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FIXED POINT RESULTS FOR UPPER ISOTONE FUNCTIONS

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Abstract: The author proves new fixed point type results for posets which are inductive and posets which are relatively inductive.

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1. Introduction

Throughout the paper, a partially ordered set will be called a poset.

In this paper the author gives some new fixed point properties of partially ordered sets. First we give some definitions.

Definition 1.1. A function from a poset $(P, \leq)$ into itself is upper isotone if

$$a \leq f(a) \text{ for some } a \in P \text{ and whenever } x \leq f(x) \leq y, \\ \text{then } f(x) \leq f(y) \text{ for all } x, y \in P. \quad (1)$$

A mapping from a poset $(P, \leq)$ into itself has a fixed point iff there exists an $x \in P$ such that $f(x) = x$.

A poset P is said to be inductive if every non empty well ordered subset of P has a least upper bound in P.

In the next section, we give some new fixed point type results.

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2. Inductive Posets and Relative Inductive Posets

We prove the following theorem.

Theorem 2.1. *Every upper isotone mapping f of an inductive poset into itself has a fixed point.*

Proof. For ordinals 0 and 1, we define $f^0(a) = a$ and $f^1(a) = f(a)$.

Since f is an upper isotone mapping from the poset P into itself, there exists an $a \in P$ such that $a \leq f(a)$. Thus in view of (1), since $a \leq f(a) \leq f(a)$ we have

$$f(a) \leq f(f(a)). \tag{2}$$

We introduce the notation

$$f(f(a)) = f^2(a). \tag{3}$$

Hence from (1) and (2), it follows that

$$a \leq f(a) \leq f^2(a). \tag{4}$$

With reasoning and generalizing the notation given above, for every natural number n (motivated by (4)) we have

$$a \leq f(a) \leq f^2(a) \leq ... \leq f^n(a), \qquad n \in \omega\,. \tag{5}$$

Next let

$$D = \{a, f(a), f^2(a), ..., f^n(a), ..., n \in \omega\}\,. \tag{6}$$

Clearly, D is a well ordered subset of the poset P and since P is an inductive poset, $lubD$ exists and is denoted by $f^\omega(a)$.

Thus,

$$lubD = f^\omega(a)\,. \tag{7}$$

Obviously,

$$f^n(a) \leq f^{n+1}(a) \leq f^\omega(a) \text{ for every natural number } n. \tag{8}$$

From (1) and (8), we have

$$f^{n+1}(a) \leq f(f^\omega(a)) = f^{\omega+1}(a) \text{ for every natural number } n\,. \tag{9}$$

But from (7) and (9), we have

$$f^\omega(a) \leq f^{\omega+1}(a)\,. \tag{10}$$

With reasoning and notation as given in (9), we have the following iteration

$f^\alpha(a)$ (whether α is a limit or a non limit ordinal) defined as:

$$f^\alpha(a) = a \text{ if } \alpha = 0\,, \tag{11}$$

$$f^\alpha(a) = f(f^{\alpha-1}(a)) \text{ if } \alpha \text{ is a nonzero ordinal and is a non limit ordinal}\,, \tag{12}$$

$$f^\alpha(a) = \sup\{f^\gamma(a) : \gamma < \alpha\} \text{ if } \alpha \text{ is a limit ordinal}\,. \tag{13}$$

Thus for every ordinal motivated by (11), (12), and (13), we have that

$$a \le f(a) \le f^2(a) \le ... \le f^\alpha(a) \le, ... \,. \tag{14}$$

Next let

$$F = \{a, f(a), f^2(a), ..., f^\alpha(a), ...\}\,. \tag{15}$$

Assume that the mapping f does not have a fixed point. Consider the elements of F namely $a, f(a), f^\alpha(a), ...$. Every term $f^\alpha(a)$ in this net can be assigned the unique ordinal α. Since the terms of the above net form a set, by the Axiom Scheme of Replacement of ZF this would imply the existence of the set of ordinal numbers yielding a contradiction. Thus the mapping f has a fixed point as desired. □

We note that in (15), the proof can also be finished by using the well known Abian/Brown [1] result which says: Let P be a chain complete poset. Let f be an increasing mapping from P into itself such that for some $a \in P$ that $a \le f(a)$. Then the mapping f has a fixed point.

We have the following corollary.

Corollary 2.2. *Let f be a mapping from a poset P into itself. Let α be an ordinal. If there exists a maximal subset of P of the form $F = \{a, f(a), f^2(a), ..., f^\alpha(a), ...,\}$, where F is an inductive poset and such that for all $x \in F$, $x \le f(x)$, then the mapping f has a fixed point.*

We give the following definition.

Definition 2.3. A function from a poset $(P, \le)$ into itself is relatively inductive if every non empty well ordered set of the poset P has an upper bound $x \in P$ such that $x \le f(x)$ and $x = f(c)$ for some $c \in P$.

In the following result, we show that the least upper bound in the previous theorem can be replaced by an upper bound.

Theorem 2.4. *Every upper isotone mapping f of a relative inductive poset into itself has a fixed point.*

Proof. Let f be an upper isotone mapping of the poset P into itself. Assume that $a \le f(a)$ for some $a \in P$. We define $f^0(a) = a$ and $f^1(a) = f(a)$.

As in the proof of the previous theorem, we obtain a sequence

$$a \leq f(a) \leq f^2(a) \leq ... \leq f^n(a), n \in \omega \,. \tag{16}$$

Let

$$D = \{a, f(a), f^2(a), ..., f^n(a), ..., \qquad n \in \omega\} \,. \tag{17}$$

Clearly the set D is a well ordered set. Thus by hypothesis, there exists an upper bound x of D such that $x \leq f(x)$. Define

$$x = f^{\omega}(a) \,. \tag{18}$$

Also by hypothesis $x = f(c)$ for some $c \in P$. Thus in view of (1) and (18), we have that

$$f^n(a) \leq f^{\omega}(a) = f(c) \leq f(f^{\omega}(a)) = f^{\omega+1}(a) \,. \tag{19}$$

With reasoning and notation as given in (18) and (19), we have the following iteration:

$$f^0(a) = a \,, \tag{20}$$

$$f^{\alpha}(a) = f(f^{\alpha-1}(a)) \text{ if } \alpha \text{ is a non limit ordinal.} \tag{21}$$

If α is a limit ordinal, define

$$f^{\alpha}(a) = x_{\alpha}, \text{ where } x_{\alpha} \text{ is an upper bound of the set } \{f^{\gamma}(a) : \gamma < \alpha\}$$
$$\text{and } x_{\alpha} \leq f(x_{\alpha}) \text{ and } x_{\alpha} = f(c_{\alpha}) \text{ for some } c_{\alpha} \in P. \tag{22}$$

Thus for every ordinal motivated by (20), (21), and (22), we have that

$$\{a \leq f(a) \leq f^2(a) \leq ... \leq f^{\alpha}(a) \leq, ... \,. \tag{23}$$

Consider the following set

$$F = \{a, f(a), f^2(a), ..., f^{\alpha}(a), ...\}. \tag{24}$$

Assume that the mapping f does not have a fixed point. Consider the elements of F; $a < f(a) < f^{\alpha}(a) < ....$ Every term $f^{\alpha}(a)$ in this net can be assigned the unique ordinal α. Since the terms of the above net form a set, by the Axiom Scheme of Replacement of ZF this would imply the existence of the set of ordinal numbers yielding a contradiction. Thus the mapping f has a fixed point as desired. □

For other results see [2], [3] and [4].

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