International Journal of Pure and Applied Mathematics
Volume 56 No. 1 2009, 87-90

ON $\mathscr{K}$-LIFTING MODULES

Fangdi Kong[1 §], Shuhui Peng[2]
[1,2]Department of Applied Mathematics
Lanzhou University of Technology
Lanzhou, 730050, P.R. CHINA
[1]e-mail: kong_fd@yahoo.cn
[2]e-mail: Pengs_h@163.com

Abstract: Let R be a ring and M a left R-module. As a proper generalization of lifting module, we introduce the concept of $\mathscr{K}$-lifting module. M is called a $\mathscr{K}$-lifting module if for every $f \in$End(M), there exists a direct summand K of M such that $K \subseteq$Ker$f$ and Ker$f/K \ll M/K$. Some properties of $\mathscr{K}$-lifting modules are given.

AMS Subject Classification: 16D99
Key Words: $\mathscr{K}$-lifting module, fully invariant submodule

1. Introduction

In this note all rings are associative with identity and all modules are unital left modules unless otherwise specified.

Let M be a module and $S \leq M$. S is called *small* in M (denoted by $S \ll M$) if for any $T \leq M, S + T = M$ implies $T = M$. For $N, L \leq M, N$ is a *supplement* of L in M if $N + L = M$ with $N \cap L \ll N$. A module M is called *supplemented* if every submodule of M has a supplement in M. On the other hand, the module M is *amply supplemented* if, for any submodules A, B of M with $M = A + B$ there exists a supplement P of A in M such that $P \leq B$. A module M is called *a weakly supplemented module* if for each submodule A of

Received: August 13, 2009
§Correspondence author

M there exists a submodule B of M such that $M = A + B$ and $A \cap B \ll M$. Let M be a module and $B \leq A \leq M$. If $A/B \ll M/B$, then B is called *a coessential submodule* of A in M. A submodule A of M is called *coclosed* if A has no proper coessential submodule. Also, we will call B *a coclosure* (or an s-closure) of A in M, if B is a coessential submodule of A and B is coclosed in M. Recall that a module M is said to be *lifting* or said to satisfy the (D_1) condition if for any submodule N of M, there exists a direct summand K of M such that $K \leq N$ and $N/K \ll M/K$, equivalently, for every submodule N of M, M has a decomposition with $M = M_1 \oplus M_2$, $M_1 \leq N$ and $M_2 \cap N$ is small in M_2. It is well known that M is lifting if and only if M is amply supplemented and every supplement submodule of M is a direct summand.

2. $\mathscr{K}$-Lifting Modules

Definition 2.1. Let M be a module. M is called a $\mathscr{K}$-lifting module if for every $f \in$End(M), there exists a direct summand K of M such that $K \subseteq$Ker$f$ and Ker$f/K \ll M/K$, equivalently for every $f \in$End(M), there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq$Ker$f$ and Ker$f \cap M_2 \ll M_2$.

Clearly, any lifting module is $\mathscr{K}$-lifting. But the converse is not true.

Example 2.2. Let $\mathbb{Z}$ be the set of integral numbers. Then $\mathbb{Z}_{\mathbb{Z}}$ is $\mathscr{K}$-lifting. But $\mathbb{Z}_{\mathbb{Z}}$ is not lifting.

Proposition 2.3. *Let M be a cyclic torsionless moudle. Then M is a $\mathscr{K}$-lifting module.*

Proof. Let $M = Rx$ and $f : M \to M$ be any endmorphism of M. If $f = 0$, then it is trivial. Let $f \neq 0$. Then for any element m of Ker$f$, there exists an element r of R such that $m = rx$ and $f(m) = f(rx) = rf(x) = 0$. Since M is torsinonless and $f(x) \neq 0$, $r = 0$ and so $m = 0$. Thus Ker$f = 0$. Hence M is a $\mathscr{K}$-lifting module. □

Proposition 2.4. *Every direct summand of a $\mathscr{K}$-lifting module is $\mathscr{K}$-lifting.*

Proof. Let M be a $\mathscr{K}$-lifting module and K a direct summand of M. Then $M = K \oplus K', K' \leq M$. Let $f \in$End(K) and $g = f \oplus 1_{K'}$, then $g \in$End(M) and Ker$g =$Ker$f$. Since M is a $\mathscr{K}$-lifting module, there exists a direct summand N of M such that $N \leq$Ker$g$ and Ker$g/N \ll M/N$. Clearly, N is a direct summand of K and $M/N = K/N \oplus ((K' + N)/N)$. Hence $\mathrm{Ker}f/N \ll K/N$. □

Theorem 2.5. *Let $M = M_1 \oplus M_2$, where M_1 and M_2 are fully invariant submodules of M. If M_1 and M_2 are both $\mathscr{K}$-lifting, then M is a $\mathscr{K}$-lifting module.*

Proof. Let $f \in \mathrm{End}(M)$. Since M_1 and M_2 are fully invariant submodules of M, we have $f_1 = f|_{M_1} : M_1 \to M_1$ and $f_2 = f|_{M_2} : M_2 \to M_2$. Then $f = f_1 \oplus f_2$ and $\mathrm{Ker}f = \mathrm{Ker}f_1 \oplus \mathrm{Ker}f_2$. Since M_1 and M_2 are both $\mathscr{K}$-lifting, there exist direct summands L_1 and L_2 such that $L_1 \leq \mathrm{Ker}f_1, L_2 \leq \mathrm{Ker}f_2, \mathrm{Ker}f_1/L_1 \ll M_1/L_1$ and $\mathrm{Ker}f_2/L_2 \ll M_2 \ll L_2$. Hence $\mathrm{Ker}f/(L_1 \oplus L_2) \ll M/(L_1 \oplus L_2)$ and $L_1 \oplus L_2$ is a direct summand of M. Thus M is a $\mathscr{K}$-lifting module. □

Proposition 2.6. *Let M be a module. Then M is $\mathscr{K}$-lifting if and only if for any $f \in \mathrm{End}(M)$, $\mathrm{Ker}f = K \oplus L$, where K is a direct summand of M and $L \ll M$.*

Proof. Let M be a $\mathscr{K}$-lifting module and $f \in \mathrm{End}(M)$. Then there exists a direct summand K of M such that $K \leq \mathrm{Ker}f$ and $\mathrm{Ker}f/K \ll M/K$. Let $M = K \oplus F, F \leq M$, then $\mathrm{Ker}f = K \oplus (\mathrm{Ker}f \cap F)$. Further, if $X \leq F$ with $(F \cap \mathrm{Ker}f) + X = F$, then $\mathrm{Ker}f + X = M$. Since $\mathrm{Ker}f/K \ll M/K$, $X + K = M$. Hence $X = F$ and $F \cap \mathrm{Ker}f \ll F$. It suffices to take $L = F \cap \mathrm{Ker}f$.

Conversely, let $f \in \mathrm{End}(M)$, $\mathrm{Ker}f = K \oplus L$, where K is a direct summand of M and $L \ll M$. Let $X \leq M$ such that $K \leq X$ and $\mathrm{Ker}f/K + X/K = M/K$. Thus $\mathrm{Ker}f$+X=M. So $K + L + X = M$ and $K + X = M$. But $K \leq X$, we have $X = M$ and $\mathrm{Ker}f/K \ll M/K$. Therefore M is $\mathscr{K}$-lifting. □

Lemma 2.7. *Let M be a module and $N \leq M$. If M/N is M-projective, then there exists a homomorphism $f \in \mathrm{End}(M)$ such that $\mathrm{Ker}f = N$.*

Proof. Let $g : M \to M/N$ be the natural projection. Since M/N is M-projective, there exists homomorphism $h : M/N \to M$ such that $gh = 1_{M/N}$. Then h is a monomorphism. Let $f = hg$, then $\mathrm{Ker}f = N$. □

Proposition 2.8. *Let M be a module. If for any $U \leq M$, M/U is M-projective, then M is lifting if and only if M is $\mathscr{K}$-lifting.*

Proof. By Lemma 2.7. □

Theorem 2.9. *Let M be a $\mathscr{K}$-lifting module and N a fully invariant submodule of M. If every factor module of M/N is M-projective, then M/N is $\mathscr{K}$-lifting.*

Proof. Let $\bar{f} \in \mathrm{End}(M/N)$ and $\mathrm{Ker}\bar{f} = T/N$. Since $M/T \cong (M/N)/(T/N)$, there exists a homomorphism $f \in \mathrm{End}(M)$ such that $\mathrm{Ker} f = T$ by Lemma 2.7. Since M is $\mathscr{K}$-lifting, there exists a direct summand K of M such that $T/K \ll M/K$. Let $M = K \oplus K', K' \leq M$. Since N is a fully invariant submodule of M, $M/N = ((K+N)/N) \oplus ((K'+N)/N)$. It is easy to see that $(K+N)/N \leq T/N$ and $T/(K+N) \ll M/(K+N)$. Hence $(T/N)/((K+N)/N) \ll (M/N)/((K+N)/N)$. Thus M/N is $\mathscr{K}$-lifting. □

References

[1] F. Kasch, *Modules and Rings*, Academic Press, London-New York (1982).

[2] D. Keskin, On lifting modules, *Comm. Algebra*, **28** (2000), 3427-3440.

[3] M.T. Kosan, δ-lifting and δ-supplemented modules, *Algebra Colloq.*, **14** (2007), 53-60.

[4] S.H. Mohamed, B.J. Müller, *Continuous and Discrete Modules*, London Math. Soc. Lecture Note Series, **147**, Cambridge University Press (1990).

[5] Y. Wang, N. Ding, Generalized supplemented modules, *Taiwanese J. Math.*, **10** (2006), 1589-1601.

[6] Y. Wang, δ-small submodules and δ-supplemented modules, *IJMM* (2007), ID:58132.

[7] R. Wisbauer, *Foundations of Module and Ring Theory*, Gordon and Breach, Philadelphia (1991).

[8] R. Wisbauer, *Foundations of Module and Ring Theory*, Gordon and Breach, Philadelphia (1991).

[9] Y. Zhou, Generalizations of perfect, semiperfect, and semiregular rings, *Algebra Colloq.*, **3** (2000), 305-318.

[10] Y. Zhou, Generalizations of perfect, semiperfect, and semiregular rings, *Algebra Colloq.*, **3** (2000), 305-318.