lambda_k : \lambda < \Lambda_k$.

2) (4.1) has two : one : zero initially positive solutions with $2k+2$ nodes, if $\lambda > \Lambda^+(k) : \lambda = \Lambda^+(k) : \lambda < \Lambda^+(k)$.

2') (4.1) has two : one : zero initially negative solutions with $2k+2$ nodes, if $\lambda > \Lambda^-(k) : \lambda = \Lambda^-(k) : \lambda < \Lambda^-(k)$.

Consequently, (4.1) has only finite many nodal solutions for any given $\lambda > 0$.

Proof. Without loss of generality, we may assume $p < q$. Let $u(x)$ be a nodal solutions, then positive/negative waves of u have same height and same wavelength. Thus for the solution $u(x)$ with $(2k+1)$ nodes; initially positive solution with $2k+2$ no des; initially negative solution with $2k+2$ nodes, then λ, u_0, v_0 defined as before should satisfy the following equations

$$\frac{a}{r}u_0^r = \frac{b}{s}v_0^s, \tag{4.5}$$

$$(k+1)(m_+(u_0) + m_-(v_0)) = 2T, \tag{4.6}$$

$$(k+2)m_+(u_0) + (k+1)m_-(v_0)) = 2T, \tag{4.7}$$

$$(k+1)(m_+(u_0) + (k+2)m_-(v_0)) = 2T. \tag{4.8}$$

Since the analysis of (4.6) to (4.8) are very similar, so we shall mainly demonstrate the case of (4.6). In order to make the analysis be symmetric with respect to u_0, v_0, we introduce an energy level parameter $t = \frac{a}{r}u_0^r$ and now study the time mapping

$$m(\lambda, t) =: m_+(u_0) + m_-(v_0) = \frac{C_3}{\sqrt[p]{\lambda}}t^{\frac{p-r}{pr}} + \frac{C_4}{\sqrt[q]{\lambda}}t^{\frac{q-r}{qr}} + \frac{D_3}{\sqrt[q]{\lambda}}t^{\frac{q-s}{qs}} + \frac{D_4}{\sqrt[p]{\lambda}}t^{\frac{p-s}{ps}}, \tag{4.9}$$

where $C_3, C_4, D_3, D_4 > 0$ depend only on $a, b, p, q, r, \alpha, \beta$, and can be written down explicitly. To save some space, we omit the details.

It follows from (4.9) that $m(\lambda, \cdot)$ is increasing (decreasing), if $r, s \le p$ $(r, s \ge q)$ and furthermore in case of strict inequality, m have even a property

$$m(\lambda, 0) = 0, \quad \text{and} \quad m(\lambda, t) \to +\infty, \quad \text{as } t \to +\infty.$$

Consequently (4.6) will have a unique solution and hence problem (4.1) has one initially positive and one initially negative nodal solutions with $(2k+1)$ nodes for any $\lambda > 0$.

If neither $r, s \le p$ nor $r, s \ge q$ holds, then at least one of the exponents of t in (4.9) is positive and one is negative, thus m have the property

$$m(\lambda, t) \to +\infty, \quad \text{as either} \quad t \to +\infty, \quad \text{or } t \to 0+.$$

We claim that in all those cases $m(\lambda, t)$ has a same property as the the function $m_+(u_0)$ in (3.3) with $p < r < q$, namely, having a unique minimum point $t_0 > 0$, decreasing on $(0, t_0)$ increasing on (t_0, ∞).

In sequel, we shall show the claim. Assume on the contrary, $m(\lambda, \cdot)$ has at least two local minimum points, which means that the equation $m_t'(\lambda, t) = 0$ has at least three solutions. There are in principle two cases, which need to be treated separately and are classified by the signs of the exponents in (4.9): 1) one exponent is positive and three exponents are negative, 2) two positive and two negative exponents, since the case with three positive exponents and one negative exponent is equivalent to case 1) by change of variable $\tau = 1/t$.

The representative situations in the cases 1) and 2) are I) $r < p < s < q$, II) $r < p < q < s$.

Let

$$\sigma_1 = \frac{1}{p'} + \frac{1}{r}, \quad \sigma_2 = \frac{1}{q'} + \frac{1}{r}, \quad \sigma_3 = \frac{1}{q'} + \frac{1}{s}, \quad \sigma_4 = \frac{1}{p'} + \frac{1}{s},$$

then $0 < \sigma_4 < \sigma_3, \sigma_1 < \sigma_2$ due to assumption $p < q$, and $m(\lambda, t)$ can be written as

$$m(\lambda, t) = \frac{C_3}{\sqrt[p]{\lambda}} t^{1-\sigma_1} + \frac{C_4}{\sqrt[q]{\lambda}} t^{1-\sigma_2} + \frac{D_3}{\sqrt[q]{\lambda}} t^{1-\sigma_3} + \frac{D_4}{\sqrt[p]{\lambda}} t^{1-\sigma_4}\,. \tag{4.10}$$

Case I) $r < p < s < q$. Then $\sigma_4 < 1 < \sigma_1, \sigma_2, \sigma_3$ and in (4.10) the first three exponents are negative and the last is positive, thus it is easy to see that the equation $m'_t(\lambda, t) = 0$ becomes

$$-\frac{C'_3}{\sqrt[p]{\lambda}} t^{-\sigma_1} - \frac{C'_4}{\sqrt[q]{\lambda}} t^{-\sigma_2} - \frac{D'_3}{\sqrt[q]{\lambda}} t^{-\sigma_3} + \frac{D'_4}{\sqrt[p]{\lambda}} t^{-\sigma_4} = 0$$

which is equivalent to

$$g_1(t) =: \frac{C'_3}{\sqrt[p]{\lambda}} t^{\sigma_4-\sigma_1} + \frac{C'_4}{\sqrt[q]{\lambda}} t^{\sigma_4-\sigma_2} + \frac{D'_3}{\sqrt[q]{\lambda}} t^{\sigma_4-\sigma_3} = \frac{D'_4}{\sqrt[p]{\lambda}}\,, \tag{4.11}$$

where $C'_3, C'_4, D'_3, D'_4 > 0$. But the function $g_1(t)$ in (4.11) is strictly decreasing and thus its graph can not intersect any horizontal line more than once. So we get a contradiction and are done in this case.

Case II) $r < p < q < s$. Then $\sigma_3, \sigma_4 < 1 < \sigma_1, \sigma_2$ and the first two exponents in (4.10) are negative and the last two are positive, therefore the equation $m'_t(\lambda, t) = 0$ becomes

$$-\frac{C'_3}{\sqrt[p]{\lambda}} t^{-\sigma_1} - \frac{C'_4}{\sqrt[q]{\lambda}} t^{-\sigma_2} + \frac{D'_3}{\sqrt[q]{\lambda}} t^{-\sigma_3} + \frac{D'_4}{\sqrt[p]{\lambda}} t^{-\sigma_4} = 0$$

which is equivalent to

$$g_2(t) =: \frac{D'_3}{\sqrt[q]{\lambda}} t^{\sigma_2-\sigma_3} + \frac{D'_4}{\sqrt[p]{\lambda}} t^{\sigma_2-\sigma_4} - \frac{C'_3}{\sqrt[p]{\lambda}} t^{\sigma_2-\sigma_1} = \frac{C'_4}{\sqrt[q]{\lambda}}\,. \tag{4.11'}$$

If we succeed to show that equation (4.11') can have at most two solutions, then we are done.

A direct computation shows

$$g'_2(t) = (\sigma_2 - \sigma_3)\frac{D'_3}{\sqrt[q]{\lambda}} t^{\sigma_2-\sigma_3-1} + (\sigma_2 - \sigma_4)\frac{D'_4}{\sqrt[p]{\lambda}} t^{\sigma_2-\sigma_4-1} - (\sigma_2 - \sigma_1)\frac{C'_3}{\sqrt[p]{\lambda}} t^{\sigma_2-\sigma_1-1}.$$

If $\sigma_1 \geq \sigma_3$, by rewriting $g'(t) = k(t) t^{\sigma_2-\sigma_1-1}$, where

$$k(t) =: (\sigma_2 - \sigma_3)\frac{D'_3}{\sqrt[q]{\lambda}} t^{\sigma_1-\sigma_3} + (\sigma_2 - \sigma_4)\frac{D'_4}{\sqrt[p]{\lambda}} t^{\sigma_1-\sigma_4} - (\sigma_2 - \sigma_1)\frac{C'_3}{\sqrt[p]{\lambda}}$$

is strictly increasing, we deduce that $g_2'(t)$ vanishes exactly once.

If $\sigma_1 < \sigma_3$, then we can rewrite $g_2'(t)$ as follows

$$g_2'(t) = l(t)t^{\sigma_2-\sigma_3-1},$$

where

$$l(t) = (\sigma_2-\sigma_3)\frac{D_3'}{\sqrt[q]{\lambda}} + (\sigma_2-\sigma_4)\frac{D_4'}{\sqrt[p]{\lambda}}t^{\sigma_3-\sigma_4} - (\sigma_2-\sigma_1)\frac{C_3'}{\sqrt[p]{\lambda}}t^{\sigma_3-\sigma_1}.$$

However, function $l(t)$ cannot vanish more than two times on $(0,+\infty)$, since $l(0)>0, l(+\infty)=+\infty$, and

$$l'(t) = \left((\sigma_3-\sigma_4)(\sigma_2-\sigma_4)\frac{D_4'}{\sqrt[p]{\lambda}}t^{\sigma_1-\sigma_4} - (\sigma_3-\sigma_1)(\sigma_2-\sigma_1)\frac{C_3'}{\sqrt[p]{\lambda}}\right)t^{\sigma_3-\sigma_1-1}$$

vanishes exactly once on $(0,+\infty)$. So this completes the proof of the claim. □

Going back to equation (4.6), let

$$M(\lambda) = \min_{t>0} m(\lambda,t),$$

we deduce that (4.6) has two solutions; one solution; no solution precisely when $(k+1)M(\lambda) < 2T; (k+1)M(\lambda) = 2T; (k+1)M(\lambda) > 2T$. Henceforth, we derive that if M is monotone, and $2T/(k+1)$ belongs to the image of $\mathbf{R}_+$, then

$$\Lambda_k = M^{-1}(2T/(k+1)), \quad k = 0,1,2,\cdots. \tag{4.12}$$

Since $m(\lambda,t)$ is decreasing in t, we deduce that even $M(\lambda)$ should be decreasing, which can be easily verified

$$\lambda_1 < \lambda_2 \quad \Rightarrow \quad M(\lambda_1) = m(\lambda_1,t_1) > m(\lambda_2,t_1) \geq \min_t m(\lambda_2,t) = M(\lambda_2).$$

To ensure that $2T/(k+1) \in M(\mathbf{R}_+)$, it is enough to show

Claim. $\lim_{\lambda\to 0+} M(\lambda) = +\infty, \quad \lim_{\lambda\to+\infty} M(\lambda) = 0.$

We shall prove the claim in two separate cases as before and first deal with the case II) $r<p<q<s$, then $\delta = q$. Let $t=t(\lambda)$ be the unique minimum point of the function $m(\lambda,\cdot)$, then we see that $M(\lambda) = m(\lambda,t(\lambda))$ and $t(\lambda)$ solves the equation (4.11'). By rewriting (4.11'), we obtain

$$\frac{t^{-\sigma_1}}{\sqrt[p]{\lambda}}\left(C_3' - D_4't^{\sigma_1-\sigma_4}\right) = \left(D_3't^{\sigma_2-\sigma_3} - C_4'\right)\frac{t^{-\sigma_2}}{\sqrt[q]{\lambda}}. \tag{4.13}$$

It follows from (4.13) that the exponent τ in the ansatz $t = (\varepsilon + \circ(1))\lambda^\tau$ in the asymptotic analysis as λ goes to zero (infinity) must be zero, because positive τ implies that terms $D_3't^{\sigma_2-\sigma_3} - C_4'$ and $C_3' - D_4't^{\sigma_1-\sigma_4}$ would have different signs whenever λ is enough small (large). So it would be a contradiction. If $\tau = 0$, then we derive easily from (4.11') that ε is determined by the equation $\varepsilon^{\sigma_1-\sigma_4} = C_3'/D_4'$ when $\lambda\to 0$; and by another equation $\varepsilon^{\sigma_2-\sigma_3} = C_4'/D_3'$ when

$\lambda \to +\infty$ and furthermore $M(\lambda)$ have asymptotic expansions

$$M(\lambda) = (C_5 + \circ(1))\frac{1}{\sqrt[p]{\lambda}} \to +\infty, \quad \text{as } \lambda \to 0+; \tag{4.14a}$$

$$M(\lambda) = (C_6 + \circ(1))\frac{1}{\sqrt[q]{\lambda}} \to 0, \quad \text{as } \lambda \to +\infty, \tag{4.14b}$$

where $C_5 = C_3\varepsilon^{1-\sigma_1} + D_4\varepsilon^{1-\sigma_4}, C_6 = C_4\varepsilon^{1-\sigma_2} + D_3\varepsilon^{1-\sigma_3}$.

Case I) $r < p < s < q$. Then $\delta = s$, and we have $M(\lambda) = m(\lambda, t(\lambda))$, where $t(\lambda)$ solves the equation (4.11). If we rewrite (4.11) as

$$C_3' t^{\sigma_4-\sigma_1} + (C_4' t^{\sigma_4-\sigma_2} + D_3' t^{\sigma_4-\sigma_3})\lambda^{\frac{1}{p}-\frac{1}{q}} = D_4'. \tag{4.15}$$

Again by the ansatz $t = (\varepsilon + \circ(1))\lambda^\tau, \lambda \to 0+$, we derive that if τ is positive, then the left in (4.15) will tend to infinity, due to $\sigma_4 - \sigma_1 < 0$; and on the other hand, if τ is negative, the left in (4.15) goes to zero due to $\frac{1}{p} - \frac{1}{q} > 0, \sigma_4 - \sigma_1, \sigma_4 - \sigma_2, \sigma_4 - \sigma_3 < 0$; Consequently, τ has to be zero, and we thereafter can proceed as in the case II) and obtain the same estimate in (4.14a).

To treat the asymptotic of $M(\lambda)$ as $\lambda \to +\infty$, we do once more the ansatz analysis $t = (\varepsilon + \circ(1))\lambda^\tau$ and infer that τ must be positive, since otherwise the left in (4.15) will go to infinity due to $\frac{1}{p} - \frac{1}{q} > 0, \sigma_4 - \sigma_2 < 0$. If τ is positive, then the first term in (4.15) tends to zero, and further the third term dominates in comparison between the second term, because $\sigma_4 - \sigma_2 < \sigma_4 - \sigma_3$. Thus τ should solve the equation

$$\frac{1}{p} - \frac{1}{q} + \tau(\sigma_4 - \sigma_3) = 0, \quad \text{i.e.,} \quad \tau = \frac{\frac{1}{p} - \frac{1}{q}}{\sigma_3 - \sigma_4} = 1,$$

and ε is then determined by $\varepsilon^{\sigma_3-\sigma_4} = D_4'/D_3'$. Whence we get from (4.10) the estimate of M as follows

$$M(\lambda) = (D_4 + D_3\varepsilon^{\sigma_3-\sigma_4} + \circ(1))\lambda^{-\frac{1}{p}+1-\sigma_4}, \quad \text{as } \lambda \to +\infty.$$

But by definition of σ_3, σ_4, it is easy to verify

$$-\frac{1}{p} + \tau(1-\sigma_4) = -\frac{1}{p} + \frac{\frac{1}{p} - \frac{1}{q}}{\sigma_3 - \sigma_4}(1-\sigma_4) = -\frac{1}{s} < 0.$$

Whence, we have established the asymptotic estimate of M

$$M(\lambda) = (C_7 + \circ(1))\lambda^{-\frac{1}{s}}, \quad \text{as } \lambda \to +\infty. \tag{4.14c}$$

It follows from (4.14b) and (4.14c) that

$$M(\lambda) = (C_8 + \circ(1))\lambda^{-\frac{1}{\delta}}, \quad M^{-1}(1/\lambda) = (C_9 + \circ(1))\lambda^{\delta}, \text{ as } \lambda \to +\infty \tag{4.14d}$$

which yields the desired estimate of Λ_k, as $k \to \infty$.

To show that $\Lambda_k^\pm \in (\Lambda_k, \Lambda_{k+1})$, we deduce similarly as above from (4.7),

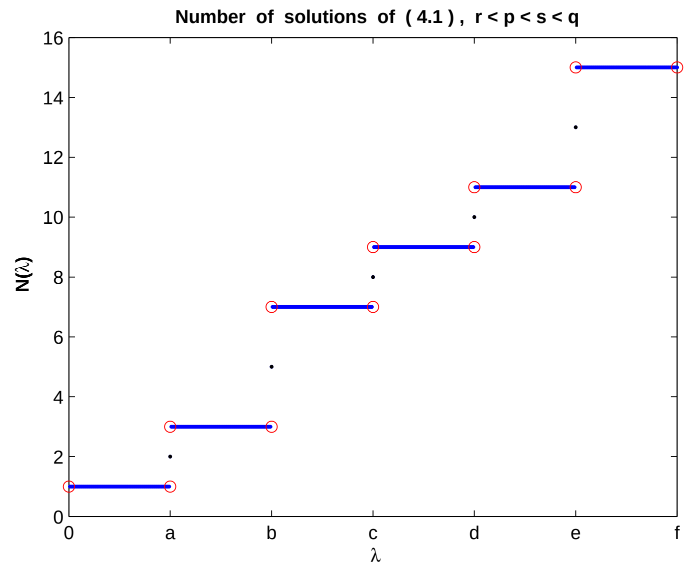

Figure 1:

(4.8) that $\Lambda_k^{\pm}$ solves

$$M_{\pm}(k, \Lambda) = 2T,$$

where $M_{\pm}(k, \lambda) = \min_{t>0} m_{\pm}(k, \lambda, t)$ and

$$m_{+}(k, \lambda, t) = (k+2)m_{+}(u_0) + (k+1)m_{-}(v_0)\,,$$
$$m_{-}(k, \lambda, t) = (k+1)m_{+}(u_0) + (k+2)m_{-}(v_0)\,.$$

However, it is clear that $m_{\pm}, M_{\pm}$ are decreasing in λ and satisfy

$$(k+1)m(\lambda, t) < m_{\pm}(k, \lambda, t) < (k+2)m(\lambda, t),$$

and which in turn imply

$$(k+1)M(\lambda) < M_{\pm}(k, \lambda) < (k+2)M(\lambda).$$

Thus we have

$$(k+1)M(\Lambda_k^{\pm}) < 2T = M_{\pm}(k, \Lambda_k^{\pm}) = 2T < (k+2)M(\Lambda_k^{\pm})$$

and by taking the inverse of M find the estimates

$$\Lambda_k^{\pm} > M^{-1}(2T/(k+1)) = \Lambda_k, \quad \Lambda_k^{\pm} < M^{-1}(2T/(k+2)) = \Lambda_{k+1}.$$

Whereas the proof is complete.

Finally, to illustrate the differences compared with the odd case, let $N(\lambda) =$

the number of solution of (4.1), then for the choice of $r < p < s < q$ we have

λ	0		Λ_0^-		Λ_1		Λ_1^-		Λ_1^+		Λ_2		Λ_2^-	$\cdots$
$N(\lambda)$	0	1	2	3	5	7	8	9	10	11	13	15	16	$\cdots$

which is visualized in Figure 1, where $a = \Lambda_0^-, b = \Lambda_1, c = \Lambda_1^-, d = \Lambda_1^+, e = \Lambda_2, f = \Lambda_2^-$.

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