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WEAKLY ALMOST CONTRA-PRECONTINUOUS FUNCTIONS

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Abstract: A new form of contra-precontinuity, which we call weak almost contra-precontinuity, is introduced. We show that this class of functions is weaker than most forms of precontinuity and contra-precontinuity, but that the class still has interesting properties. Conditions are established under which the weakly almost contra-precontinuous image of a space is nearly compact. Finally, properties related to the graphs of these functions and other basic properties of weakly almost contra-precontinuous functions are investigated.

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1. Introduction

The concepts of contra-continuity and its variations have been extensively studied. Dontchev [5] introduced the notion of contra-continuity in 1996. Jafari and Noiri [7] continued this line of research by developing contra precontinuity in 2002. Almost contra-precontinuity was introduced by Ekici [6] in 2004. Recently Baker [4] developed weak contra-precontinuity. The purpose of this note is to further extend this line of research by introducing the notion of weak almost contra-precontinuity. It is shown that weak almost contra-precontinuity is implied by almost precontinuity, almost contra-precontinuity, and weak contra precontinuity and that weak almost contra-precontinuity implies slight precontinuity. The basic properties of this class of functions are developed. For

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example, it is established that weakly almost contra-precontinuous functions with an Urysohn codomain satisfy two types of closed graph property. Finally, conditions are established under which the image of a strongly S-preclosed space under a weak almost contra-precontinuous function is nearly compact.

2. Preliminaries

The symbols X and Y represent topological spaces with no separation properties assumed unless explicitly stated. All sets are considered to be subsets of topological spaces. The closure and interior of a set A are signified by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A set A is regular open (respectively, preopen [9]) if $A = \mathrm{Int}(\mathrm{Cl}(A))$ (respectively, $A \subseteq \mathrm{Int}(\mathrm{Cl}(A))$. A set A is regular closed (respectively, preclosed) if the complement of A is regular open (respectively, preopen). The preclosure of a set A, denoted by $\mathrm{pCl}(A)$, is the intersection of all preclosed sets containing A.

Definition 1. A function $f : X \to Y$ is said to be contra-continuous (see [5]) if $f^{-1}(V)$ is closed for every open subset V of Y.

Definition 2. A function $f : X \to Y$ is said to be weakly contra-continuous (see [3]) (respectively, weakly contra-precontinuous (see [4])) provided that, whenever $A \subseteq V \subseteq Y$, A is closed in Y, and V is open in Y, then $\mathrm{Cl}(f^{-1}(A)) \subseteq f^{-1}(V)$ (respectively, $\mathrm{pCl}(f^{-1}(A)) \subseteq f^{-1}(V)$).

Definition 3. A function $f : X \to Y$ is said to be precontinuous (see [9]) (respectively, almost precontinuous (see [10])) if $f^{-1}(V)$ is preopen for every open (respectively, regular open) subset V of Y.

Definition 4. A function $f : X \to Y$ is said to be contra-precontinuous (see [7]) (respectively, almost contra-precontinuous (see [6])) if $f^{-1}(V)$ is preclosed for every open (respectively, regular open) subset V of Y.

Definition 5. A function $f : X \to Y$ is said to be slightly precontinuous (see [2]) provided that, for every $x \in X$ and every clopen subset V of Y containing $f(x)$, there exists a preopen subset U of X with $x \in U$ and $f(U) \subseteq V$.

Obviously a function is slightly precontinuous if and only if inverse images of clopen sets are preopen.

3. Weakly Almost Contra-Precontinuous Functions

We define a function $f : X \to Y$ to be weakly almost contra-precontinuous provided that for every regular open subset V of Y and every regular closed subset A of Y with $A \subseteq V$, we have $\mathrm{pCl}(f^{-1}(A)) \subseteq f^{-1}(V)$. Since $\mathrm{pCl}(f^{-1}(S)) = S \cup \mathrm{Cl}(\mathrm{Int}(S))$, we have the following result.

Theorem 3.1. *A function $f : X \to Y$ is weakly almost contra-precontinuous if and only if, whenever A is a regular closed subset of Y, V is a regular open subset of Y, and $A \subseteq V$, then $Cl(Int(f^{-1}(A))) \subseteq f^{-1}(V)$.*

From [4] we have that weak contra-precontinuous implies weak contra-precontinuity and obviously weak contra-precontinuity implies weak almost contra-precontinuity.

Theorem 3.2. *If the function $f : X \to Y$ is almost contra-precontinuous, then f is weakly almost contra-precontinuous.*

Proof. Assume $f : X \to Y$ is almost contra-precontinuous. Let $A \subseteq V \subseteq Y$, where A is regular closed in Y and V is regular open in Y. Since f is almost contra-precontinuous, $f^{-1}(V)$ is preclosed and therefore $\mathrm{pCl}(f^{-1}(A)) \subseteq \mathrm{pCl}(f^{-1}(V)) = f^{-1}(V)$. Thus$f$ is weakly almost contra-precontinuous. □

Theorem 3.3. *If $f : X \to Y$ is almost precontinuous, then f is weakly almost contra-precontinuous.*

Proof. Suppose $f : X \to Y$ is almost precontinuous. Let $A \subseteq V \subseteq Y$, where A is regular closed in Y and V is regular open in Y. Since f is almost precontinuous, $f^{-1}(A)$ is preclosed and thus $\mathrm{pCl}(f^{-1}(A)) = f^{-1}(A) \subseteq f^{-1}(V)$. Therefore f is weakly almost contra-precontinuous. □

Theorem 3.4. *If the function $f : X \to Y$ is weakly almost contra-precontinuous, then f is slightly precontinuous.*

Proof. Let $f : X \to Y$ be weakly almost contra-precontinuous and let V be a clopen subset of Y. Then we have $V \subseteq V$, where V is regular open and regular closed. Since f is weakly almost contra-precontinuous, $\mathrm{pCl}(f^{-1}(V)) \subseteq f^{-1}(V)$ and hence $f^{-1}(V)$ is preclosed. Thus f is slightly precontinuous. □

Corollary 3.5. (see [4]) *If $f : X \to Y$ is weakly contra-precontinuous, then f is slightly precontinuous.*

Thus weak almost contra-continuity is between weak contra-precontinuity and slight precontinuity. We now have the following implications:

$$\begin{array}{ccccc} & & \text{weakly contra-continuous} & & \\ & & \Downarrow & & \\ & & \text{weakly contra-precontinuous} & & \\ & & \Downarrow & & \\ \text{almost precont.} & \Rightarrow & \text{weakly almost contra-precont.} & \Rightarrow & \text{sl. precont.} \\ & & \Uparrow & & \\ & & \text{almost contra-precontinuous} & & \end{array}$$

The examples that follow establish that none of these implications are reversible. From [4] we have that weak contra-precontinuity does not imply weak contra-continuity.

Example 3.6. Let X denote the real numbers and let $\sigma = \{X, \emptyset, \{0\}\}$ and let τ be the usual topology on X. Let $f : (X, \sigma) \to (X, \tau)$ be the identity mapping. Since (X, τ) is connected, f is slightly precontinuous. However, f is not weakly almost contra-precontinuous. To see this note that, if $A = [0, 1]$ and $V = (-2, 2)$, then A is regular closed in (X, τ) and V is regular open in (X, τ) with $A \subseteq V$, but $\mathrm{pCl}(f^{-1}(A)) \nsubseteq f^{-1}(V)$.

Example 3.7. Let $X = \{a, b, c\}$ have the topologies $\sigma = \{X, \emptyset, \{a\}\}$ and $\tau = \{X, \emptyset, \{c\}, \{a, c\}, \{b, c\}\}$. Assume $f : (X, \sigma) \to (X, \tau)$ is the identity mapping. Since the only regular open sets in (X, τ) are X and $\emptyset$, f is weakly almost contra-precontinuous. However, f is not weakly contra-precontinuous, since for $A = \{a\}$ and $V = \{a, c\}$ in the space (X, τ), $\mathrm{pCl}(f^{-1}(A)) \nsubseteq f^{-1}(V)$.

In order to show that weak almost contra-precontinuity does not imply either almost precontinuity or almost contra precontinuity, it suffices to show that almost contra-precontinuity and almost precontinuity are independent.

Example 3.8. Let $X = \{a, b, c\}$ have the topology $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$. The identity mapping $f : X \to X$ is almost precontinuous but not almost contra-precontinuous. Note that $A = \{a\}$ is regular open in X, but $f^{-1}(A)$ is not preclosed.

Example 3.9. Let $X = \{a, b, c\}$ have the topologies $\sigma = \{X, \emptyset, \{c\}\}$ and $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$. The identity mapping $f : (X, \sigma) \to (X, \tau)$ is almost contra-precontinuous. However, since $A = \{a\}$ is regular open in (X, τ), but $f^{-1}(A)$ is not preopen in (X, σ), f fails to be almost precontinuous.

Recall that space is extremally disconnected (briefly an ED space) provided that the closures of open sets are open.

Theorem 3.10. *If the function $f : X \to Y$ is weakly almost contra-precontinuous and Y is an ED space, then f is almost precontinuous.*

Proof. Let V be a regular closed subset of Y. Since Y is an ED space, V is clopen and hence V is also regular open. Then, since, f is weakly almost contra-precontinuous, $\mathrm{pCl}(f^{-1}(V)) \subseteq f^{-1}(V)$, which proves that $f^{-1}(V)$ is preclosed. Thus f is almost precontinuous. □

Definition 6. A function is said to be almost weakly continuous (see [8]) provided that $f^{-1}(V) \subseteq \mathrm{Int}(\mathrm{Cl}(f^{-1}(\mathrm{Cl}(V))))$ for every open subset V of Y.

Popa and Noiri [13] proved the following characterization of almost weak continuity.

Theorem 3.11. (see [13]) *A function $f : X \to Y$ is almost weakly continuous if and only if $pCl(f^{-1}(V)) \subseteq f^{-1}(Cl(V))$ for every open subset V of Y.*

As an immediate consequence of Theorem 3.11, we have the following result.

Corollary 3.12. *If $f : X \to Y$ is almost precontinuous, then f is almost weakly continuous.*

Corollary 3.13. *If the function $f : X \to Y$ is weakly almost contra-precontinuous and Y is an ED space, then f is almost weakly continuous.*

Corollary 3.14. (see [4]) *If $f : X \to Y$ is weakly contra-precontinuous and Y is an ED space, then f is almost weakly continuous.*

4. Properties

We begin by investigating the preservation of properties related to compactness.

Definition 7. A space X is said to be nearly compact (see [14]) provided that every open cover of X has a finite subfamily such that the interiors of the closures cover X or equivalently if every cover of X by regular open sets has a finite subcover.

Definition 8. A space X is called a P_Σ-space (see [15]) provided that every open set is the union of regular closed sets.

Definition 9. A space is said to be strongly S-preclosed (see [4]) if every cover of X by preclosed sets has a finite subcover.

Theorem 4.1. *Let $f : X \to Y$ be a weakly almost contra-precontinuous surjection and let Y be a P_Σ-space. If X is strongly S-preclosed, then Y is nearly compact.*

Proof. Let $\mathcal{C}$ be a cover of Y by regular open sets. Let $y \in Y$ and let $V_y \in \mathcal{C}$ such that $y \in V_y$. Since Y is a P_Σ-space, there exists a regular closed set A_y such that $y \in A_y \subseteq V_y$. Since f is weakly almost contra-precontinuous, $\mathrm{pCl}(f^{-1}(A_y)) \subseteq f^{-1}(V_y)$ It follows that $\{\mathrm{pCl}(f^{-1}(A_y)) : y \in Y\}$ is a cover of X by preclosed sets. Since X is strongly S-preclosed, there exists a finite subcover $\{\mathrm{pCl}(f^{-1}(A_{y_i})) : i = 1, 2, \ldots, n\}$. It then follows that $\{V_{y_i} : i = 1, 2, \ldots, n\}$ is a finite subcover of $\mathcal{C}$, which proves that Y is nearly compact. □

We now consider various properties related to the graph of a weakly almost contra-precontinuous function. Recall that the graph of a function $f : X \to Y$ is the subset $G(f) = \{(x, f(x)) : x \in X\}$ of the product space $X \times Y$.

Definition 10. The graph, $G(f)$, of a function $f : X \to Y$ is said to be P-regular (see [6]) provided that, for every $(x, y) \in X \times Y - G(f)$, there exist a preclosed set U in X containing x and regular open set V in Y containing y such that $(U \times V) \cap G(f) = \emptyset$.

Theorem 4.2. *If the function $f : X \to Y$ is weakly almost contra-precontinuous and Y is Urysohn, then $G(f)$ is P-regular.*

Proof. Let $(x, y) \in X \times Y - G(f)$. Then, since $y \neq f(x)$ and Y is Urysohn, there exist open sets V and W in Y such that $y \in V$ and $f(x) \in W$ and $\mathrm{Cl}(V) \cap \mathrm{Cl}(W) = \emptyset$. Then we see that $\mathrm{Cl}(W) \subseteq Y - \mathrm{Cl}(V)$, $\mathrm{Cl}(W)$ is regular closed, and $Y - \mathrm{Cl}(V)$ is regular open. Since f is weakly almost contra-precontinuous $\mathrm{pCl}(f^{-1}(\mathrm{Cl}(W))) \subseteq f^{-1}(Y - \mathrm{Cl}(V))$. It then follows that $(x, y) \in \mathrm{pCl}(f^{-1}(\mathrm{Cl}(W))) \times \mathrm{Int}(\mathrm{Cl}(V)) \subseteq X \times Y - G(f)$, which proves that $G(f)$ is P-regular. □

Definition 11. A function $f : X \to Y$ has a strongly contra-closed graph (see [11]), provided that for every $(x, y) \in X \times Y - G(f)$, there exist a preopen set U in X containing x and regular closed set V in Y containing y such that $(U \times V) \cap G(f) = \emptyset$.

Theorem 4.3. *If $f : X \to Y$ is weakly almost contra-precontinuous and Y is Urysohn, then $G(f)$ is strongly contra-preclosed.*

Proof. Let $(x, y) \in X \times Y - G(f)$. Then there exist open sets V and W in Y such that $y \in V$ and $f(x) \in W$ and $\mathrm{Cl}(V) \cap \mathrm{Cl}(W) = \emptyset$. Therefore $\mathrm{Cl}(V) \subseteq Y - \mathrm{Cl}(W)$. Since $\mathrm{Cl}(V)$ is regular closed, $Y - \mathrm{Cl}(W)$ is regular open, and f is weakly almost contra-precontinuous, we have that $\mathrm{pCl}(f^{-1}(\mathrm{Cl}(V))) \subseteq f^{-1}(Y - \mathrm{Cl}(W))$. It then follows that $(x, y) \in (X - \mathrm{pCl}(f^{-1}(\mathrm{Cl}(V)))) \times \mathrm{Cl}(V) \subseteq X \times Y - G(f)$ and hence $G(f)$ is strongly contra-preclosed. □

Finally we consider relationships between weak almost contra-precontinuity and a form of generalized preclosed set.

Definition 12. A set A is said to be generalized preregular closed (briefly gpr-closed) (see [1]) provided that $\mathrm{pCl}(A) \subseteq U$ whenever $A \subseteq U$ and U is regular open.

Definition 13. A function $f : X \to Y$ is called approximately preregular irresolute (briefly apr-irresolute) (see [1]) if $\mathrm{pCl}(A) \subseteq f^{-1}(V)$ whenever V is regular open, A is gpr-closed, and $A \subseteq f^{-1}(V)$.

Definition 14. A function $f : X \to Y$ is said to be almost gpr-continuous provided that $f^{-1}(A)$ is gpr-closed for every regular closed subset A of X.

Theorem 4.4. *If $f : X \to Y$ is weakly almost contra-precontinuous and images of gpr-closed sets are regular closed, then f is apr-irresolute.*

Proof. Let V be a regular open subset of Y and let A be a gpr-closed subset of X such that $A \subseteq f^{-1}(V)$. Then $f(A)$ is regular closed and $f(A) \subseteq V$. Since f is weakly almost contra-precontinuous, $\mathrm{pCl}(f^{-1}(f(A))) \subseteq f^{-1}(V)$. Therefore $\mathrm{pCl}(A) \subseteq f^{-1}(V)$ and hence f is apr-irresolute. □

Theorem 4.5. *If $f : X \to Y$ is almost gpr-continuous and apr-irresolute, then f is weakly almost contra-precontinuous.*

Proof. Assume $A \subseteq V \subseteq Y$, where A is regular closed in Y and V is regular open in Y. Since f is almost gpr-continuous, $f^{-1}(A)$ is gpr-closed. Then, since $f^{-1}(A) \subseteq f^{-1}(V)$ and f is apr-irresolute, $\mathrm{pCl}(f^{-1}(A)) \subseteq f^{-1}(V)$, which proves that f is weakly almost contra-precontinuous. □

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