International Journal of Pure and Applied Mathematics

Volume 56 No. 1 2009, 91-96

FIXED POINTS ON THE CLOSURE OF OPEN, CONVEX AND BOUNDED SETS

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Abstract: Let X be a reflexive and separable Banach space, U a nonempty, open, convex and bounded subset of X containing the origin O and and $f : \overline{U} \to X$ a continuous mapping in the weak topology of X. The boundary condition, here introduced, has the following statement: There exists $p > 1$ such that for all $x \in \partial U$ one has

$$\|x - fx\|^p \geq \|fx\|^p - \|x\|^p$$

or

$$\|x - fx\| > \|fx\| - \|x\|.$$

This paper contains some fixed points theorems under the boundary conditions, giving an extension of previous results concerning a ball B_r to a general closed and bounded sets in a reflexive and separable Banach space X.

AMS Subject Classification: 47H10, 54H25
Key Words: reflexive and separable Banach space, fixed points theorems

1. Introduction

A continuous map $f : \overline{U} \to H$, where U is a nonempty open, convex and bounded subset of a Banach space X of finite dimension, generally have no fixed points. To ensure the existence of fixed points it is necessary to add suitable conditions. In this paper our aim is to consider additional boundary

Received: August 15, 2009

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conditions. Let X be a Banach space and $U \subset X$ is a nonempty, open and bounded set containing the origin O. We denote by $S_r = \{x \in X : \|x\| < r\}$ the open ball with center at O and radius $r > 0$ and $B_r = \overline{S_r}$ the closed ball.

Altman [1] proved the following:

Theorem 1. *Let H be a separable Hilbert space. Assume that $f : B_r \to H$ is a weakly closed mapping. If $f(B_r)$ is a bounded set and the following condition is satisfied*

$$(fx, x) \le \|x\|^2, \tag{1.1}$$

for all $x \in \partial B_r$, then f has a fixed point.

Later Shinbrot [6] proved a similar result for a weakly continuous mapping, without requiring that $f(B_r)$ is bounded.

Theorem 2. *Let H be a separable Hilbert space. Assume that $f : H \to H$ is a weakly closed mapping. If there exists $r > 0$ such that*

$$(fx, x) \le \|x\|^2, \tag{1.1}$$

for all $x \in \partial B_r$, then f has a fixed point.

Theorem 3. (see [3]) *Let H be a reflexive and separable Banach space and let B_r be the closed ball of radius r of X. Assume that $f : B_r \to H$ is a continuous mapping in the weak topology of X and satisfying the following condition*

$$\|x - fx\|^p > \|fx\|^p - \|x\|^p, \tag{1.2}$$

for all $x \in \partial B_r$. Then f has a fixed point.

Theorem 4. (see [4]) *Let H be a reflexive and separable Banach space and let B_r be the closed ball of radius r of X. Assume that $f : B_r \to H$ is a continuous mapping in the weak topology of X and satisfying the following condition*

$$\|x - fx\| > \|fx\| - \|x\|, \tag{1.2}$$

for all $x \in \partial B_r$. Then f has a fixed point.

In this paper, using topological degree argument, we shall prove the following result.

Theorem 5. *Let H be a reflexive and separable Banach space and let U be a nonempty, open, convex and bounded subset of X containing the origin O. Assume that $f : \overline{U} \to H$ is a continuous mapping in the weak topology of X and satisfying the following condition:*

There exists $p > 1$ such that

$$\|x - fx\|^p \ge \|fx\|^p - \|x\|^p\,, \tag{1.2.a}$$

or

$$\|x - fx\| > \|fx\| - \|x\|, \tag{1.2.b}$$

for all $x \in \partial U$. Then f has a fixed point.

Remark 1. In the previous results one has the more restrictive hypothesis $\overline{U} = B_r$ and a metric approach has been used.

Remark 2. If the condition (1.2a) is verified for $p = 2$ we have

$$\|x - fx\|^2 \geq \|fx\|^2 - \|x\|^2, \tag{1.1}$$

that in the special case where X is a Hilbert space becomes the condition (1.1) of Altman and Shinbrot.

2. Main Result

In this section we deal with a general bounded and open subset U of a reflexive and separable Banach space containing 0. We report, without proof, the following elementary result.

Lemma. *Let $p > 1$. Then for all $0 < t < 1$ the following inequality*

$$(1-t)^p < 1 - t^p \tag{2.1}$$

holds.

The proof is performed in two steps. In the first step we assume that X is a finite dimensional Banach space, while in the second step we assume that X is infinite dimensional reflexive and separable Banach space.

Proof of Theorem 5. Step 1. We note that, since X is finite dimensional, then the mapping $f : \overline{U} \to X$ turns out to be a compact mapping. We will use a topological degree argument to prove that f has a fixed point. For $t \in [0, 1]$ and $x \in \overline{U}$ define the homotopy mapping

$$F : [0, 1] \times \overline{U} \to X$$

by

$$F(t, x) = tf(x). \tag{2.2}$$

It is immediate verify that F is compact. Remark that for any $t \in [0, 1]$ the Leray-Schauder degree

$$deg(x - F(t, x), U, 0) \tag{2.2}$$

is well defined if and only if the equation

$$F(t, x) - x = 0 \tag{2.3}$$

has no solution on $(t,x) \in [0,1] \times \partial U$. Indeed, suppose by contradiction, that there exists

$$(t_0, x_0) \in [0,1] \times \partial U,$$

$$x_0 = F(t_0, x_0) = t_0 f(x_0). \tag{2.4}$$

We have two cases.

If (1.2a) holds, we have

$$\|f(x_0) - x_0\|^p = \|f(x_0) - t_0 f(x_0)\|^p = (1-t)^p \|f(x_0)\|^p, \tag{2.5}$$

hence

$$\|f(x_0) - x_0\|^p = \frac{(1-t_0)^p}{t_0^p} \|x_0\|^p. \tag{2.6}$$

On the other hand, we have

$$\|f(x_0) - x_0\|^p = \|x_0\|^p t_0^{-p},$$

hence

$$\|f(x_0) - x_0\|^p = \frac{(1-t_0^p)}{t_0^p} \|x_0\|^p \geq \|fx_0\|^p - \|x_0\|^p = \|x_0\| t_0^{-p} - \|x_0\|^p, \tag{2.7}$$

henceforth

$$\|f(x_0) - x_0\|^p = \frac{1-t_0^p}{t_0^p} \|f(x_0)\|^p. \tag{2.8}$$

In view of condition (1.2), from (2.6) and (2.8) we get

$$(1-t_0)^p \geq 1 - t_0^p, \tag{2.9}$$

which cannot occur for any $t \in (0,1)$ as stated in Lemma 2. Moreover if $t_0 = 0$ then $x_0 = 0 \in U$. Finally if $t_0 = 1$, then $f(x_0) = x_0$, which is excluded by hypothesis. Therefore for all $(t,x) \in [0,1] \times \partial C$ it follows that

$$F(t,x) \neq x \tag{2.10}$$

and therefore the LS degree $d(x - F(t,x), U, 0)$ is well defined.

Analogously if (1.2b) holds, then we have

$$\|fx_0 - x_0\| = \|t_0^{-1} x_0 - x_0\| = (t_0^{-1} - 1)\|x_0\| > \|fx_0\| - \|x_0\|$$

which implies

$$t_0^{-1}\|x_0\| > \|fx_0\|,$$

thus we reach a contradiction.

Since the homotopy map is compact and f has no fixed points on ∂U, we can compute the degree using the following chain of equalities

$$deg(x - F(t,x), U, 0) = deg(x - tf(x), U, 0) = deg(x, U, 0) = 1. \tag{2.11}$$

This implies that there exists $x \in \overline{U}$ such that $f(x) = x$.

Step 2. Since X is a reflexive and separable Banach space, there exists a sequence X_n of finite dimensional subspace of X such that $\overline{\cup X_n} = X$. Moreover we can define a projection operator

$$P_n : X \to X_n, \tag{2.2}$$

where $P_n(x) = z, \quad \|x - z\| = \min_{y \in X_n} \|x - y\|$. Since f is continuous in the weak topology on X the restriction of $P_n f$ to subspace X_n is continuous (strong and weak convergence are identical in finite dimensional space).

Now we verify that $P_n f$ satisfies the boundary condition (1.2a) and (1.2b). Indeed

$$\|x - P_n f x\|^p = \|x - f x\|^p \geq \|f(x)\|^p - \|x\|^p = \|P_n f(x)\|^p - \|x\|^p, \quad (2.13.a)$$

or

$$\|x - P_n f x\| = \|x - P_n f x\| > \|f x\| - \|x\| = \|P_n f x\|^p - \|x\|, \qquad (2.13.b)$$

for all $x \in \partial U$. Then there exists $x_n \in X_n \cap \overline{U}$ such that $P_n f(x_n) = x_n$.

Thus the sequence $(x_n)_{n \in N}$ is contained in $\overline{U}$. Now taking into account that $\overline{U}$ is weakly compact, there exists a subsequence (x_{nk}) weakly convergent to some $x_* \in U$. Being f weakly continuous, the sequence $f(x_{nk})$ is weakly convergent to $f(x_*)$. Hence we get

$$x_{nk} = P - n_k f(x_{nk}) \to f(x_*), \tag{2.14}$$

while $x_{nk} \to x_*$: Finally we have $f(x_*) = x_*$, i.e. f has a fixed point on $\overline{U}$. $\square$

References

[1] M. Altman, A fixed point theorem in Hilbert space, *Bull. Polon. Sci.*, **5** (1957), 19-22.

[2] F.E. Browder, Fixed points theorems for noncompact mappings in Hilbert spaces, *Proc. Nat. Acad. Sci. USA*, **43** (1965), 1272-1276.

[3] D. Delbosco, Fixed points on a ball under a general boundary condition, *Ind. J. Math.*, **39** (1997), 207-212.

[4] D. Delbosco, G. Viola, Existence of fixed points for mappings on a ball of reflexive and separable Banach space, *Int. J. Pure Appl. Math.*, **32** (2006), 275-282.

[5] C. Kuratowski, Sur les espaces completes, *Fund. Math.*, **15** (1930), 301-309.

[6] Ky Fan, Extensions of two fixed point therems of F.E. Browder, *Math. Z.*, **112** (1969), 234-240.