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WRONG TAILS AND RIGHT VALUES

Ragnhild Johanne Rensaa[1 §], Haakon Waadeland[2]
[1]R&D Group for Electromechanical Systems
Institute for Information, Energy and Space Technology
Narvik University College
P.O. Box 385, Narvik, N-8505, NORWAY
e-mail: rjr@hin.no
[2]Department of Mathematical Sciences
Norwegian University of Science and Technology
Trondheim, N-7491, NORWAY
e-mail: haakonwa@math.ntnu.no

Abstract: The purpose of the present paper is to introduce an alternative method for computing approximants of a continued fraction. Although the method is based upon established theory for tails of continued fractions – right tails and wrong tails – it has never been presented in this form. The main idea is to illustrate how wrong tails can be used in the computation. This is a consequence of the stability of the algorithm for wrong tails.

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1. Introduction

In the present paper we are going to deal with continued fractions of the form

$$\mathop{\mathrm{K}}_{m=1}^{\infty}\left(\frac{a_m}{1}\right), \quad a_n \in \mathbb{C}, a_n \neq 0,\ n = 1, 2, 3, \dots . \tag{1}$$

We shall assume that they converge, which means that $\lim_{n\to\infty} f_n = f \in \widehat{\mathbb{C}}$ exists. Here f_n are the *approximants*. For the general theory of continued fractions we refer to Jones and Thron [4] and Lorentzen and Waadeland [5].

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§Correspondence author

Remark. A convenient way of representing continued fractions is by way of linear fractional transformations. With

$$s_m(w) = \frac{a_m}{1+w}$$

we find

$$S_n(w) = s_1 \cdot s_2 \cdots s_n(w) = \frac{a_1}{1} + \frac{a_2}{1} + \cdots + \frac{a_n}{1+w}.$$

We have in particular $f_n = S_n(0)$. In many cases it is much more natural to study $S_n(w_n)$ for some convenient w_n rather than $S_n(0)$. Often very much is gained in speed of convergence by such a transition. An extensive theory has been developed throughout the last 30-40 years (see e.g. Lorentzen and Waadeland [5] with further references). The most recent steps have rather successfully been aiming at high-precision, reliable evaluation of certain special functions. We shall not go into that here, merely refer to Cuyt et al [1], [2].

The concept of *tail* is an important one in the theory of continued fractions:

Definition 1. (Sequence of Tails) Let $g^{(0)}$ be an arbitrary complex number. The sequence $\left\{g^{(n)}\right\}$, defined recursively in (2), is called a sequence of tails of (1) if for all $n \in N_0$ and all a_n

$$g^{(n)} = \frac{a_{n+1}}{1+g^{(n+1)}} \quad n = 0, 1, 2, ... \tag{2}$$

If $g^{(0)} = f^{(0)} := f$, and hence $g^{(n)} = f^{(n)}$, then the tails are called *right tails,* else *wrong tails*. The main purpose of the paper is to illustrate how wrong tails can be used in the computation of sequences of approximants of continued fractions. Before we do this we present two examples to illustrate the stability of wrong tails.

Example 2. It is well known that the continued fraction

$$\underset{m=1}{\overset{\infty}{\mathrm{K}}} \left(\frac{m(m+2)}{1} \right)$$

converges to the value 1, actually that $f^{(n)} = n + 1$ for all $n \in \mathbb{N}$. We take the tail number 0 to be

$$g^{(0)} = 1 + h, \tag{3}$$

where h is a small number $\neq 0$. Let $\{g^{(m)}\}$ denote the sequence of (wrong) tails given by the initial condition (3) and the recurrence relation (2). We find

$$g^{(1)} - 2 = \frac{-3h}{1+h}, \quad g^{(2)} - 3 = \frac{6h}{1 - \frac{1}{2}h}.$$

It then follows by induction on (2) that

$$g^{(2k)} - (2k+1) = \frac{(k+1)(2k+1)h}{1 - \frac{k}{2}h},$$

$$g^{(2k+1)} - (2k+2) = \frac{-(k+1)(2k+3)h}{1 + (1 + \frac{k}{2})h},$$

showing that a start near $f^{(0)}$ does not imply a tail sequence near $\{f^{(m)}\}$. Moreover, it follows that

$$\lim_{n\to\infty} \frac{g^{(n)}}{n+1} = -1$$

for all $h \neq 0$, illustrating the stability of wrong tails.

Example 3. A different example is the one-periodic continued fraction

$$\mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{2}{1}\right). \tag{4}$$

Any sequence $\{g^{(m)}\}$ of tails of (4) is given by $g^{(0)}$ and the recurrence relation

$$g^{(m)} = \frac{2}{1 + g^{(m+1)}}.$$

Since $u = 2/(1+u)$ has the solutions $u = 1$ and $u = -2$, two particular tail value sequences are

$$1, 1, 1, \ldots, 1, \ldots \quad \text{and} \quad -2, -2, -2, \ldots, -2, \ldots, \tag{5}$$

the first one being the sequence of right tail values, the last one of wrong values. Now take $g^{(0)} = 1 + h$, where $h \neq 0, -3$. We find

$$g^{(1)} = \frac{1-h}{1+h},\ g^{(2)} = \frac{1+3h}{1-h},$$

and as in Example 2

$$g^{(2k)} = \frac{1 + \frac{2^{2k+1}+1}{3}h}{1 - \frac{2^{2k}-1}{3}h}, \quad g^{(2k+1)} = \frac{1 - \frac{2^{2k+2}-1}{3}h}{1 + \frac{2^{2k+1}+1}{3}h}.$$

From this follows that $g^{(2k)}$ and $g^{(2k+1)}$ both tend to -2 when $k \to \infty$, i.e. the tail sequence approaches the particular sequence of wrong tails in (5).

2. Tails as Tools

What we have seen in the two examples in Section 1 is an instability of the sequence of right tail values. An "almost good" value of the continued fraction does not lead to "almost good" values of the tails. Nevertheless, wrong tails are useful in different ways. We only mention one, first presented in Waadeland [6], and here somewhat rephrased:

Let

$$\kappa_n := -\frac{1+g^{(n)}}{g^{(n)}}. \tag{6}$$

Then the following holds:

Theorem 4. *If*

$$S_n := 1 + \kappa_1 + \kappa_1\kappa_2 + \cdots + \kappa_1\kappa_2\cdots\kappa_n\,,$$

then

$$f_n = g^{(0)}\left(1 - 1/S_n\right).$$

Observe that we get the same f_n regardless of the value of $g^{(0)}$. Illustration for $n = 1$:

$$S_1 := 1 + \kappa_1 = 1 - \frac{1+g^{(1)}}{g^{(1)}} = -\frac{1}{g^{(1)}},$$

$$g^{(0)}(1 - \frac{1}{S_1}) = g^{(0)}(1 + g^{(1)}) = a_1 = f_1.$$

The theorem is proved in Waadeland [6]. But we shall here present an alternative proof of the theorem by using Euler's transformation. We will use the result to derive an algorithm for how to find the sequence of approximants f_n.

Proof. We use the following recurrence relation

$$\kappa_{m+1} = \frac{1}{1 + a_{m+1}(\kappa_m + 1)} - 1, \quad m = 1, 2, \cdots, \tag{7}$$

where

$$\kappa_1 = \frac{a_1}{r - a_1}$$

is obtained from (6) and (2). Here $r = g^{(0)}$. We use formula (2.3.28), in Jones and Thron [4], p. 37, which for our purpose and with our notation takes the form

$$1 + \kappa_1 + \cdots + \kappa_1\kappa_2\cdots\kappa_n = 1 + \frac{\kappa_1}{1} \; + \; \frac{-\kappa_2}{1+\kappa_2} \; + \; \frac{-\kappa_3}{1+\kappa_3} \; + \cdots + \; \frac{-\kappa_n}{1+\kappa_n}.$$

It follows immediately that

$$1 + \kappa_2 + \cdots + \kappa_2\kappa_2\cdots\kappa_n = \frac{1}{1} \; + \; \frac{-\kappa_2}{1+\kappa_2} \; + \; \frac{-\kappa_3}{1+\kappa_3} \; + \cdots + \; \frac{-\kappa_n}{1+\kappa_n},$$

leading to

$$\frac{1}{S_n} = 1 + \frac{-\kappa_1}{1+\kappa_1} \; + \; \frac{-\kappa_2}{1+\kappa_2} \; + \cdots + \; \frac{-\kappa_n}{1+\kappa_n}. \tag{8}$$

Use of (7), (8), the relation

$$a_m = g^{(m-1)}(1 + g^{(m)})$$

and an equivalence transformation leads to the result

$$g^{(0)}\,(1 - 1/S_n) = \operatorname*{K}_{m=1}^{n} \left(\frac{a_m}{1} \right). \qquad \square$$

Observe that the argument is independent of the value of $g^{(0)}$.

3. The Algorithm

We recall that $g^{(0)}$ is arbitrary. To make the notations more convenient, we put $r = g^{(0)}$. Let κ_m be given as in (7). Then the sequence of approximants f_m may be produced by the following scheme:

$$\begin{aligned}
&\kappa_1 = \frac{a_1}{r - a_1} \\
&L_1 = \kappa_1 \\
&S_1 = 1 + L_1 \\
&f_1 = r(1 - 1/S_1) \\
&\kappa_2 = \frac{1}{1 + a_2(\kappa_1 + 1)} - 1 \\
&L_2 = L_1 \times \kappa_2 \\
&S_2 = S_1 + L_2 \\
&f_2 = r(1 - 1/S_2) \\
&\vdots \\
&\kappa_{m+1} = \frac{1}{1 + a_{m+1}(\kappa_m + 1)} - 1 \\
&L_{m+1} = L_m \times \kappa_{m+1} \\
&S_{m+1} = S_m + L_{m+1} \\
&f_{m+1} = r(1 - 1/S_{m+1}) \\
&\vdots
\end{aligned}$$

The stability of the algorithm with respect to r follows immediately from the independence of r. Observe also the continued fraction formula

$$\kappa_{m+1}+1=\frac{1}{1}+\frac{a_{m+1}}{1}+\frac{a_m}{1}+\cdots+\frac{a_2}{1+a_2(1+\kappa_0)},\quad m\geq 1,\tag{9}$$

where $\kappa_0+1=-1/g^{(0)}$. The equality (9) is crucial in Waadeland [6] and was established there by a proof based upon certain formulas where $A_n-g^{(0)}B_n$ and B_n are expressed in terms of tails, Jakobsen and Waadeland [3], Propositions 1 and 2, Wall [7], p. 47.

Example 5. As a non-trivial example take the continued fraction

$$\mathop{\mathrm{K}}_{m=1}^{\infty}\left(\frac{m^2}{1}\right).$$

With $r=2/5$, the scheme above leads to the κ_m-values from κ_1 to κ_{10},

$$-\frac{5}{3},-\frac{8}{5},-\frac{27}{22},-\frac{40}{29},-\frac{275}{246},-\frac{174}{133},-\frac{287}{268},-\frac{304}{237},-\frac{1809}{1730},-\frac{790}{617},$$

and with $r=1/2$ to the very different κ_m-values,

$$-2,-\frac{4}{3},-\frac{3}{2},-\frac{8}{7},-\frac{25}{18},-\frac{14}{13},-\frac{49}{36},-\frac{208}{199},-\frac{729}{530},-\frac{1990}{1937},$$

but in both cases to the same sequence of approximants

$$1,\quad\frac{1}{5},\quad\frac{5}{7},\quad\frac{13}{47},\quad\frac{23}{37},\quad\frac{101}{319},\quad\frac{307}{533},\quad\frac{641}{1879},\quad\frac{893}{1627},\quad\frac{7303}{20417}.$$

To implement the algorithm numerically, *Maple* is used.

Algorithm (Approximant)

Input

chosen value of r

the numerator a_n in the partial fraction

the number M of terms to be computed

Ouput

the M^{th} approximant a[M]

Compute

```
r:=value:
contfrac := proc(m, alist)
  local k,l,s,f:
  if (m>1) then
    k,l,s,f:=contfrac(m-1, alist);
```

```
        k[m]:=1/(1+alist[m]*(k[m-1]+1))-1;
        l[m]:=l[m-1]*k[m];
        s[m]:=s[m-1]+l[m];
        f[m]:=r*(1-1/s[m]);
        return k,l,s,f;
      elif (m=1) then
        k[m]:=alist[m]/(r-alist[m]);
        l[m]:=k[m];
        s[m]:=l[m]+1;
        f[m]:=r*(1-1/s[m]);
        return k,l,s,f;
      end if;
    end proc:
    Digits:=120:
    a_n:= numerator in the partial fraction ;
    M:=number of terms to be computed:
    alist:=[seq(a_n, n=1..M)]:
    k,l,s,f:=contfrac(M, alist):
    seq(simplify(k[i]), i=1..M): seq(simplify(l[i]), i=1..M):
    seq(simplify(s[i]), i=1..M): seq(simplify(f[i]), i=1..M):
    a:=seq(evalf(simplify(f[i])), i=1..M):
    a[M];
```

Example 6. For the convergent continued fraction $\operatorname*{K}_{m=1}^{\infty}\left(\frac{m(m+2)}{1}\right)$, i.e. $a_n = n(n+2)$ in the algorithm, approximants produced by the algorithm are given in Table 1. The convergence is indicated by the mean values of the approximants included in the table.

Example 7. For the convergent continued fraction $\operatorname*{K}_{m=1}^{\infty}\left(\frac{m^2}{1}\right)$, i.e. $a_n = n^2$ in the algorithm, the results are displayed in Table 2.

Example 8. With $\operatorname*{K}_{m=1}^{\infty}\left(\frac{m^3}{1}\right)$, i.e. $a_n = n^3$ in the algorithm, approximants produced are given in Table 3. The continued fraction diverges. However, the even and odd approximants converge separately (see Jones and Thron [4], p.

M	Approximants a[M] even	odd	Mean value $g_M = (f_M + f_{M-1})/2$
⋮	⋮	⋮	⋮
96	0.960000000		
97		1.040816327	1.000408163
98	0.960784314		1.000800320
99		1.040000000	1.000392157
100	0.961538462		1.000769231
⋮	⋮	⋮	⋮
	0.996000000		
997		1.004008016	1.000004008
998	0.996007984		1.000008000
999		1.004000000	1.000003992
1000	0.996015936	.	1.000007968

Table 1: Approximants and mean values when $a_n = n(n+2)$

M	Approximants a[M] even	odd
⋮	⋮	⋮
97		0.4533380335
98	0.4323114407	
99		0.4531247122
100	0.4325145899	
⋮	⋮	⋮
997		0.4437380420
998	0.4416545870	
999		0.4437359555
1000	0.4416566632	

Table 2: Approximants when $a_n = n^2$

106, Lorentzen and Waadeland [5], p. 131). The table shows $g_M = (f_M + f_{M-1})/2$.

M	Approximants a[M] even	odd	Mean value $g_M = (f_M + f_{M-1})/2$
⋮	⋮	⋮	⋮
996	0.2584116633		
997		0.5116126972	0.3850121803
998	0.2584180905		0.3850153938
999		0.5116026930	0.3850103917
1000	0.2584244981	.	0.3850135955

Table 3: Approximants and mean values when $a_n = n^3$

4. A Different Example

We conclude this paper by showing an example of a different type, but where we can use the same algorithm slightly modified. In the tables we here present the relative truncation error rather than the value itself.

Example 9. a) The logarithmic function has a formal series expansion

$$\mathrm{Log}(1+z) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} z^k, |z| < 1,$$

and an S-fraction in z of the form

$$\mathrm{Log}(1+z) = \frac{z}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{a_m z}{1}\right), \quad \text{where} \quad a_m = \left\lfloor \frac{m}{2} \right\rfloor \frac{2m-1+(-1)^m}{4m(m-1)}.$$

(Lorentzen and Waadeland [5], p. 17). Table 4 gives values for $z = 0.2$ (left) and $z = 6$ (right).

Approximants for particular values when $z = 0.2$ are

$$f_{19} = 0.1823215567939546262117180 35,$$
$$f_{20} = 0.1823215567939546262117180 25.$$

Since the continued fraction has only positive terms we have $f_{2n} < f < f_{2n-1}$, in particular $f_{20} < f < f_{19}$, and hence

$$\mathrm{Log}(1+0.2) \simeq 0.1823215567939546262117180,$$

correctly rounded in the 25-th place (Lorentzen and Waadeland [5], pp. 96-97). Approximants for particular values when $z = 6$ are

$$f_{19} = 1.9459110,$$
$$f_{20} = 1.9459098,$$

from which it follows that we have $\mathrm{Log}(1+6) \simeq 1.9459$, correctly rounded in

$M+1$	$\left\lvert \frac{f-f_n}{f} \right\rvert$	$M+1$	$\left\lvert \frac{f-f_n}{f} \right\rvert$
2	0.00276092	2	0.2291525
3	0.00017214	3	0.2333560
4	0.00000611	4	0.0512646
5	$3.40e-7$	5	0.0386156
⋮	⋮	⋮	⋮
17	$2.54e-23$	17	$2.17e-6$
18	$1.09e-24$	18	$8.48e-7$
19	$5.25e-26$	19	$4.38e-7$
20	$2.26e-27$	20	$1.74e-7$

Table 4: $a_n = z \left\lfloor \frac{n}{2} \right\rfloor \frac{2n-1+(-1)^n}{4n(n-1)}$

$M+1$	$\left\lvert \frac{f-f_n}{f} \right\rvert$	$M+1$	$\left\lvert \frac{f-f_n}{f} \right\rvert$
2	0.000137484	2	0.6716542287522
3	0.000001385	3	1.0020374901748
4	$1.38e-8$	4	0.4070349703730
5	$1.37e-10$	5	0.4258552653999
⋮	⋮	⋮	⋮
17	$1.11e-34$	17	$6.73e-3$
18	$1.09e-36$	18	$4.80e-3$
19	$1.07e-38$	19	$3.46e-3$
20	$1.05e-40$	20	$2.48e-3$

Table 5: $a_n = z^2 \frac{(n-1)^2}{(2n-3)(2n-1)}$

the 4^{th} place.

b) The Arcus tangent function has a formal series expansion

$$\text{Arctan} z = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1} z^{2k-1}, \quad |z| < 1,$$

and an S-fraction in z^2 of the form

$$z\text{Arctan}(1+z) = \frac{z^2}{1} + \underset{m=2}{\overset{\infty}{\mathrm{K}}} \left(\frac{a_m z^2}{1} \right) \text{ where } a_m = \frac{(m-1)^2}{(2m-3)(2m-1)}.$$

Table 5 gives values for $z = 0.2$ (left) and $z = 6$ (right).

Approximants for particular values when $z = 0.2$ are

$$f_{19} = 0.19739555984988075837004976519479029344970,$$
$$f_{20} = 0.19739555984988075837004976519479029344756,$$

from which it follows as in the log-case that we have

$$\mathrm{Arctan}(0.2) \simeq 0.197395559849880758370049765194790293445,$$

correctly rounded in the 38-th place. Approximants for particular values when $z = 6$ are

$$f_{19} = 1.410510621, \qquad f_{20} = 1.402166760.$$

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