International Journal of Pure and Applied Mathematics
Volume 56 No. 1 2009, 97-109

GENERALIZED DIFFERENCE OPERATOR OF THE FOURTH KIND AND ITS APPLICATIONS IN NUMBER THEORY (PART - I)

M. Maria Susai Manuel[1 §], V. Chandrasekar[2], G. Britto Antony Xavier[3], R. Pugalarasu[4], S. Elizabeth[5]

[1,2,3,4]Department of Mathematics
Sacred Heart College
Tirupattur, Tamil Nadu, 635 601, INDIA
[1]e-mail: manuelmsm_03@yahoo.co.in

[5]Department of Mathematics
Auxilium College
Vellore, Tamil Nadu, INDIA

Abstract: In this paper, the authors extend the theory of the generalized difference operator Δ_ℓ to the Generalized difference operator of the fourth kind $\Delta_{\ell_1,\ell_2,\ell_3,\ell_4}$ for the positive reals ℓ_1, ℓ_2, ℓ_3 and ℓ_4 and present the discrete version of Leibnitz Theorem, binomial theorem, Newton's formula with reference to $\Delta_{\ell_1,\ell_2,\ell_3,\ell_4}$. Also by defining its inverse, we establish a few formulae for the sum of third partial sums P^3S^n, for $n = 1, 2, 3...$ of the higher powers of the arithmetic progressions in number theory. Appropriate examples are provided to illustrate the results.

AMS Subject Classification: 39A12
Key Words: generalized difference operator, generalized polynomial factorial, partial sums

1. Introduction

The theory of difference operator Δ for the function $u(k)$ defined as $\Delta u(k) = u(k+1) - u(k)$ is established in [1, 11, 3] and the theory of generalized difference

Received: August 17, 2009

§Correspondence author

operator of first, second and third kinds defined as

$$\Delta_\ell u(k) = u(k+\ell) - u(k), \quad k \in [0,\infty), \ell \in (0,\infty), \tag{1}$$

$$\Delta_{\ell,m} u(k) = u(k+\ell+m) - [u(k+\ell) + u(k+m)] + u(k), \tag{2}$$

$$\begin{aligned}\Delta_{\ell_1,\ell_2,\ell_3} u(k) &= u(k+\ell_1+\ell_2+\ell_3) - [u(k+\ell_1+\ell_2) + u(k+\ell_1+\ell_3) \\ &\quad + u(k+\ell_2+\ell_3)] + [u(k+\ell_1) + u(k+\ell_2) + u(k+\ell_3)] - u(k)\end{aligned} \tag{3}$$

are developed in [4], [9] and [10], respectively. By extending the study for sequences of complex numbers and ℓ to be real, some new qualitative properties like rotatory, expanding and shrinking, spiral and web like were studied for the solutions of difference equations involving Δ_ℓ. The results obtained can be found in [4]-[8].

The formula for the value of the sum of the n-th powers of an arithmetic progression S^n, where

$$S^n = j^n + (j+\ell)^n + (j+2\ell)^n + \cdots + (j+k\ell)^n$$

is established in [4], formula for PS^n, where

$$PS^n = j^n + \overline{j^n + (j+\ell)^n} + \cdots + \overline{j^n + (j+\ell)^n + \cdots + (j+k\ell)^n}$$

is established in [9] and similarly P^2S^n is presented in [10].

Hence, in this paper, we derive the formulae for finding the value of third partial sums P^3S^n, where

$$\begin{aligned} P^3S^n &= j^n + \left\{ j^n + \left[j^n + \overline{j^n + (j+\ell)^n} \right] \right\} \\ &+ \left\{ j^n + \left[j^n + \overline{j^n + (j+\ell)^n} \right] + \left[j^n + \overline{j^n + (j+\ell)^n} + \overline{j^n + (j+\ell)^n + (j+2\ell)^n} \right] \right\} \\ &+ \cdots + \left\{ j^n + \left[j^n + \overline{j^n + (j+\ell)^n} \right] + \left[j^n + \overline{j^n + (j+\ell)^n} + \overline{j^n + (j+\ell)^n + (j+2\ell)^n} \right] \right. \\ &\quad + \cdots + \left[j^n + \overline{j^n + (j+\ell)^n} + \overline{j^n + (j+\ell)^n + (j+2\ell)^n} \right. \\ &\quad\quad \left. \left. + \ldots + \overline{j^n + (j+\ell)^n + \cdots + (j+k\ell)^n} \right] \right\}, \end{aligned}$$

for $n = 1, 2, 3, \ldots.$ using $\Delta_{\ell,\ell,\ell,\ell}$ and Stirling numbers of the second kind.

Throughout this paper, we make use of the following assumptions:

(i) r and n are positive integers and ℓ_1, ℓ_2, ℓ_3 and ℓ_4 are positive reals,

(ii) n^* is the largest non negative integer such that $k - n^*\ell \geq 0$,

(iii) c_i, $i = 0, 1, 2, \ldots$ are constants,

(iv) $rC_i = \frac{r!}{(r-i)!i!}$ where $0! = 1$, $r! = 1.2.3\ldots r$,

(v) $[x]$ is the integer part of x.

2. Basic Definitions and Examples

In this section, we present the discrete versions of Leibnitz and binomial theorems with reference to $\Delta_{\ell_1,\ell_2,\ell_3,\ell_4}$. Suitable examples are provided to illustrate the results.

Definition 2.1. Let $u : [0,\infty) \to \mathbb{C}$ be any complex valued function on $[0,\infty)$. We define the generalized difference operator of the fourth kind, $\Delta_{\ell_1,\ell_2,\ell_3,\ell_4}$ for $u(k)$ as

$$\Delta_{\ell_1,\ell_2,\ell_3,\ell_4}u(k) = u(\ell_1+\ell_2+\ell_3+\ell_4) - [u(k+\ell_1+\ell_2+\ell_3)+u(k+\ell_1+\ell_2+\ell_4) + u(k+\ell_1+\ell_3+\ell_4)+u(k+\ell_2+\ell_3+\ell_4)] + [u(k+\ell_1+\ell_2)+u(k+\ell_1+\ell_3) + u(k+\ell_1+\ell_4)+u(k+\ell_2+\ell_3)+u(k+\ell_2+\ell_4)+u(k+\ell_3+\ell_4)] - (u(k+\ell_1)+u(k+\ell_2)+u(k+\ell_3)+u(k+\ell_4))+u(k). \quad (4)$$

Lemma 2.2. *If E^ℓ is the usual shift operator defined as $E^\ell u(k) = u(k+\ell)$, then the following are simple to derive. For $\ell_j, j = 1,2,3,4,$ we obtain:*

(i) $$\Delta_{\ell_1,\ell_2,\ell_3,\ell_4} = E^{\ell_1+\ell_2+\ell_3+\ell_4} - \left(E^{\ell_1+\ell_2+\ell_3} + E^{\ell_1+\ell_2+\ell_4} + E^{\ell_1+\ell_3+\ell_4} + E^{\ell_2+\ell_3+\ell_4}\right) + \left(E^{\ell_1+\ell_2} + E^{\ell_1+\ell_3} + E^{\ell_1+\ell_4} + E^{\ell_2+\ell_3} + E^{\ell_2+\ell_4} + E^{k+\ell_3+\ell_4}\right) - \left(E^{\ell_1} + E^{\ell_2} + E^{\ell_3} + E^{\ell_4}\right) + 1. \quad (5)$$

(ii) $$\Delta_{\ell_1,\ell_2,\ell_3,\ell_4} = \Delta_{\ell_1+\ell_2+\ell_3+\ell_4} - (\Delta_{\ell_1+\ell_2+\ell_3} + \Delta_{\ell_1+\ell_2+\ell_4} + \Delta_{\ell_1+\ell_3+\ell_4} + \Delta_{\ell_2+\ell_3+\ell_4}) + (\Delta_{\ell_1+\ell_2} + \Delta_{\ell_1+\ell_3} + \Delta_{\ell_1+\ell_4} + \Delta_{\ell_2+\ell_3} + \Delta_{\ell_2+\ell_4} + \Delta_{\ell_3+\ell_4}) - (\Delta_{\ell_1} + \Delta_{\ell_2} + \Delta_{\ell_3} + \Delta_{\ell_4}). \quad (6)$$

(iii) $$\Delta_{\ell_1,\ell_2,\ell_3,\ell_4} = \Delta_{\ell_1}\Delta_{\ell_2}\Delta_{\ell_3}\Delta_{\ell_4}. \quad (7)$$

(iv) $$\Delta_{\ell_1,\ell_2,\ell_3,\ell_4} = \prod_{j=1}^{4}\left(\sum_{i=1}^{\ell_j} \ell_j c_i \Delta^i\right). \quad (8)$$

Definition 2.3. The second order of the generalized difference operator of the fourth kind is $\Delta^2_{\ell_1,\ell_2,\ell_3,\ell_4} = \Delta_{\ell_1,\ell_2,\ell_3,\ell_4}(\Delta_{\ell_1,\ell_2,\ell_3,\ell_4})$ and in general, the n-th order of the generalized difference operator of the fourth kind is defined as $\Delta^n_{\ell_1,\ell_2,\ell_3,\ell_4} = \Delta_{\ell_1,\ell_2,\ell_3,\ell_4}\left(\Delta^{n-1}_{\ell_1,\ell_2,\ell_3,\ell_4}\right)$.

Remark 2.4. For the positive integers p, q,

$$\Delta^p_{\ell_1,\ell_2,\ell_3,\ell_4}\Delta^q_{\ell_1,\ell_2,\ell_3,\ell_4} = \Delta^q_{\ell_1,\ell_2,\ell_3,\ell_4}\Delta^p_{\ell_1,\ell_2,\ell_3,\ell_4}.$$

As a consequence of Definition 2.3, the following results can be obtained easily.

Lemma 2.5. *(i)* *If $P_{4p-1}(k) = c_{4p-1}k^{4p-1} + c_{4p-2}k^{4p-2} + \cdots + c_1k^1 + c_0$ is any polynomial in k of degree $4p-1$ then, $\Delta^p_{\ell_1,\ell_2,\ell_3,\ell_4} P_{4p-1}(k) = 0$.*

(ii) *If m and n are positive integers and ℓ is a real number, then*

$$\Delta^n_{\ell,\ell,\ell,\ell} k^m = \begin{cases} 0, & \text{if} \quad m < 4n; \\ m!\ell^m, & \text{if} \quad m = 4n. \end{cases} \tag{9}$$

(iii) If $P_k = c_0k^{4n} + c_1k^{4n-1} + c_2k^{4n-2} + \cdots + c_n$ is any polynomial in k of degree $4n$, then

$$\Delta^n_{\ell,\ell,\ell,\ell} P_k = c_0(4n!)\ell^{4n}. \tag{10}$$

(iv) For the shift operator E,

$$\Delta^r_{\ell_1,\ell_2,\ell_3,\ell_4} = \prod_{j=1}^{4} \Big(\sum_{i=0}^{r} \ (-1)^i \ rC_i E^{\ell_j(r-i)} \Big) \tag{11}$$

which is equivalent to

$$\Delta^r_{\ell_1,\ell_2,\ell_3,\ell_4} u(k) = \prod_{j=1}^{4} \left(\sum_{i=0}^{r} \ (-1)^i \ rC_i u\big(k + \ell_j(r-i)\big) \right). \tag{12}$$

(v) If $\ell_j = \sum_{i=1}^{n} \ell_{j_i}$, $j = 1, 2, 3, 4$, then

$$\Delta_{\ell_1,\ell_2,\ell_3,\ell_4} = \prod_{j=1}^{4} \left[\prod_{i=1}^{n} \Big(\Delta_{\ell_{j_i}} + 1 \Big) - 1 \right].$$

(vi) (a) $\Delta_{n\ell_1,n\ell_2,n\ell_3,n\ell_4} = E^{n(\ell_1+\ell_2+\ell_3+\ell_4)} - (E^{n(\ell_1+\ell_2+\ell_3)} + E^{n(\ell_1+\ell_3+\ell_4)} +$

$$E^{n(\ell_1+\ell_2+\ell_4)} + E^{n(\ell_2+\ell_3+\ell_4)}) + (E^{n(\ell_1+\ell_2)} + E^{n(\ell_1+\ell_3)} + E^{n(\ell_1+\ell_4)} + E^{n(\ell_2+\ell_3)}$$
$$+E^{n(\ell_2+\ell_4)} + E^{n(\ell_3+\ell_4)}) - (E^{n(\ell_1)} + E^{n(\ell_2)} + E^{n(\ell_3)} + E^{n(\ell_4)}) + 1. \tag{13}$$

(b) $\Delta_{n\ell_1,n\ell_2,n\ell_3,n\ell_4} = (1+\Delta_{\ell_1+\ell_2+\ell_3+\ell_4})^n - [(1+\Delta_{\ell_1+\ell_2+\ell_3})^n + (1+\Delta_{\ell_1+\ell_3+\ell_4})^n + (1 + \Delta_{\ell_1+\ell_2+\ell_4})^n + (1 + \Delta_{\ell_2+\ell_3+\ell_4})^n] + [(1 + \Delta_{\ell_1+\ell_2})^n + (1 + \Delta_{\ell_1+\ell_3})^n + (1 + \Delta_{\ell_1+\ell_4})^n + (1 + \Delta_{\ell_2+\ell_3})^n + (1 + \Delta_{\ell_2+\ell_4})^n +$

$$(1 + \Delta_{\ell_3+\ell_4})^n] - [(1 + \Delta_{\ell_1})^n + (1 + \Delta_{\ell_2})^n + (1 + \Delta_{\ell_3})^n + (1 + \Delta_{\ell_4})^n] + 1. \tag{14}$$

(c) $\Delta_{n\ell_1,n\ell_2,n\ell_3,n\ell_4} = \sum_{r=0}^{n} nC_r \Big\{ \Delta^r_{\ell_1+\ell_2+\ell_3+\ell_4} - [\Delta^r_{\ell_1+\ell_2+\ell_3} + \Delta^r_{\ell_1+\ell_3+\ell_4}$
$+\Delta^r_{\ell_1+\ell_2+\ell_4} + \Delta^r_{\ell_2+\ell_3+\ell_4}] + [\Delta^r_{\ell_1+\ell_2} + \Delta^r_{\ell_1+\ell_3} + \Delta^r_{\ell_1+\ell_4} + \Delta^r_{\ell_2+\ell_3} + \Delta^r_{\ell_2+\ell_4}$

$$+\Delta^r_{\ell_3+\ell_4}] - [\Delta^r_{\ell_1} + \Delta^r_{\ell_2} + \Delta^r_{\ell_3} + \Delta^r_{\ell_4}] \Big\} + 1. \tag{15}$$

(d) $\Delta^n_{\ell_1,\ell_2,\ell_3,\ell_4} = \sum_{r=0}^{r} (-1)^r nC_r \Delta^{n-r}_{\ell_1+\ell_2+\ell_3+\ell_4} \Big\{ \sum_{i=0}^{r} (-1)^i nC_i (\Delta_{\ell_1+\ell_2+\ell_3}$

$$+\Delta_{\ell_1+\ell_2+\ell_4}+\Delta_{\ell_1+\ell_2+\ell_4}+\Delta_{\ell_2+\ell_3+\ell_4})^{r-i}\Big\{\sum_{j=0}^{i}(-1)^j iC_j[\Delta_{\ell_1+\ell_2}+\Delta_{\ell_3+\ell_4}+$$

$$\Delta_{\ell_1+\ell_3}+\Delta_{\ell_1+\ell_4}+\Delta_{\ell_2+\ell_4}+\Delta_{\ell_2+\ell_3}]^{i-j}(\Delta_{\ell_1}+\Delta_{\ell_2}+\Delta_{\ell_3}+\Delta_{\ell_4})^i\Big\}\Big\}. \quad (16)$$

$$(e)\quad \Delta^n_{\ell_1,\ell_2,\ell_3,\ell_4}=\prod_{j=1}^{4}\left(\sum_{i=0}^{n-1}(-1)^i nC_i\Delta_{(n-i)\ell_j}\right). \quad (17)$$

The following is the discrete version of Leibnitz Theorem according to $\Delta_{\ell_1,\ell_2,\ell_3,\ell_4}$.

Theorem 2.6. *For the functions $u:[0,\infty)\to\mathbb{C}$, $v:[0,\infty)\to\mathbb{C}$,*

$$\Delta^n_{\ell_1,\ell_2,\ell_3,\ell_4}(u(k)v(k))=\Delta^n_{\ell_1}(\Delta^n_{\ell_2}(\Delta^n_{\ell_3}(u(k)\Delta^n_{\ell_4}v(k))))+nC_1\Delta^n_{\ell_1}(\Delta^n_{\ell_2}$$

$$\Delta^{n-1}_{\ell_4}v(k+\ell))))+nC_2\Delta^n_{\ell_1}(\Delta^n_{\ell_2}(\Delta^n_{\ell_3}(\Delta^2_{\ell_4}(u(k))\Delta^{n-2}_{\ell_4}(v(k+2\ell)))))$$

$$+\cdots+nC_n\Delta^n_{\ell_1}(\Delta^n_{\ell_2}(\Delta^n_{\ell_3}(\Delta^n_{\ell_4}(u(k))(v(k+n\ell))))). \quad (18)$$

Proof. The proof follows from the generalized version of the Leibnitz Theorem (see [4]) and (7). □

The following is the direct consequence of 13 and 17.

Lemma 2.7. *For the usual shift operator E, we have*

$$E^{n(\ell_1+\ell_2+\ell_3+\ell_4)}-(E^{n(\ell_1+\ell_2+\ell_3)}+E^{n(\ell_1+\ell_3+\ell_4)}+E^{n(\ell_1+\ell_2+\ell_4)}$$

$$+E^{n(\ell_2+\ell_3+\ell_4)})+(E^{n(\ell_1+\ell_2)}+E^{n(\ell_1+\ell_3)}+E^{n(\ell_1+\ell_4)}+E^{n(\ell_2+\ell_3)}+E^{n(\ell_2+\ell_4)}$$

$$+E^{n(\ell_3+\ell_4)})-(E^{n\ell_1}+E^{n\ell_2}+E^{n\ell_3}+E^{n\ell_4})=\sum_{r=0}^{n}nC_r\Big\{\sum_{i=0}^{r-1}(-1)^i rC_i$$

$$[\Delta_{(r-i)(\ell_1+\ell_2+\ell_3+\ell_4)}-(\Delta_{(r-i)(\ell_1+\ell_2+\ell_3)}+\Delta_{(r-i)(\ell_1+\ell_2+\ell_4)}$$

$$+\Delta_{(r-i)(\ell_1+\ell_3+\ell_4)}+\Delta_{(r-i)(\ell_2+\ell_3+\ell_4)})+(\Delta_{(r-i)(\ell_1+\ell_2)}+\Delta_{(r-i)(\ell_1+\ell_3)}$$

$$+\Delta_{(r-i)(\ell_1+\ell_4)}+\Delta_{(r-i)(\ell_2+\ell_3)}+\Delta_{(r-i)(\ell_2+\ell_4)}+\Delta_{(r-i)(\ell_3+\ell_4)})$$

$$-(\Delta_{(r-i)\ell_1}+\Delta_{(r-i)\ell_2}+\Delta_{(r-i)\ell_3}+\Delta_{(r-i)\ell_4})]\Big\}. \quad (19)$$

Theorem 2.8. *If p is a positive integer, then*

$$(k+n(\ell_1+\ell_2+\ell_3+\ell_4))^p-[(k+n(\ell_1+\ell_2+\ell_3))^p+(k+n(\ell_1+\ell_2+\ell_4))^p+(k+n(\ell_1$$

$$+\ell_3+\ell_4))^p+(k+n(\ell_2+\ell_3+\ell_4))^p]+[(k+n(\ell_1+\ell_2))^p+(k+n(\ell_1+\ell_3))^p$$

$$+(k+n(\ell_1+\ell_4))^p+(k+n(\ell_2+\ell_3))^p+(k+n(\ell_2+\ell_4))^p+(k+n(\ell_3+\ell_4))^p]$$

$$-[(k+n(\ell_1))^p+(k+n(\ell_2))^p+(k+n(\ell_3))^p+(k+n(\ell_4))^p]$$

$$
\begin{aligned}
&= \sum_{r=0}^{n} nC_r \Big\{ \Big\{ (k+r(\ell_1+\ell_2+\ell_3+\ell_4))^p - [(k+r(\ell_1+\ell_2+\ell_3))^p + (k+r(\ell_1+\ell_2+\ell_4))^p \\
&+ (k+r(\ell_1+\ell_3+\ell_4))^p + (k+r(\ell_2+\ell_3+\ell_4))^p] + [(k+r(\ell_1+\ell_2))^p + (k+r(\ell_1 \\
&+\ell_3))^p + (k+r(\ell_1+\ell_4))^p + (k+r(\ell_2+\ell_3))^p + (k+r(\ell_2+\ell_4))^p + (k+r(\ell_3+\ell_4))^p] \\
&-[(k+r(\ell_1))^p + (k+r(\ell_2))^p + (k+r(\ell_3))^p + (k+r(\ell_4))^p] \Big\} - r\, c_1 \Big\{ (k+(r-1)(\ell_1+\ell_2 \\
&+\ell_3+\ell_4))^p - [(k+(r-1)(\ell_1+\ell_2+\ell_3))^p + (k+(r-1)(\ell_1+\ell_2+\ell_4))^p + (k+(r-1)(\ell_1 \\
&\quad + \ell_3+\ell_4))^p + (k+(r-1)(\ell_2+\ell_3+\ell_4))^p] + [(k+(r-1)(\ell_1+\ell_2))^p \\
&\quad + (k+(r-1)(\ell_1+\ell_3))^p + (k+(r-1)(\ell_1+\ell_4))^p + (k+(r-1)(\ell_2+\ell_3))^p \\
&\quad + (k+(r-1)(\ell_2+\ell_4))^p + (k+(r-1)(\ell_3+\ell_4))^p] - [(k+(r-1)(\ell_1))^p \\
&\quad + (k+(r-1)(\ell_2))^p + (k+(r-1)(\ell_3))^p + (k+(r-1)(\ell_4))^p \Big\} + \cdots \\
&+(-1)^{r-1} r\, c_{r-1} \Big\{ (k+\ell_1+\ell_2+\ell_3+\ell_4)^p - [(k+\ell_1+\ell_2+\ell_3)^p + (k+\ell_1+\ell_2+\ell_4)^p + (k \\
&+\ell_1+\ell_3+\ell_4)^p + (k+\ell_2+\ell_3+\ell_4)^p] + [(k+\ell_1+\ell_2)^p + (k+\ell_1+\ell_3)^p + (k+\ell_1+\ell_4)^p \\
&\quad + (k+\ell_2+\ell_3)^p + (k+\ell_2+\ell_4)^p + (k+\ell_3+\ell_4)^p] \\
&\quad - [(k+\ell_1)^p + (k+\ell_2)^p + (k+\ell_3)^p + (k+\ell_4)^p] \Big\} \Big\}. \qquad (20)
\end{aligned}
$$

Proof. The proof follows by operating (19) on $u(k) = k^p$. □

3. Generalized Polynomial Factorial of the Fourth Kind

In this section, we establish the relation among the generalized polynomial factorial, the polynomial k^n and discrete version of Newton's formula on $\Delta_{\ell,\ell,\ell,\ell}$.

Definition 3.1. The generalized polynomial factorial in k of the fourth kind is defined as

$$
\begin{aligned}
k^{(n)}_{\ell_1,\ell_2,\ell_3,\ell_4} &= \Big[(k+\ell_1+\ell_2+\ell_3)^{(n)}_{\ell_4} + (k+\ell_1+\ell_2+\ell_4)^{(n)}_{\ell_3} + (k+\ell_1+\ell_3+\ell_4)^{(n)}_{\ell_2} \\
&+(k+\ell_2+\ell_3+\ell_4)^{(n)}_{\ell_1} \Big] - \Big[(k+\ell_1+\ell_2)^{(n)}_{\ell_3} + (k+\ell_1+\ell_2)^{(n)}_{\ell_4} + (k+\ell_1+\ell_3)^{(n)}_{\ell_2} \\
&+(k+\ell_1+\ell_3)^{(n)}_{\ell_4} + (k+\ell_1+\ell_4)^{(n)}_{\ell_2} + (k+\ell_1+\ell_4)^{(n)}_{\ell_3} + (k+\ell_2+\ell_3)^{(n)}_{\ell_1} \\
&+(k+\ell_2+\ell_3)^{(n)}_{\ell_4} + (k+\ell_2+\ell_4)^{(n)}_{\ell_1} + (k+\ell_2+\ell_4)^{(n)}_{\ell_3} + (k+\ell_3+\ell_4)^{(n)}_{\ell_1} \\
&+(k+\ell_3+\ell_4)^{(n)}_{\ell_2} \Big] + \Big[(k+\ell_1)^{(n)}_{\ell_2} + (k+\ell_1)^{(n)}_{\ell_3} + (k+\ell_1)^{(n)}_{\ell_4} + (k+\ell_2)^{(n)}_{\ell_1} \\
&+(k+\ell_2)^{(n)}_{\ell_3} + (k+\ell_2)^{(n)}_{\ell_4} + (k+\ell_3)^{(n)}_{\ell_1} + (k+\ell_3)^{(n)}_{\ell_2} + (k+\ell_3)^{(n)}_{\ell_4} + (k+\ell_4)^{(n)}_{\ell_1}
\end{aligned}
$$

$$(k+\ell_4)^{(n)}_{\ell_2} + (k+\ell_4)^{(n)}_{\ell_3}\Big] - \Big[(k)^{(n)}_{\ell_1} + (k)^{(n)}_{\ell_2} + (k)^{(n)}_{\ell_3} + (k)^{(n)}_{\ell_4}\Big]. \tag{21}$$

Using the Stirling numbers of the first kind s^n_r, the following can be easily obtained.

Lemma 3.2. *For any real t, we have*

$$\sum_{r=1}^{n} s^n_r t^{n-r} \Delta_{\ell_1,\ell_2,\ell_3,\ell_4} k^r = \Delta_{\ell_1,\ell_2,\ell_3,\ell_4} k^{(n)}_t \tag{22}$$

and

$$\begin{aligned}
\Delta^m_{\ell_1,\ell_2,\ell_3,\ell_4} k^{(n)}_t &= \sum_{r=1}^{n} s^n_r t^{n-r}\Big\{\sum_{j=0}^{m}(-1)^j mC_j\Big[\sum_{k=0}^{m}(-1)^k mC_k\Big[\sum_{p=0}^{m}(-1)^p \\
&\quad mC_p(k+\ell_4 m+\ell_3(m-j)+\ell_2(m-k)+\ell_1(m-p))^r\Big]\Big]\Big\} \\
&\quad -mC_1\sum_{r=1}^{n} s^n_r t^{n-r}\Big\{\sum_{j=0}^{m}(-1)^j mC_j\Big[\sum_{k=0}^{m}(-1)^k\ mc_k\Big[\sum_{p=0}^{m}(-1)^p mC_p \\
&\quad (k+\ell_4(m-1)+\ell_3(m-j)+\ell_2(m-k)+\ell_1(m-p))^r\Big]\Big]\Big\} + \cdots + \\
&\quad +(-1)^i mC_i\sum_{r=1}^{n} s^n_r t^{n-r}\Big\{\sum_{j=0}^{m}(-1)^j mC_j\Big[\sum_{k=0}^{m}(-1)^k mC_k\Big[\sum_{p=0}^{m}(-1)^p mC_p \\
&\quad (k+\ell_4(m-i)+\ell_3(m-j)+\ell_2(m-k)+\ell_1(m-p))^r\Big]\Big]\Big\} + \cdots + \\
&\quad +(-1)^m\sum_{r=1}^{n} s^n_r t^{n-r}\Big\{\sum_{j=0}^{m}(-1)^j mC_j\Big[\sum_{k=0}^{m}(-1)^k mC_k\Big[\sum_{p=0}^{m}(-1)^p mC_p \\
&\quad (k+\ell_3(m-j)+\ell_2(m-k)+\ell_1(m-p))^r\Big]\Big]\Big\}.
\end{aligned} \tag{23}$$

Proof. The proof follows from (7) and the relation

$$\Delta^m_\ell k^{(n)}_t = \sum_{r=0}^{m}\big[(-1)^r mC_r\sum_{i=1}^{n} s^n_i t^{n-i}(k+(m-r)\ell)^i\big]. \qquad \square$$

Lemma 3.3. *If $t=\ell_1$, then*

$$\begin{aligned}
\Delta_{\ell_1,\ell_2,\ell_3,\ell_4} k^{(n)}_t &= n\ell_1\Big\{(k+\ell_2+\ell_3+\ell_4)^{(n-1)}_{\ell_1} - [(k+\ell_2+\ell_3)^{(n-1)}_{\ell_1} \\
&\quad + (k+\ell_2+\ell_4)^{(n-1)}_{\ell_1} + (k+\ell_3+\ell_4)^{(n-1)}_{\ell_1}] + [(k+\ell_2)^{(n-1)}_{\ell_1} + (k+\ell_3)^{(n-1)}_{\ell_1} \\
&\quad + (k+\ell_4)^{(n-1)}_{\ell_1}] - (k)^{(n-1)}_{\ell_1}\Big\}.
\end{aligned} \tag{24}$$

Proof. The proof follows from (4), (21) and the polynomial factorial of the first kind $k_t^{(n-1)}$. □

Lemma 3.4. *For the polynomial factorial $k^{(n)}_{\ell_1,\ell_2,\ell_3,\ell_4}$, we have*

$$\Delta_{\ell_1,\ell_2,\ell_3,\ell_4} k^{(n)}_{\ell_1,\ell_2,\ell_3,\ell_4} = n\ell_1 \Delta^2_{\ell_2,\ell_3,\ell_4} k^{(n-1)}_{\ell_1} + n\ell_2 \Delta^2_{\ell_1,\ell_3,\ell_4} k^{(n-1)}_{\ell_2} + n\ell_3 \Delta^2_{\ell_1,\ell_2,\ell_4} k^{(n-1)}_{\ell_3} + n\ell_4 \Delta^2_{\ell_1,\ell_2,\ell_3} k^{(n-1)}_{\ell_4}. \quad (25)$$

Proof. The proof follows from (7) and $\Delta_\ell k_\ell^{(n)} = n\ell k_\ell^{(n-1)}$. □

Corollary 3.5. *If $\ell_1 = \ell_2 = \ell_3 = \ell_4 = \ell$ then,*

$$\Delta_{\ell,\ell,\ell,\ell} \; k^{(n)}_{\ell,\ell,\ell,\ell} = (n\ell)^{(4)}_\ell \; k^{(n-4)}_{\ell,\ell,\ell,\ell}. \quad (26)$$

The following theorem is the generalized version of Newton's formula with reference to $\Delta_{\ell,\ell,\ell,\ell}$.

Theorem 3.6. *Let $f(k)$ be a polynomial in k of degree $3n$. Then $f(k)$ can be expressed as*

$$f(k) = f(0) + \frac{\Delta_{\ell,\ell,\ell,\ell} f(0)}{4!\ell^4} k_\ell^{(4)} + \frac{\Delta^2_{\ell,\ell,\ell,\ell} f(0)}{8!\ell^8} k_\ell^{(8)} + \cdots + \frac{\Delta^n_{\ell,\ell,\ell} f(0)}{(4n)!\ell^{4n}} k_\ell^{(4n)}. \quad (27)$$

Proof. Assume that

$$f(k) = a_0 + a_1 k_\ell^{(4)} + a_2 k_\ell^{(8)} + \cdots + a_n k_\ell^{(4n)}. \quad (28)$$

The coefficients $a_i, i = 1, 2, 3, \cdots, n$ are determined from the relation

$$\Delta^r_{\ell,\ell,\ell} f(0) = a_r (4r)! \ell^{4r}. \quad (29)$$

The proof then follows from (28) and (29). □

Corollary 3.7. *Let $f(k)$ be a polynomial in k of degree $3n$. Then $f(k-t)$ can be expressed as*

$$f(t) + \frac{\Delta_{\ell,\ell,\ell,\ell} f(t)}{4!\ell^4}(k-t)_\ell^{(4)} + \cdots + \frac{\Delta^n_{\ell,\ell,\ell,\ell} f(t)}{(4n)!\ell^{4n}}(k-t)_\ell^{(4n)}. \quad (30)$$

Proof. Replacing 0 by t and k by $(k-t)$ in (27) we get the result as desired. □

4. Inverse of Generalized Difference Operator of the Fourth Kind and its Applications

In this section, we define the inverse $\Delta^{-1}_{\ell_1,\ell_2,\ell_3,\ell_4}$ and present some results using the inverse which will be used to find $P^3 S^n$.

Definition 4.1. The inverse of the generalized difference operator of the fourth kind denoted by $\Delta^{-1}_{\ell_1,\ell_2,\ell_3,\ell_4}$ is defined as follows. If $\Delta_{\ell_1,\ell_2,\ell_3,\ell_4}(z(k)) = y(k)$, then

$$z(k) = \Delta^{-1}_{\ell_1,\ell_2,\ell_3,\ell_4} y(k) + c_{3j}\left(\frac{k^{(3)}_{\ell_3}}{3!\ell_3^3}\right) + c_{2j}\left(\frac{k^{(2)}_{\ell_2}}{2\ell_2^2}\right) + c_{1j}\left(\frac{k^{(1)}_{\ell_1}}{\ell_1}\right) + c_{0j}, \tag{31}$$

where c_{0j}, c_{1j}, c_{3j}, c_{4j}'s are constants and $j = k - \left[\frac{k}{\ell}\right]\ell$.

Lemma 4.2. *If $k \in [n\ell, \infty)$, then*

$$\Delta^{-1}_{\ell_1,\ell_2,\ell_3,\ell_4} k^{(n)}_{\ell_1,\ell_2,\ell_3,\ell_4} = \frac{k^{(n+1)}_{\ell_1}}{\ell_1(n+1)} + \frac{k^{(n+1)}_{\ell_2}}{\ell_2(n+1)} + \frac{k^{(n+1)}_{\ell_3}}{\ell_3(n+1)} + \frac{k^{(n+1)}_{\ell_4}}{\ell_4(n+1)} + c_{3j}\left(\frac{k^{(3)}_{\ell_3}}{3!\ell_3^3}\right) + c_{2j}\left(\frac{k^{(2)}_{\ell_2}}{2\ell_2^2}\right) + c_{1j}\left(\frac{k^{(1)}_{\ell_1}}{\ell_1}\right) + c_{0j}. \tag{32}$$

Proof. The proof follows from (31) and the relation

$$\Delta_\ell(k^{(n+1)}_\ell + c) = \ell(n+1)k^{(n)}_\ell.$$

□

Theorem 4.3. *If $j = k - \left[\frac{k}{\ell}\right]\ell$, then there exits constants c_{0j}, c_{1j}, c_{2j} and c_{3j} such that*

$$\Delta^{-1}_{\ell,\ell,\ell,\ell} u(k) = \sum_{t=2}^{n^*}\sum_{s=1}^{n^*}\sum_{r=1}^{n^*}\sum_{q=0}^{n^*} u(k - t\ell - s\ell - r\ell - q\ell) + c_{3j}\left(\frac{k^{(3)}_{\ell_3}}{3!\ell_3^3}\right) + c_{2j}\left(\frac{k^{(2)}_{\ell_2}}{2\ell_2^2}\right) + c_{1j}\left(\frac{k}{\ell}\right) + c_{0j}. \tag{33}$$

Proof. The proof follows by the relation $\Delta_{\ell,\ell,\ell,\ell}\Big\{ \sum\limits_{t=2}^{n^*}\sum\limits_{s=1}^{n^*}\sum\limits_{r=1}^{n^*}\sum\limits_{q=0}^{n^*} u(k - t\ell - s\ell - r\ell - q\ell) + c_{3j}\left(\frac{k^{(3)}_{\ell_3}}{3!\ell_3^3}\right) + c_{2j}\left(\frac{k^{(2)}_{\ell}}{2\ell^2}\right) + c_{1j}\left(\frac{k}{\ell}\right) + c_{0j}\Big\} = u(k)$. □

Lemma 4.4. *If $\lambda \neq 1, k \geq 4\ell$ and P_k is any function of k, then*

$$\sum_{t=2}^{n^*}\sum_{s=1}^{n^*}\sum_{r=1}^{n^*}\sum_{q=0}^{n^*} \lambda^{k-t\ell-s\ell-r\ell-q\ell} P_{k-t\ell-s\ell-r\ell-q\ell} = \frac{\lambda^k}{(\lambda^\ell - 1)^4}\left\{1 - \frac{\lambda^\ell \Delta_\ell}{(\lambda^\ell - 1)} + \frac{\lambda^{2\ell}\Delta_\ell^2}{(\lambda^\ell - 1)^2} + \cdots\right\}^4 P_k + c_{3j}\left(\frac{k^{(3)}_{\ell_3}}{3!\ell_3^3}\right) + c_{2j}\left(\frac{k^{(2)}_{\ell_2}}{2\ell_2^2}\right) + c_{1j}\left(\frac{k}{\ell}\right) + c_{0j}. \tag{34}$$

Proof. Let $\Delta_{\ell,\ell,\ell,\ell}\lambda^k F_k = \lambda^k P_k$, where $P_k = (\lambda^\ell E^\ell - 1)^4 F_k$. Operating both sides by $\Delta^{-1}_{\ell,\ell,\ell,\ell}$, we obtain

$$\Delta^{-1}_{\ell,\ell,\ell,\ell}\lambda^k P_k = \lambda^k F_k + c_{3j}\left(\frac{k^{(3)}_{\ell_3}}{3!\ell_3^3}\right) + c_{2j}\left(\frac{k^{(2)}_{\ell}}{2\ell^2}\right) + c_{1j}\left(\frac{k}{\ell}\right) + c_{0j}$$

$$= \lambda^k(\lambda^\ell E^\ell - 1)^{-4} P_k + c_{3j}\left(\frac{k^{(3)}_{\ell_3}}{3!\ell_3^3}\right) + c_{2j}\left(\frac{k^{(2)}_{\ell}}{2\ell^2}\right) + c_{1j}\left(\frac{k}{\ell}\right) + c_{0j}.$$

Now the proof follows from (33) and the binomial theorem. □

Lemma 4.5. *The relation between $\Delta^{-1}_{\ell,\ell,\ell,\ell}$ and Δ^{-1} is*

$$\sum_{p=0}^{\ell-1}\sum_{q=0}^{\ell-1}\sum_{r=0}^{\ell-1}\sum_{s=0}^{\ell-1}\Delta^{-1}_{\ell,\ell,\ell,\ell}u(k+p+q+r+s) = \Delta^{-1}\left(\Delta^{-1}\left(\Delta^{-1}\left(\Delta^{-1}u(k)\right)\right)\right)$$

$$+c_{3j}\left(\frac{k^{(3)}_{\ell_3}}{6\ell_3^3}\right) + c_{2j}\left(\frac{k^{(2)}_{\ell_2}}{2\ell_2^2}\right) + c_{1j}\left(\frac{k}{\ell}\right) + c_{0j}.$$

Proof. The proof follows from $\sum_{i=0}^{\ell-1}\Delta_\ell^{-1}u(k+i) = \Delta^{-1}u(k) + c$ and (33). □

The following two lemmas are easy deductions.

Lemma 4.6. *If S^n_r's are the Stirling numbers of the second kind, then*

$$(k+3\ell)^n - 3(k+2\ell)^n + 3(k+\ell)^n - k^n = \frac{1}{4}\sum_{r=1}^{n} S^n_r \ell^{n-r} k^{(r)}_{\ell,\ell,\ell,\ell}. \tag{35}$$

Lemma 4.7. *For the polynomial factorial of the fourth kind $k^{(n)}_{\ell,\ell,\ell,\ell}$, we have*

$$(i) \qquad \Delta^{-1}_{\ell,\ell,\ell,\ell}k^{(n)}_\ell = \frac{k^{(n+7)}_{\ell,\ell,\ell,\ell}}{4(n+1)...(n+7)\ell^7} + \frac{c_{3j}k^{(3)}_\ell}{3!\ell^3} + \frac{c_{2j}k^{(2)}_\ell}{2\ell^2} + \frac{c_{1j}k}{\ell} + c_{0j}, \tag{36}$$

$$(ii) \qquad k^n = \frac{1}{4}\sum_{r=1}^{n} S^n_r \ell^{n-r}\Delta^{-1}_{\ell,\ell,\ell,\ell}k^{(r)}_{\ell,\ell,\ell,\ell} + \frac{c_{3j}k^{(3)}_\ell}{3!\ell^3} + \frac{c_{2j}k^{(2)}_\ell}{2\ell^2} + \frac{c_{1j}k}{\ell} + c_{0j}. \tag{37}$$

The following theorem is the general rule to find the value of P^3S^n, where S^n is the sum of n-th powers of an arithmetic progression.

Theorem 4.8. *If $k \in [5\ell, \infty)$, then*

$$\sum_{t=2}^{n^*}\sum_{s=1}^{n^*}\sum_{r=1}^{n^*}\sum_{q=0}^{n^*}(k - t\ell - s\ell - r\ell - q\ell)^n = \sum_{p=1}^{n}\frac{S^n_p \ell^{n-(p+4)} k^{(p+4)}_\ell}{\prod_{i=1}^{4}(p+i)}$$

$$-[c_{3j}\left(\frac{k_\ell^{(3)}}{3!\ell^3}\right)+c_{2j}\left(\frac{k_\ell^{(2)}}{2\ell^2}\right)+c_{1j}\left(\frac{k}{\ell}\right)+c_{0j}], \quad (38)$$

where $c'_{ij}s$ for $i=3,2,1,0$ are obtained by solving the system of four equations

$$\sum_{t=2}^{n^*}\sum_{s=1}^{n^*}\sum_{r=1}^{n^*}\sum_{q=0}^{n^*}((n+a)-t-s-r-q)\ell+j)^n=\sum_{p=1}^{n}\frac{S_p^n\ell^{n-(p+4)}((n+a)\ell+j)_\ell^{(p+4)}}{\prod\limits_{i=1}^{4}(p+i)}$$

$$-[c_{3j}\frac{((n+a)\ell+j)_\ell^{(3)}}{3!\ell^3}+c_{2j}\frac{((n+a)\ell+j)_\ell^{(2)}}{2\ell^2}+c_{1j}\frac{((n+a)\ell+j)}{\ell}+c_{0j}], \quad (39)$$

for $a=3,4,5,6$.

Proof. From (7) and (33), we have

$$\sum_{t=2}^{n^*}\sum_{s=1}^{n^*}\sum_{r=1}^{n^*}\sum_{q=0}^{n^*}(k-t\ell-s\ell-r\ell-q\ell)^n+c_{3j}\left(\frac{k_\ell^{(3)}}{6\ell^3}\right)+c_{2j}\left(\frac{k_\ell^{(2)}}{2\ell^2}\right)$$

$$+c_{1j}\left(\frac{k}{\ell}\right)+c_{0j}=\sum_{p=1}^{n}\frac{S_p^n\ell^{n-(p+4)}k_\ell^{(p+4)}}{\prod\limits_{i=1}^{4}(p+i)}. \quad (40)$$

Replace k by $(n+a)\ell+j$ for $a=3,4,5,6$ in (40) and solve the system of four equations with four unknowns. The proof follows by (39). □

Example 4.9. We shall find the values of $S^5=2^5+6^5+10^5+14^5+\cdots+54^5$, PS^5, the sum of all partial sums of s^5, P^2S^5, sum of all partial sums of PS^5, P^3S^5, sum of all partial sums of P^2S^5, where

$$PS^5=2^5+\overline{2^5+6^5}+\overline{2^5+6^5+10^5}+\cdots+\overline{2^5+6^5+10^5+\cdots+54^5},$$

$$P^2S^5=2^5+\left[2^5+\overline{2^5+6^5}\right]+\left[2^5+\overline{2^5+6^5}+\overline{2^5+6^5+10^5}\right]+\cdots$$
$$+\left[2^5+\overline{2^5+6^5}+\overline{2^5+6^5+10^5}+\cdots+\overline{2^5+6^5+\cdots+54^5}\right],$$

$$P^3S^5=2^5+\left[2^5+(2^5+\overline{2^5+6^5})\right]+\left[2^5+\left(2^5+\overline{2^5+6^5}\right)+\left(2^5+\overline{2^5+6^5}\right.\right.$$
$$\left.\left.+\overline{2^5+6^5+10^5}\right)\right]+\cdots+\left[2^5+(2^5+(2^5+\overline{2^5+6^5}))+\left(2^5+(2^5\right.\right.$$
$$\left.+\overline{2^5+6^5}+\overline{2^5+6^5+10^5})\right)+\cdots+\left(2^5+(2^5+(2^5+\overline{2^5+6^5}))\right.$$
$$\left.\left.+\cdots+\left(2^5+\left(2^5+\overline{2^5+6^5}\right)+\cdots+54^5\right)\right)\right].$$

Solution. In (38) take $n=5, j=2, \ell=4$ and $k=70$. By taking $k=34,38,42,46$ in (40), we obtain four simultaneous equations as given below:

$$\sum_{t=2}^{n^*}\sum_{s=1}^{n^*}\sum_{r=1}^{n^*}\sum_{q=0}^{n^*}(34-4t-4s-4r-4q)^5+c_{3j}\left(\frac{34_4^{(3)}}{(6)4^3}\right)+c_{2j}\left(\frac{34_4^{(2)}}{(2)4^2}\right)$$

$$+ c_{1j}\big(\frac{34}{4}\big) + c_{0j} = \sum_{p=1}^{5} \frac{S_p^5 4^{5-(p+4)} 34_4^{(p+4)}}{(p+1)(p+2)(p+3)(p+4)}, \quad (41)$$

$$\sum_{t=2}^{n^*}\sum_{s=1}^{n^*}\sum_{r=1}^{n^*}\sum_{q=0}^{n^*}(38-4t-4s-4r-4q)^5 + c_{3j}\big(\frac{38_4^{(3)}}{(6)4^3}\big) + c_{2j}\big(\frac{38_4^{(2)}}{(2)4^2}\big)$$

$$+ c_{1j}\big(\frac{38}{4}\big) + c_{0j} = \sum_{p=1}^{5} \frac{S_p^5 4^{5-(p+4)} 38_4^{(p+4)}}{(p+1)(p+2)(p+3)(p+4)}, \quad (42)$$

$$\sum_{t=2}^{n^*}\sum_{s=1}^{n^*}\sum_{r=1}^{n^*}\sum_{q=0}^{n^*}(42-4t-4s-4r-4q)^5 + c_{3j}\big(\frac{42_4^{(3)}}{(6)4^3}\big) + c_{2j}\big(\frac{42_4^{(2)}}{(2)4^2}\big)$$

$$+ c_{1j}\big(\frac{42}{4}\big) + c_{0j} = \sum_{p=1}^{5} \frac{S_p^5 4^{5-(p+4)} 42_4^{(p+4)}}{(p+1)(p+2)(p+3)(p+4)}, \quad (43)$$

$$\sum_{t=2}^{n^*}\sum_{s=1}^{n^*}\sum_{r=1}^{n^*}\sum_{q=0}^{n^*}(46-4t-4s-4r-4q)^5 + c_{3j}\big(\frac{46_4^{(3)}}{(6)4^3}\big) + c_{2j}\big(\frac{46_4^{(2)}}{(2)4^2}\big)$$

$$+ c_{1j}\left(\frac{46}{4}\right) + c_{0j} = \sum_{p=1}^{5} \frac{S_p^5 4^{5-(p+4)} 46_4^{(p+4)}}{(p+1)(p+2)(p+3)(p+4)}. \quad (44)$$

By solving equations (41), (42), (43) and (44) we get constants c_{ij}'s, for $i = 0, 1, 2, 3$ and the equation (38) yields

$$P^3S^5 = 16627027070.$$

Similarly one can find P^2S^5, PS^5 and S^5 using $\Delta_{\ell,\ell,\ell}$, $\Delta_{\ell,\ell}$, Δ_ℓ (see [4], [9], [10]).

Acknowledgments

The research is supported by University Grants Commission, New Delhi.

References

[1] R.P. Agarwal, *Difference Equations and Inequalities*, Marcel Dekker, New York (2000).

[2] Saber N. Elaydi, *An Introduction to Difference Equations*, Second Edition, Springer (1999).

[3] Walter G. Kelley, Allan C. Peterson, *Difference Equations, An Introduction with Applications*, Academic Press (1991).

[4] M. Maria Susai Manuel, G. Britto Antony Xavier, E. Thandapani, Theory of generalized difference operator and its applications, *Far East Journal of Mathematical Sciences*, **20**, No. 2 (2006), 163-171.

[5] M. Maria Susai Manuel, G. Britto Antony Xavier, E. Thandapani, Qualitative properties of solutions of certain class of difference equations, *Far East Journal of Mathematical Sciences*, **23**, No. 3 (2006), 295-304.

[6] M. Maria Susai Manuel, G. Britto Antony Xavier, E. Thandapani, Generalized Bernoulli polynomials through weighted Pochhammer symbols, *Far East Journal of Applied Mathematics*, **26**, No. 3 (2007), 321-333.

[7] M. Maria Susai Manuel, A. George Maria Selvam, G. Britto Antony Xavier, Rotatory and boundedness of solutions of certain class of difference equations, *International Journal of Pure and Applied Mathematics*, **33**, No. 3 (2006), 333-343.

[8] M. Maria Susai Manuel, G. Britto Antony Xavier, Recessive, dominant and spiral behaviours of solutions of certain class of generalized difference equations, *International Journal of Differential Equations and Applications*, **10**, No. 4 (2007), 423-433.

[9] M. Maria Susai Manuel, G. Britto Antony Xavier, V. Chandrasekar, Generalized difference operator of the second kind and its application to number theory, *International Journal of Pure and Applied Mathematics*, **47**, No. 1 (2008), 127-140.

[10] M. Maria Susai Manuel, G. Britto Antony Xavier, V. Chandrasekar, R. Pugalarasu, S. Elizabeth, On generalized difference operator of third kind and its applications in number theory, *International Journal of Pure and Applied Mathematics*, **53**, No. 1 (2009), 69-81.

[11] Ronald E. Mickens, *Difference Equations*, Van Nostrand Reinhold Company, New York (1990).