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# ON REBONATO AND JÄCKEL'S PARAMETRIZATION METHOD FOR FINDING NEAREST CORRELATION MATRICES

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**Abstract:** Portfolio risk forecasts are often made by estimating an asset or factor correlation matrix. However, estimation difficulties or exogenous constraints can lead to correlation matrix candidates that are not positive semidefinite (psd). Therefore, practitioners need to reimpose the psd property with the minimum possible correction. Rebonato and Jäckel (2000) raised this question and proposed an approach; in this paper we improve on that approach by introducing a more geometric perspective on the problem.

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## 1. Introduction

Forecasts of correlations matrices are important for portfolio risk management, but they often suffer from estimation problems such as spurious outliers in the data, nonsynchronous data, questionable relevance of past history, and varying levels of confidence bounds. Also, risk managers may want to alter some entries of an estimated correlation matrix to account for information not incorporated into the estimation algorithm itself. The result can be a proposed "target"

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correlation matrix that, while visibly symmetric with unit diagonal, is not in fact positive semidefinite (psd) because of the presence of negative eigenvalues.

Non-psd matrices cannot arise as true correlation matrices. Moreover, they cannot be tolerated by the manager because they correspond to the existence of portfolios with negative risk, which, for example, will spoil portfolio optimization routines. This means the manager must correct the target matrix.

As a simple example, consider the following $3 \times 3$ correlation matrix, which could represent correlations of three assets, indices, or market factors:

$$C = \begin{pmatrix} 1 & 0.9 & 0.7 \\ 0.9 & 1 & 0.4 \\ 0.7 & 0.4 & 1 \end{pmatrix}.$$

Suppose, given some additional information, the manager wishes to adjust the correlation between variables 2 and 3 from 0.4 to 0.3.

We then get a target matrix

$$\tilde{C} = \begin{pmatrix} 1 & 0.9 & 0.7 \\ 0.9 & 1 & 0.3 \\ 0.7 & 0.3 & 1 \end{pmatrix},$$

The eigenvalues of $C$ are (2.35, 0.61, 0.03), and the eigenvalues of $\tilde{C}$ are (2.3, 0.71, -0.007). Hence the latter is not a valid correlation matrix.

Rebonato and Jäckel in [7] proposed a method of correcting the target by finding the nearest true correlation matrix, where "nearest" can be interpreted quite broadly (see below). For an $n \times n$ target, the method involves parametrizing the space of true correlation matrices using $N = n(n-1)$ real angle variables, expressing the distance to the target matrix as a function of these $N$ variables, and then applying a standard unconstrained minimization routine to minimize that distance.

The purpose of this note is to describe how to improve this method by costlessly reducing the number of variables from $N$ to $N/2$. This can represent a substantial savings in computation time for moderate values of $n$, since time is exponential in the dimension of the problem. In the process, we give a geometric description which is helpful in understanding the characteristics of the problem, and is useful for analyzing variants of the original question.

Since their paper was published in 2000, a variety of other approaches to this problem have been published. We comment briefly on them at the end of this note.

## 2. Finding the Nearest Valid Correlation Matrix

### 2.1. Positive Semi-Definiteness

It is helpful to review the standard terminology and properties. An $n \times n$ matrix $A$ is defined to be *positive semi-definite (psd)* if it is symmetric and has non-negative eigenvalues.

The following properties of an $n \times n$ matrix $A$ are equivalent:

1. $A$ is psd;
2. $A$ is symmetric and $x^T Ax \geq 0$ for all $x \in R^n$;
3. $A$ is the covariance matrix for some $n$-dimensional random vector;
4. $A = BB^T$ for some $n \times n$ matrix $B$.

### 2.2. Optimization Problem

We need some more notation:

**Definition 1.** Let $\mathrm{Corr}(n)$ denote the set of $n \times n$ psd matrices with 1's on diagonal, and $C(n)$ denote the $n \times n$ symmetric matrices with 1's on diagonal, and all elements in $[-1, 1]$.

This means that $\mathrm{Corr}(n)$ is the set of all valid $n \times n$ correlation matrices, and $C(n)$ is the superset of $\mathrm{Corr}(n)$ containing the $n \times n$ matrices that look to the naked eye like correlation matrices (targets).

With this notation, we can now state the following precise version of the manager's problem:

**Optimization Problem.** For a given target $\tilde{C} \in C(n)$, find $C \in \mathrm{Corr}(n)$ closest to $\tilde{C}$ in terms of some suitable norm. That is, minimize the objective

$$g(C) = ||C - \tilde{C}||^2$$

as $C$ ranges over the space $\mathrm{Corr}(n)$ of true correlation matrices.

The norm can be chosen at the discretion of the user to to emphasize the most important correlations, if needed. A simple default equally weighted choice is the Frobenius norm $||X||_F = tr(XX^T)$. However, weighted versions of this norm are equally permitted.

## 3. The Quotient Topology of Corr($n$)

### 3.1. Equivariance

It is very helpful to understand more of the structure of Corr($n$). Let $B(n)$ denote the space of $n \times n$ matrices with rows of unit length. As follows from the previous discussion, and as Rebonato and Jäckel (2000) point out,

**Corollary 2.** *$C \in \mathrm{Corr}(n)$ if and only if $C = BB^T$ for some $B \in B(n)$.*

With this in mind, there is a natural mapping

$$F : B(n) \to \mathrm{Corr}(n)$$

defined by $F(B) = BB^T$.

Rebonato and Jäckel's proposal for solving the optimization problem can be described as follows: parametrize $B(n)$ with $n(n-1)$ real angle variables (each of $n$ rows can range over the $(n-1)$-dimensional unit sphere in $R^n$), and use $F$ to transfer the parametrization to Corr($n$). The resulting unconstrained objective may then be optimized with standard nonlinear optimization tools.

However, there is further symmetry in this problem. Let $\mathrm{O}(n)$ denote the orthogonal group, i.e. the set of matrices $O$ such that $OO^T = I$.

**Lemma 3.** *Suppose $B_1, B_2$ are two $n \times n$ matrices. Then $B_1 B_1^T = B_2 B_2^T$ if and only if there exists $O \in \mathrm{O}(n)$ with $B_2 = B_1 O$.*

This means that $F$ has the following "equivariance property": $F(B_1) = F(B_2)$ if and only if $B_1 = B_2 O$ for some orthogonal matrix $O$. So $F^{-1}(\mathrm{C})$ is the set $\{BO : O \in \mathrm{O}(n)\}$ for any $B$ such that $BB^T = C$. If we think of $\mathrm{O}(n)$ as a group acting on $B(n)$ by right multiplication, then another way to say it is the $F$-preimage of any point is an $\mathrm{O}(n)$–orbit, and we can write the topological equivalence

$$\mathrm{Corr}(n) \cong B(n)/\,\mathrm{O}(n).$$

To minimize the objective $g$, we do not have to parametrize $B(n)$, but merely the smaller quotient space $B(n)/\,\mathrm{O}(n)$, described concretely next.

### 3.2. Topology of Corr($n$)

We know that

$$B(n) \cong S^{n-1} \times \cdots \times S^{n-1} \qquad (n \text{ times}).$$

To describe the quotient, we find a good representative in each $\mathrm{O}(n)$-equvalence class (orbit).

By multiplying any $B \in B(n)$ on the right by orthogonal matrices, we may arrange the following:

1. Rotate the 1-st row to axis 1;
2. Rotate the 2-nd row to (12)-plane, keeping axis 1 fixed;
3. Rotate the 3-rd row to (123)-subspace, keeping the (12)-plane fixed;
4. etc.

We get (e.g. $4 \times 4$):

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \\ * & * & * & * \end{pmatrix},$$

where the $i$-th row is a unit vector with the last $n-i$ entries equal to zero.

If we denote by $S^k$ the $k$-dimensional unit sphere, the result is an element of the triangular product of spheres $S(n)$, defined by

$$S(n) \equiv S^0 \times S^1 \times \cdots \times S^{n-1}.$$

We have not quite yet found a unique representative in each equivalence class, because in each factor of $S(n)$ there are two representatives of the same element of $\mathrm{Corr}(n)$: upper and lower.

**Definition 4.** (Triangular Product of Half-Spheres) Let $\hat{S}^k$ denote the $k$-dimensional upper half-sphere (with boundary).

Define

$$\hat{S}(n) \equiv \hat{S}^0 \times \hat{S}^1 \times \cdots \times \hat{S}^{n-1}.$$

This is homeomorphic to a cell of dimension $n(n-1)/2$, and

$$\mathrm{Corr}(n) \cong \hat{S}(n)$$

For optimization, we may now parametrize $\hat{S}(n)$ and use the natural map $f$.

### 3.3. Numerical Optimization Framework

The dimension of $\mathrm{Corr}(n)$ is $n(n-1)/2$, which is equal to the dimension of $S(n)$ and $\hat{S}(n)$. To avoid constraints, we may use $S(n)$ instead of $\hat{S}(n)$ in optimization. We minimize

$$\min ||C - BB^T||^2 = \min ||C - f(B)||^2,$$

where $B$ ranges over the triangular product of spheres $S(n)$ (parametrized by sines and cosines).

**Notes.** The natural map $f$, restricted to $\hat{S}(n)$, is the inverse of the Cholesky decomposition.

The Cholesky decomposition is not special – it is just a convenient way to describe the quotient space

$$S^{n-1} \times \cdots \times S^{n-1}/O(n).$$

Any other way to choose representatives would also work.

### 3.4. Back to the 3 × 3 Example

Recall

$$\tilde{C} = \begin{pmatrix} 1 & 0.9 & 0.7 \\ 0.9 & 1 & 0.3 \\ 0.7 & 0.3 & 1 \end{pmatrix}.$$

Nonlinear least squares optimization over 3 dimensional space of angle parameters gives us:

$$\hat{C} = \begin{pmatrix} 1 & 0.895 & 0.697 \\ 0.895 & 1 & 0.303 \\ 0.697 & 0.303 & 1 \end{pmatrix}$$

with eigenvalues (2.29, .707, 0).

## 4. Constrained Optimization Problem

The same ideas work when there are diagonal block constraints. For example,

**Problem 2.** Given $C \in C(n)$, with diagonal block $\Theta \in \mathrm{Corr}(k)$, $k < n$, find closest $\hat{C} \in \mathrm{Corr}(n)$ preserving the block $\Theta$

$$C = \left(\begin{array}{cc} \Theta & * \\ \hline * & * \end{array}\right).$$

We want to find a smooth space of representatives $B$ such that $BB^T$ is a correlation matrix with block $\Theta$. Let $S(n,k)$ be the set of lower triangular $k \times n$ matrices with unit rows ($k \leq n$). Fix $A \in S(n,k)$ such that $AA^T = \Theta$. Let

$$B = \begin{pmatrix} A_{k\times n} \\ \hline E_{(n-k)\times n} \end{pmatrix}$$

as $E$ ranges over the space $S^k \times \cdots \times S^{n-1}$ (that is, the $i$-th row of $E$ is an $n$-dimensional unit vector $e_i$ such that $e_{ij} = 0$ for $j = k+i, \ldots, n$).

The dimension of this space is $\frac{n(n-1)}{2} - \frac{k(k-1)}{2}$. We optimize as before.

As a check on the dimension count, note $C$ is of the form

$$C = \begin{pmatrix} \Theta & M(k, n-k) \\ M(n-k, k) & \mathrm{Corr}(n-k) \end{pmatrix}.$$

Positive definite matrices are open in the space of symmetric matrices, so dimension is

$$k(n-k) + \frac{(n-k)(n-k-1)}{2} = \frac{n(n-1)}{2} - \frac{k(k-1)}{2}.$$

## 5. Concluding Remarks

The closest correlation matrix to a given candidate may be obtained by nonlinear least squares optimization over the space $\mathrm{Corr}(n)$ of $n \times n$ correlation matrices, which is nicely homeomorphic to a triangular product of half-spheres. $\mathrm{Corr}(n)$ may be easily parametrized by angle variables, and constraining a fixed diagonal block is easily accommodated. At the mild cost of doubling the search space (dimension unchanged), we may formulate the problem as an unconstrained optimization of a nonlinear real-valued function of $n(n-1)/2$ real angle variables.

Note, however, that we have an unconstrained but nonconvex problem, as formulated here. The space of covariance matrices forms a convex cone; positive semidefinite least squares methods are available for fast optimization for large $n$, e.g. [4], [5], [8].

For applications to portfolio risk, there are often natural constraints that do not permit a convex approach, so these nonconvex problems arise naturally (see [1] and [3]).

For related geometric approaches to parametrizing $\mathrm{Corr}(n)$, see [2] and [6].

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