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SOME CONSTRUCTIONS OF k-SUPER MEAN GRAPHS

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Abstract: Let G be a (p,q) graph and $f : V(G) \to \{1, 2, 3, \ldots, p+q\}$ be an injection. For each edge $e = uv$, let $f^*(e) = (f(u)+f(v))/2$ if $f(u)+f(v)$ is even and $f^*(e) = (f(u)+f(v)+1)/2$ if $f(u)+f(v)$ is odd. Then f is called a super mean labeling if $f(V) \cup \{f^*(e) : e \in E(G)\} = \{1, 2, 3, \ldots, p+q\}$. A graph that admits a super mean labeling is called a super mean graph. Let G be a (p,q) graph and $f : V(G) \to \{1, 2, 3, \ldots, p+q+k-1\}$ be an injection. For each edge $e = uv$, let $f^*(e) = \left\lceil \frac{f(u)+f(v)}{2} \right\rceil$. Then f is called a k-super mean labeling if $f(V) \cup \{f^*(e) : e \in E(G)\} = \{k, k+1, k+2, \ldots, p+q+k-1\}$. A graph that admits a k-super mean labeling is called a k-super mean graph. In this paper we present super mean labeling of $C_m \cup C_n$ and T_p-tree and also we construct some k-super mean graphs.

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1. Introduction

By a graph we mean a finite, simple and undirected one. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$ respectively. The union of two graphs G_1 and G_2 is the graph $G_1 \cup G_2$ with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. The disjoint union of m copies of the graph G is denoted by mG. Let T be a tree and u_0 and v_0 be two adjacent vertices in T. Let there be two pendent vertices u and v in T such that the length of u_0-u path is equal to the length of v_0-v path. If the edge u_0v_0 is deleted from T and u, v are joined by an edge uv, then such a transformation of T is called an elementary parallel transformation (or an ept) and the edge u_0v_0 is called a transformable edge. If by a sequence of ept's T can be reduced to a path then T is called a T_p-tree (transformed tree) and any such sequence regarded as a composition of mappings (ept's) denoted by P is called a parallel transformation of T. The path, the image of T under P is denoted as $P(T)$. $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. Terms and notations not defined here are used in the sense of Harary [2].

2. Preliminary Results

The concept of super mean labeling was introduced in [6]. B. Gayathri et al have introduced k-super mean labeling of graphs in [1]. Let G be a (p, q) graph and $f : V(G) \to \{1, 2, 3, \ldots, p + q + k - 1\}$ be an injection. For each edge $e = uv$, let $f^*(e) = \left\lceil \frac{f(u)+f(v)}{2} \right\rceil$. Then f is called a k-super mean labeling if $f(V) \cup \{f^*(e) : e \in E(G)\} = \{k, k + 1, k + 2, \ldots, p + q + k - 1\}$. A graph that admits a k-super mean labeling is called a k-super mean graph. A 1-super mean graph is a super mean graph. We make reference to the following results.

Remark 2.1. (see [6]) C_4 is not a super mean graph.

Theorem 2.2. (see [6]) *If G_1 and G_2 are two super mean graphs, then $G_1 \cup G_2$ is also a super mean graph.*

Theorem 2.3. (see [7]) *C_n is a super mean graph except for $n = 4$.*

Theorem 2.4. (see [6]) *The graph $C_3 \cup C_n$ is a super mean graph for all $n \geq 3$.*

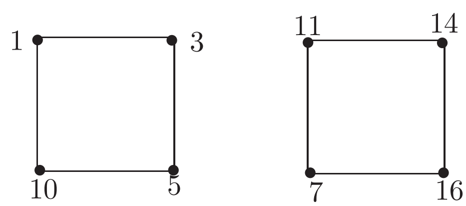
3. Super Mean Graph

Theorem 3.1. *The graph $C_m \cup C_n$ is a super mean graph for all $m \geq 3$, $n \geq 3$.*

Proof. If $m \neq 4$ and $n \neq 4$ then by Theorem 2.2 and Theorem 2.3, $C_m \cup C_n$ is a super mean graph. Suppose that at least one of m, n is 4. Let $m = 4$ and let u_1, u_2, u_3, u_4 be the vertices of C_4. We consider the following two cases.

Case (i). n is odd. Let $C_n = (v_1 v_2 v_3 \dots v_n)$. By Theorem 2.4, $C_m \cup C_3$ is a super mean graph. For $n \geq 5$, define $f : V(C_4 \cup C_n) \to \{1, 2, 3, \dots, p + q = 2n + 8\}$ by $f(u_1) = 1$; $f(u_2) = 3$; $f(u_3) = 5$; $f(u_4) = 11$; $f(v_1) = 7$; $f(v_2) = 10$; $f(v_3) = 14$; $f(v_{3+i}) = 14 + 2i$ for $1 \leq i \leq 2$; $f(v_{5+i}) = 18 + 4i$ for $1 \leq i \leq \frac{n-5}{2}$; $f\left(v_{\frac{n+7}{2}}\right) = 2n+5$; $f\left(v_{\frac{n+7}{2}+i}\right) = 2n+5-4i$ for $1 \leq i \leq \frac{n-7}{2}$. Clearly the induced edge labels are distinct and $f(V) \cup \{f^*(e) : e \in E\} = \{1, 2, 3, \dots, 2n+8\}$. Hence f is a super mean labeling of $C_4 \cup C_n$.

Case (ii). n is even. When $n = 4$, a super mean labeling of $C_4 \cup C_4$ is given below:



For $n \geq 6$, define $f : V(C_4 \cup C_n) \to \{1, 2, 3, \dots, p+q = 2n+8\}$ $f(v_i) = 2i-1$ for $1 \leq i \leq \frac{n+4}{2}$; $f(v_n) = 2n + 8$; $f(v_{n-i}) = 2n + 8 - 2i$ for $1 \leq i \leq \frac{n-6}{2}$; $f(u_1) = n+4$; $f(u_2) = n+7$; $f(u_3) = n+13$; $f(u_4) = n+11$. Clearly the induced edge labels are distinct and $f(V) \cup \{f^*(e) : e \in E\} = \{1, 2, 3, \dots, 2n + 8\}$. Therefore f is a super mean labeling of $C_4 \cup C_n$. Hence $C_m \cup C_n$ is a super mean graph for all $m \geq 3$, $n \geq 3$. □

Example 3.2. A super mean labeling of $C_4 \cup C_8$ is given in Figure 1.

Corollary 3.3. *The graph $mC_n, n \neq 4$ is a super mean graph for all $m \geq 1$.*

Proof. It follows from Theorem 2.2 and Theroem 3.1. □

Corollary 3.4. *The graph mC_4 is a super mean graph for all even values of m.*

Proof. It follows from Theorem 2.2 and Theroem 3.1. □

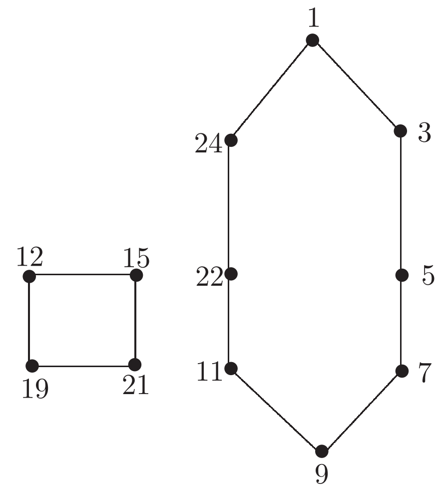


Figure 1:

Theorem 3.5. *Every T_p-tree T is a super mean graph.*

Proof. Let T be a T_p-tree with n vertices. By the definition of a T_p-tree there exists a parallel transformation P of T such that for the path $P(T)$ we have (i) $V(P(T)) = V(T)$ and (ii) $E(P(T)) = (E(T) - E_d) \cup E_p$, where E_d is the set of edges deleted from T and E_p is the set of edges newly added through the sequence $P = (P_1, P_2, \ldots, P_k)$ of the epts P used to arrive at the path $P(T)$. Clearly E_d and E_p have the same number of edges.

Now denote the vertices of $P(T)$ successively as $v_1, v_2, v_3, \ldots, v_n$ starting from one pendant vertex of $P(T)$ right up to other. The labeling f defined by $f(v_i) = 2v_i - 1$ for $1 \leq i \leq n$ is the super mean labeling of the path $P(T)$.

Let $v_i v_j$ be an edge in T for some indices i and $j, 1 \leq i \leq j \leq n$ and let P_1 be the ept that deletes this edge and adds the edge $v_{i+t} v_{j-t}$ where t is the distance of v_i from v_{i+t} as also the distance of v_j from v_{j-t}. Let P be a parallel transformation of T that contains P_1 as one of the constituent epts. Since $v_{i+t} v_{j-t}$ is an edge in the path $P(T)$, it follows that $i+t+1 = j-t$ which implies $j = i + 2t + 1$. Therefore i and j are of opposite parity, that is i is odd and j is even or viceversa. The value of the edge $v_i v_j$ is given by,

$$f^*(v_i v_j) = f^*(v_i v_{i+2t+1}) = \left\lceil \frac{f(v_i) + f(v_{i+2t+1})}{2} \right\rceil = 2(i+t) \qquad (1)$$

and

$$f^*(v_{i+1} v_{j-t}) = f^*(v_{i+t} v_{i+t+1}) = \left\lceil \frac{f(v_{i+t}) + f(v_{i+t+1})}{2} \right\rceil = 2(i+t). \qquad (2)$$

Therefore from (1) and (2) $f^*(v_iv_j) = f^*(v_{i+t}v_{j-t})$. Hence f is a super mean labeling of T_p-tree T. □

Example 3.6. A super mean labeling of a T_P tree with 14 vertices is given in Figure 2.

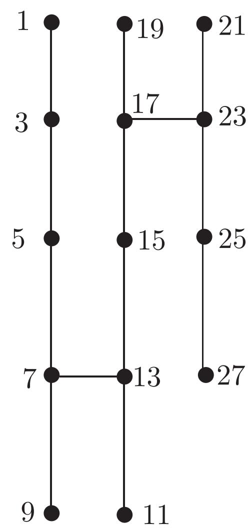


Figure 2:

4. Construction of Some k-Super Mean Graphs

In this section we construct some k-super mean graphs.

Theorem 4.1. *Let $G_i(p_i, q_i)$ be k_i-super mean graphs for $1 \leq i \leq n$ with $k_1 \geq k_2 \geq k_3 \geq \cdots \geq k_n$ and $p_1+q_1 = p_2+q_2 = p_3+q_3 = \cdots = p_n+q_n = p+q$ and let u_i, v_i be the vertices labeled $k_i, p+q+k_i-1$ respectively of $G_i (1 \leq i \leq n)$. Then the graph G which is made by joining v_1 to u_2 by an edge and u_2 to v_3 to u_4 and so on until we reach the last graph G_n is a k_1-super mean graph.*

Proof. Add the number $(p+q+1)(i-1)+k_1-k_i$ to all vertex labels of each k_i-super mean graph G_i for $1 < i \leq n$. We note that the vertex labels remain distinct, the edge labels of the graph G_1 will remain fixed and the edge labels of the graph $G_i, 1 < i \leq n$ will increase by the number $(p+q+1)(i-1)+k_1-k_i$ and the edge (bridge) between G_i and G_{i+1} will get the label

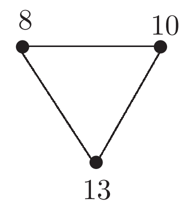
$$\frac{1}{2}\left[p+q+k_i-1+(i-1)(p+q+1)+k_1-k_i+k_{i+1}+i(p+q+1)\right.$$

$$+k_1 - k_{i+1}] = \frac{1}{2}\left[p + q + k_i - 1 + (2i-1)(p+q+1) + 2k_1 - k_i\right]$$
$$= i(p+q+1) + k_1 - 1.$$

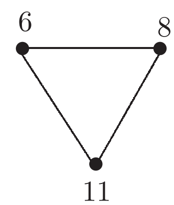
Therefore the edge labels of the whole graph G are distinct. Also the union of the induced edge labels and vertex labels of G results the set $\{k_1, k_1+1, k_1+2, \ldots, n(p+q+1)+k_1-2\}$. Hence G is a k_1-super mean graph. □

Example 4.2. Let $n = 4$ and $G_i = C_3$ for $i = 1, 2, 3, 4$. Then $p_i + q_i = 6$ for each i. Let $k_1 = 8$, $k_2 = 6$, $k_3 = 3$ and $k_4 = 1$.

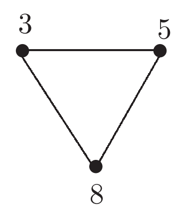
An 8-super mean labeling of C_3 is



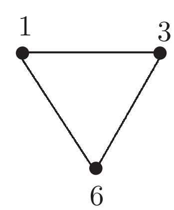
A 6-super mean labeling of C_3 is



A 3-super mean labeling of C_3 is



A 1-super mean labeling of C_3 is



The graph G obtained by the above construction is given in Figure 3.

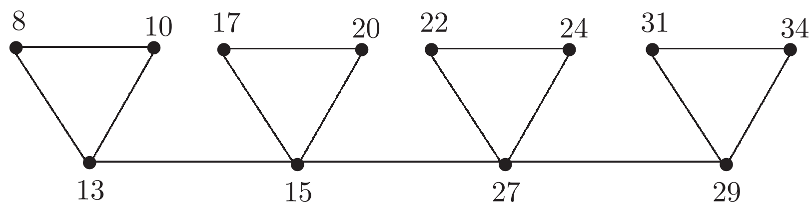


Figure 3:

Corollary 4.3. *If $G_1(p_1, q_1), G_2(p_2, q_2), G_3(p_3, q_3), \ldots, G_n(p_n, q_n)$ are the k-super mean graphs with $p_1 + q_1 = p_2 + q_2 = p_3 + q_3 = \cdots = p_n + q_n = p + q$ and let u_i, v_i be the vertices labeled $k, k+p+q-1$ respectively of $G_i (1 \le i \le n)$. Then the graph G which is made by joining v_1 to u_2 by an edge and u_2 to v_3 and v_3 to u_4 and so on until we reach the last graph G_n is again a k-super mean graph.*

Proof. Proof follows from Theorem 4.1, by taking $k_1 = k_2 = k_3 = \cdots = k_n = k$. □

Theorem 4.4. *Let $G_i(p_i, q_i)$ be k_i-super mean graphs for $1 \le i \le n$ with $k_1 \ge k_2 \ge k_3 \ge \cdots \ge k_n$ and $p_1 + q_1 = p_2 + q_2 = p_3 + q_3 = \cdots = p_n + q_n = p + q$. Suppose that every graph G_i has a vertex w_i with label $\left\lceil \frac{p_i+q_i+2k_i-2}{2} \right\rceil$. Then the graph G obtained by joining w_i to w_{i+1} for $1 \le i \le n-1$ by an edge is again a k_1-super mean graph.*

Proof. Add the number $(p+q+1)(i-1)+k_1-k_i$ to all vertex labels of each k_i-super mean graph G_i for $1 < i \le n$. We note that the vertex labels remain distinct, the edge labels of the graph G_1 will remain fixed and the edge labels of the graph $G_i, 1 < i \le n$ will increase by $(p+q+1)(i-1)+k_1-k_i$ and the edge (bridge) between G_i and G_{i+1} will get the label

$$\left\lceil \frac{\left\lceil \frac{p+q+2k_i-2}{2} \right\rceil + (i-1)(p+q+1) + k_1 - k_i + i(p+q+1) + k_1 - k_{i+1} + \left\lceil \frac{p+q+2k_{i+1}-2}{2} \right\rceil}{2} \right\rceil$$
$$= i(p+q+1) + k_1 - 1.$$

Therefore the edge labels of the whole graph G are distinct. Also the union of the induced edge labels and the vertex labels of G results the set $\{k_1, k_1 + 1, k_1 + 2, \ldots, n(p+q+1) + k_1 - 2\}$. Hence G is a k_1-super mean graph. □

Example 4.5. Let $m = 3$ and $G_i = P_5$ for $i = 1, 2, 3$. Then $p_i + q_i = 9$. Let

$k_1 = 5$, $k_2 = 3$ and $k_3 = 2$. The graph G obtained by the above construction is given in Figure 4.

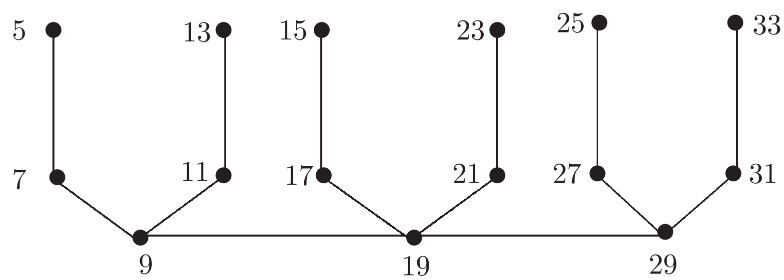


Figure 4:

Corollary 4.6. *If $G_1(p_1, q_1)$, $G_2(p_2, q_2)$, $G_3(p_3, q_3), \ldots, G_n(p_n, q_n)$ are the k-super mean graphs with $p_1 + q_1 = p_2 + q_2 = p_3 + q_3 = \cdots = p_n + q_n = p + q$ and suppose that every graph G_i has a vertex w_i with label $\left\lceil \frac{p_i+q_i+2k_i-2}{2} \right\rceil$ for $1 \leq i \leq n$. Then the graph G that is obtained by joining w_i to w_{i+1} for $1 \leq i \leq n-1$ by an edge is again a k-super mean graph.*

Proof. Proof follows from Theorem 4.4, by taking $k_1 = k_2 = k_3 = \cdots = k_n = k$. □

Theorem 4.7. *Let $G_i(p_i, q_i)$ be k_i-super mean graphs for $1 \leq i \leq n$ with $k_1 \geq k_2 \geq k_3 \geq \cdots \geq k_n$ and $p_1 + q_1 = p_2 + q_2 = p_3 + q_3 = \cdots = p_n + q_n = p + q$ and let u_i, v_i be the vertices labeled $k_i, p+q+k_i-1$ respectively of $G_i (1 \leq i \leq n)$. Then the graph G which is made my joining u_1 to v_2 by an edge and v_2 to u_3 and u_3 to v_4 and so on until we reach the last graph G_n is again a k_1-super mean graph.*

Proof. Similar to Theorem 4.1 □

Theorem 4.8. *Let $G_i(p_i, q_i)$ be k_i-super mean graphs for $1 \leq i \leq n$ where n is odd and $k_1 \geq k_2 \geq k_3 \geq \cdots \geq k_n$ and $p_1 + q_1 = p_2 + q_2 = p_3 + q_3 = \cdots = p_n + q_n = p + q$ and let u_i, v_i be the vertices labeled $k_i, p + q + k_i - 1$ respectively of $G_i (1 \leq i \leq n)$. Then the graph G which is made my joining v_1 to u_2 by an edge and u_2 to v_3 and v_3 to u_4 and so on until we reach the last graph G_n and joining v_n to v_1 is again a k_1-super mean graph.*

Proof. Add the number $(p+q+1)(i-1)+k_1-k_i$ to all vertex labels of each k_i-super mean graph G_i for $1 < i \leq \frac{n+1}{2}$ and add the number $(p+q+1)(i-1)+k_1-k_i+1$ to all vertex labels of each k_i-super mean graph G_i for $\frac{n+3}{2} \leq i \leq n$. We note that the vertex labels remain distinct, the edge labels of the graph G_1

will remain fixed. Also for $1 < i \leq \frac{n+1}{2}$, the edge labels of the graph G_i will increase by the number $(p+q+1)(i-1)+k_1-k_i$ and for $\frac{n+3}{2} \leq i \leq n$, the edge labels of the graph G_i will increase by the number $(p+q+1)(i-1)+k_1-k_i+1$. The edge (bridge) between G_i and $G_{i+1}, 1 \leq i \leq \frac{n-1}{2}$, will get the label

$$\frac{1}{2}\left[p+q+k_i-1+(i-1)(p+q+1)+k_1-k_i+k_{i+1}+i(p+q+1)+k_1 -k_{i+1}\right] = \frac{1}{2}\left[p+q+k_i-1+(2i-1)(p+q+1)+2k_1-k_i\right] = i(p+q+1)+k_1-1$$

and the edge between G_i and G_{i+1}, $\frac{n+3}{2} \leq i \leq n-1$, will get the label

$$\frac{1}{2}\left[p+q+k_i-1+(i-1)(p+q+1)+k_1-k_i+k_{i+1}+i(p+q+1)+k_1 -k_{i+1}+1\right] = i(p+q+1)+k_1.$$

The edge between $G_{\frac{n+1}{2}}$ and $G_{\frac{n+3}{2}}$ has the label

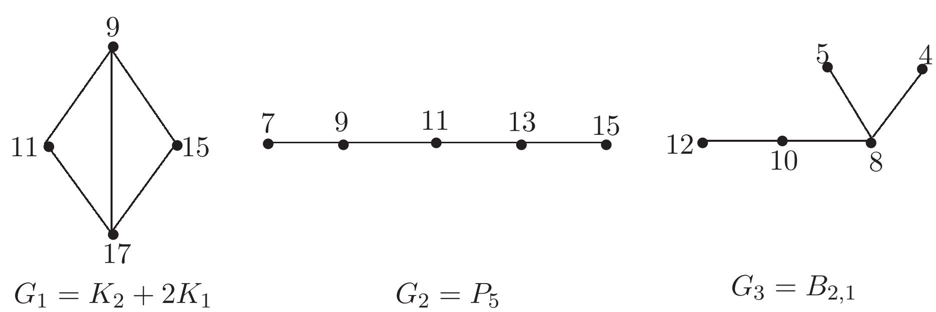
$$\left\lceil \frac{k_{\frac{n+1}{2}}+p+q-1+\left(\frac{n-1}{2}\right)(p+q+1)+k_1-k_{\frac{n+1}{2}}+k_{\frac{n+3}{2}}+\left(\frac{n+1}{2}\right)(p+q+1)+k_1-k_{\frac{n+3}{2}}+1}{2} \right\rceil = \left\lceil \frac{(n+1)(p+q)+n+2k_1}{2} \right\rceil$$

and the edge between G_n and G_1 has the label

$$\left\lceil \frac{k_n+p+q-1+(n-1)(p+q+1)+k_1-k_n+1+k_1+p+q-1}{2} \right\rceil = \left\lceil \frac{(n+1)(p+q)+n+2k_1-2}{2} \right\rceil.$$

Therefore the edge labels of the whole graph G are distinct. Also the union of the induced edge labels and vertex labels of G results the set $\{k_1, k_1+1, k_1+2, \ldots, n(p+q+1)+k_1-1\}$. Hence G is a k_1-super mean graph. □

Example 4.9. Let $m = 3$ and $G_1 = K_2 + 2K_1$, $G_2 = P_5$, $G_3 = B_{2,1}$. Then $p_i + q_i = 9$. Let $k_1 = 9, k_2 = 7$ and $k_3 = 4$. Consider a k_i-super mean labeling of G_i, $i = 1, 2, 3$ as follows:



The 9-super mean labeling a graph G obtained by the above construction is given in Figure 5.

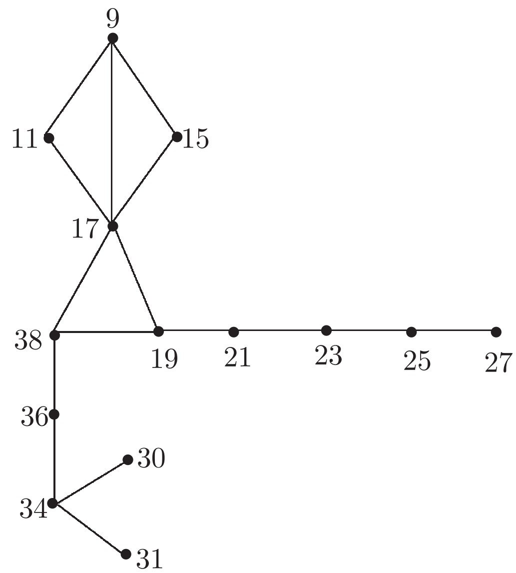


Figure 5:

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