International Journal of Pure and Applied Mathematics
Volume 56 No. 1 2009, 111-122

GENERALIZED DIFFERENCE OPERATOR OF THE FOURTH KIND AND ITS APPLICATIONS IN NUMBER THEORY (PART-II)

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Abstract: As the theory of the generalized difference operator of first kind Δ_ℓ, second Δ_{ℓ_1,ℓ_2} and third $\Delta_{\ell_1,\ell_2,\ell_3}$ have been developed in [4], [9], [10], [11], in this paper, the authors extend the theory of Δ_ℓ to the fourth kind operator $\Delta_{\ell_1,\ell_2,\ell_3,\ell_4}$ for the positive reals ℓ_1, ℓ_2, ℓ_3 and ℓ_4 by presenting some results on generalized polynomial factorials of third and fourth kinds, generalized Leibnitz Theorem, Newton's formula and formula to find third partial sums of products of n consecutive terms of arithmetic and arithmetic-geometric progression by defining its inverse operators $\Delta^{-1}_{\ell_1,\ell_2,\ell_3,\ell_4}$.

AMS Subject Classification: 39A12
Key Words: generalized difference operator, generalized factorial, partial sums

1. Introduction

The theory of the difference operator Δ for the function $u(k)$ defined as

$$\Delta u(k) = u(k+1) - u(k), \qquad k \in \mathbb{N}, \tag{1}$$

where $\mathbb{N} = \{0, 1, 2, 3, \cdots\}$ is established in [1], [12]-[3] and the theory of gener-

Received: August 18, 2009

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alized difference operator of first kind defined as

$$\Delta_\ell u(k) = u(k+\ell) - u(k), \quad k \in [0,\infty), \ell \in (0,\infty), \tag{2}$$

and the second kind defined as

$$\Delta_{\ell,m} u(k) = u(k+\ell+m) - [u(k+\ell) + u(k+m)] + u(k), \tag{3}$$

where $k \in [0,\infty), \ell, m \in (0,\infty)$ are developed in [4] and [9] respectively. Recently, we took up the definition of Δ_ℓ as given in (2) and developed the theory of difference equations in a different direction and obtained some interesting results in the application of number theory. By extending the study for sequences of complex numbers and ℓ to be real, some new qualitative properties like rotatory, expanding and shrinking, spiral and web like were studied for the solutions of difference equations involving Δ_ℓ. The results obtained can be found in [4]-[8].

The formulae for C_n, the sum of products of n consecutive terms of A.P., PC_n, the sum of all partial sums from C_n, P^2C_n, the sum of all partial sums from PC_n, nS^n, the sum of arithmetic-geometric progression, PnS^n, the sum of all partial sums of nS^n and P^2nS^n, the sum of all partial sums of PnS^n have been derived in [4], [9], [10], [11], where

$$C_n = \prod_{r=0}^{n-1}(i+r\ell) + \prod_{r=1}^{n}(i+r\ell) + \prod_{r=2}^{n+1}(i+r\ell) + \cdots + \prod_{r=k}^{n+k-1}(i+r\ell),$$

$$PC_n = \prod_{r=0}^{n-1}(i+r\ell) + \overline{\prod_{r=0}^{n-1}(i+r\ell)} + \cdots + \overline{\prod_{r=0}^{n-1}(i+r\ell) + \prod_{r=1}^{n}(i+r\ell) + \cdots + \prod_{r=k}^{n+k-1}(i+r\ell)},$$

$$P^2C_n = \prod_{r=0}^{n-1}(i+r\ell) + \left[\prod_{r=0}^{n-1}(i+r\ell) + \overline{\prod_{r=0}^{n-1}(i+r\ell) + \prod_{r=1}^{n}(i+r\ell)}\right] + \cdots + \left[\prod_{r=0}^{n-1}(i+r\ell) + \overline{\prod_{r=0}^{n-1}(i+r\ell) + \prod_{r=1}^{n}(i+r\ell)} + \cdots + \overline{\prod_{r=0}^{n-1}(i+r\ell) + \prod_{r=1}^{n}(i+r\ell) + \cdots + \prod_{r=k}^{n+k-1}(i+r\ell)}\right],$$

$$nS^n = ja^j + (j+\ell)a^{j+\ell} + (j+2\ell)a^{j+2\ell} + \cdots + (j+n\ell)a^{j+n\ell},$$

$$PnS^n = ja^j + \overline{ja^j + (j+\ell)a^{j+\ell}} + \cdots + \overline{ja^j + \cdots + (j+n\ell)a^{j+n\ell}},$$

$$\begin{aligned}P^2nS^n = {} & ja^j + [ja^j + \overline{ja^j + (j+\ell)a^{j+\ell}}] + [ja^j + \overline{ja^j + (j+\ell)a^{j+\ell}} \\ & + \overline{ja^j + (j+\ell)a^{j+\ell} + (j+2\ell)a^{j+2\ell}}] + \cdots + [ja^j + \overline{ja^j + (j+\ell)a^{j+\ell}} \\ & + \cdots + \overline{ja^j + (j+\ell)a^{j+\ell} + (j+2\ell)a^{j+2\ell} + \cdots + (j+n\ell)a^{j+n\ell}}].\end{aligned}$$

Hence in this paper, we develop some significant results on $\Delta_{\ell_1,\ell_2,\ell_3,\ell_4}$ and derive formulae for finding the values of P^3C_n, the sum of all partial sums from P^2C_n and P^3nS^n, the sum of all partial sums of P^2nS^n, where

$$\begin{aligned}P^3C_n = {} & \prod_{r=0}^{n-1}(i+r\ell) + \Bigg(\prod_{r=0}^{n-1}(i+r\ell) + \Bigg[\prod_{r=0}^{n-1}(i+r\ell) \\ & + \overline{\prod_{r=0}^{n-1}(i+r\ell) + \prod_{r=1}^{n}(i+r\ell)}\Bigg]\Bigg) + \cdots + \Bigg(\prod_{r=0}^{n-1}(i+r\ell) \\ & + \Bigg[\prod_{r=0}^{n-1}(i+r\ell) + \overline{\prod_{r=0}^{n-1}(i+r\ell) + \prod_{r=1}^{n}(i+r\ell)}\Bigg] + \Bigg[\prod_{r=0}^{n-1}(i+r\ell) \\ & + \overline{\prod_{r=0}^{n-1}(i+r\ell) + \prod_{r=1}^{n}(i+r\ell)} + \overline{\prod_{r=0}^{n-1}(i+r\ell) + \prod_{r=1}^{n}(i+r\ell) + \prod_{r=2}^{n+1}(i+r\ell)}\Bigg] \\ & + \cdots + \Bigg[\prod_{r=0}^{n-1}(i+r\ell) + \overline{\prod_{r=0}^{n-1}(i+r\ell) + \prod_{r=1}^{n}(i+r\ell)} + \cdots \\ & + \overline{\prod_{r=0}^{n-1}(i+r\ell) + \prod_{r=1}^{n}(i+r\ell) + \cdots + \prod_{r=k}^{n+k-1}(i+r\ell)}\Bigg]\Bigg),\end{aligned}$$

$$\begin{aligned}P^3nS^n = {} & ja^j + \left(ja^j + \overline{ja^j + (j+\ell)a^{j+\ell}}\right) + \cdots \\ & + \Big(ja^j + [ja^j + \overline{ja^j + (j+\ell)a^{j+\ell}}] + [ja^j + \overline{ja^j + (j+\ell)a^{j+\ell}} \\ & + \overline{ja^j + (j+\ell)a^{j+\ell} + (j+2\ell)a^{j+2\ell}}] + \cdots + [ja^j + \overline{ja^j + (j+\ell)a^{j+\ell}} \\ & + \cdots + \overline{ja^j + (j+\ell)a^{j+\ell} + (j+2\ell)a^{j+2\ell} + \cdots + (j+n\ell)a^{j+n\ell}}]\Big),\end{aligned}$$

using $\Delta_{\ell,\ell,\ell,\ell}$.

Throughout this paper, we use the following assumptions:

(i) r and n are positive integers and ℓ_1, ℓ_2, ℓ_3 and ℓ_4 are positive reals;

(ii) n^* is the largest non negative integer such that $k - n^*\ell \geq 0$;

(iii) $c, c_0, c_1, c_2, \ldots$ are constants;

(iv) $rC_i = \frac{r!}{(r-i)!i!}$ where $0! = 1$, $r! = 1.2.3...r$;

(v) $[x]$ is integer part of x.

2. Preliminares

Definition 2.1. Let $u : [0, \infty) \to \mathbb{C}$ be any complex valued function on $[0, \infty)$. We define the generalized difference operator of the fourth kind for $u(k)$ as

$$\begin{aligned}\Delta_{\ell_1,\ell_2,\ell_3,\ell_4} u(k) = u(k+\ell_1+\ell_2+\ell_3+\ell_4) - [u(k+\ell_1+\ell_2+\ell_3) + u(k+\ell_1+\ell_2 \\ +\ell_4) + u(k+\ell_1+\ell_3+\ell_4) + u(k+\ell_2+\ell_3+\ell_4)] + [u(k+\ell_1+\ell_2) \\ + u(k+\ell_1+\ell_3) + u(k+\ell_1+\ell_4) + u(k+\ell_2+\ell_3) + u(k+\ell_2+\ell_4) + u(k+\ell_3 \\ +\ell_4)] - [u(k+\ell_1) + u(k+\ell_2) + u(k+\ell_3) + u(k+\ell_4)] + u(k). \quad (4)\end{aligned}$$

Lemma 2.2. *If E is the usual shift operator defined as $E^\ell u(k) = u(k+\ell)$, then the following is simple to derive. If $\ell_j, j = 1, 2, 3, 4$ are positive reals, we obtain*

(i) $$\begin{aligned}\Delta_{\ell_1,\ell_2,\ell_3,\ell_4} = E^{\ell_1+\ell_2+\ell_3+\ell_4} - \Big(E^{\ell_1+\ell_2+\ell_3} + E^{\ell_1+\ell_2+\ell_4} + E^{\ell_1+\ell_3+\ell_4} \\ +E^{\ell_2+\ell_3+\ell_4}\Big) + \Big(E^{\ell_1+\ell_2} + E^{\ell_1+\ell_3} + E^{\ell_1+\ell_4} + E^{\ell_2+\ell_3} + E^{\ell_2+\ell_4} \\ +E^{k+\ell_3+\ell_4}\Big) - \Big(E^{\ell_1} + E^{\ell_2} + E^{\ell_3} + E^{\ell_4}\Big) + 1. \quad (5)\end{aligned}$$

(ii) $$\begin{aligned}\Delta_{\ell_1,\ell_2,\ell_3,\ell_4} = \Delta_{\ell_1+\ell_2+\ell_3+\ell_4} - (\Delta_{\ell_1+\ell_2+\ell_3} + \Delta_{\ell_1+\ell_2+\ell_4} + \Delta_{\ell_1+\ell_3+\ell_4} \\ +\Delta_{\ell_2+\ell_3+\ell_4}) + (\Delta_{\ell_1+\ell_2} + \Delta_{\ell_1+\ell_3} + \Delta_{\ell_1+\ell_4} + \Delta_{\ell_2+\ell_3} + \Delta_{\ell_2+\ell_4} \\ +\Delta_{\ell_3+\ell_4}) - (\Delta_{\ell_1} + \Delta_{\ell_2} + \Delta_{\ell_3} + \Delta_{\ell_4}). \quad (6)\end{aligned}$$

(iii) $$\Delta_{\ell_1,\ell_2,\ell_3,\ell_4} = \Delta_{\ell_1}\Delta_{\ell_2}\Delta_{\ell_3}\Delta_{\ell_4}. \quad (7)$$

(iv) $$\begin{aligned}\Delta_{\ell_1+\ell_2+\ell_3+\ell_4} = \Big(\sum_{i=1}^{\ell_1} \ell_1\ c_i\ \Delta^i\Big)\Big(\sum_{j=1}^{\ell_2} \ell_1\ c_j\ \Delta^j\Big)\Big(\sum_{k=1}^{\ell_3} \ell_1\ c_k\ \Delta^k\Big) \\ \Big(\sum_{m=1}^{\ell_4} \ell_4\ c_m\ \Delta^m\Big). \quad (8)\end{aligned}$$

Definition 2.3. The generalized polynomial factorial in k of the fourth kind is defined as

$$k^{(n)}_{\ell_1,\ell_2,\ell_3,\ell_4} = \big[(k+\ell_1+\ell_2+\ell_3)^{(n)}_{\ell_4} + (k+\ell_1+\ell_2+\ell_4)^{(n)}_{\ell_3} + (k+\ell_1+\ell_3+\ell_4)^{(n)}_{\ell_2}$$
$$+(k+\ell_2+\ell_3+\ell_4)^{(n)}_{\ell_1}\big] - \big[(k+\ell_1+\ell_2)^{(n)}_{\ell_3} + (k+\ell_1+\ell_2)^{(n)}_{\ell_4} + (k+\ell_1+\ell_3)^{(n)}_{\ell_2}$$
$$+(k+\ell_1+\ell_3)^{(n)}_{\ell_4} + (k+\ell_1+\ell_4)^{(n)}_{\ell_2} + (k+\ell_1+\ell_4)^{(n)}_{\ell_3} + (k+\ell_2+\ell_3)^{(n)}_{\ell_1}$$
$$+(k+\ell_2+\ell_3)^{(n)}_{\ell_4} + (k+\ell_2+\ell_4)^{(n)}_{\ell_1} + (k+\ell_2+\ell_4)^{(n)}_{\ell_3} + (k+\ell_3+\ell_4)^{(n)}_{\ell_1}$$
$$+(k+\ell_3+\ell_4)^{(n)}_{\ell_2}\big] + \big[(k+\ell_1)^{(n)}_{\ell_2} + (k+\ell_1)^{(n)}_{\ell_3} + (k+\ell_1)^{(n)}_{\ell_4} + (k+\ell_2)^{(n)}_{\ell_1}$$
$$+(k+\ell_2)^{(n)}_{\ell_3} + (k+\ell_2)^{(n)}_{\ell_4} + (k+\ell_3)^{(n)}_{\ell_1} + (k+\ell_3)^{(n)}_{\ell_2} + (k+\ell_3)^{(n)}_{\ell_4} + (k+\ell_4)^{(n)}_{\ell_1}$$
$$(k+\ell_4)^{(n)}_{\ell_2} + (k+\ell_4)^{(n)}_{\ell_3}\big] - \big[(k)^{(n)}_{\ell_1} + (k)^{(n)}_{\ell_2} + (k)^{(n)}_{\ell_3} + (k)^{(n)}_{\ell_4}\big]. \tag{9}$$

Definition 2.4. The inverse of the generalized difference operator of the fourth kind denoted by $\Delta^{-1}_{\ell_1,\ell_2,\ell_3,\ell_4}$ is defined as follows. If $\Delta_{\ell_1,\ell_2,\ell_3,\ell_4} z(k) = y(k)$, then

$$z(k) = \Delta^{-1}_{\ell_1,\ell_2,\ell_3,\ell_4} y(k) + c_{3j}\Big(\frac{k^{(3)}_{\ell_3}}{3!\ell_3^3}\Big) + c_{2j}\Big(\frac{k^{(2)}_{\ell_2}}{2\ell_2^2}\Big) + c_{1j}\Big(\frac{k^{(1)}_{\ell_1}}{\ell_1}\Big) + c_{0j}, \tag{10}$$

where c_{0j}, c_{1j}, c_{3j} and c_{4j}'s are constants which depend on $j = k - \big[\frac{k}{\ell}\big]\ell$.

Theorem 2.5. *If $j = k - \big[\frac{k}{\ell}\big]\ell$, then there exits constants c_{0j}, c_{1j}, c_{2j} and c_{3j} such that*

$$\Delta^{-1}_{\ell,\ell,\ell,\ell} u(k) = \sum_{t=2}^{n_t^*}\sum_{s=1}^{n_s^*}\sum_{r=1}^{n_r^*}\sum_{q=0}^{n_q^*} u(k - t\ell - s\ell - r\ell - q\ell)$$
$$+ c_{3j}\left(\frac{k^{(3)}_{\ell}}{3!\ell^3}\right) + c_{2j}\left(\frac{k^{(2)}_{\ell}}{2\ell^2}\right) + c_{1j}\left(\frac{k}{\ell}\right) + c_{0j}, \tag{11}$$

where $n_q^* = \big[\frac{k-t\ell-s\ell-r\ell}{\ell}\big]$, $n_r^* = \big[\frac{k-t\ell-s\ell}{\ell}\big]$, $n_s^* = \big[\frac{k-t\ell}{\ell}\big]$ and $n_t^* = \big[\frac{k}{\ell}\big]$.

Proof. The proof follows by the relation

$$\Delta_{\ell,\ell,\ell,\ell}\Big\{\sum_{t=2}^{n_t^*}\sum_{s=1}^{n_s^*}\sum_{r=1}^{n_r^*}\sum_{q=0}^{n_q^*} u(k-(t+s+r+q)\ell)$$
$$+ c_{3j}\Big(\frac{k^{(3)}_{\ell}}{3!\ell^3}\Big) + c_{2j}\Big(\frac{k^{(2)}_{\ell}}{2\ell^2}\Big) + c_{1j}\Big(\frac{k}{\ell}\Big) + c_{0j}\Big\} = u(k). \quad \square$$

3. Main Results

The following theorem is the discrete version of the generalized Leibnitz Theorem according to $\Delta_{\ell,\ell,\ell,\ell}$

Theorem 3.1. *If $u : [0,\infty) \to \mathbb{C}$ and $v : [0,\infty) \to \mathbb{C}$ are any two functions, then*

$$\Delta^n_{\ell,\ell,\ell,\ell}(u(k)v(k)) = \Delta^n_{\ell,\ell,\ell}(u(k)\Delta^n_\ell v(k)) + nC_1\Delta^n_{\ell,\ell,\ell}(\Delta_\ell u(k)$$
$$\Delta^{n-1}_\ell v(k+\ell)) + nC_2\Delta^n_{\ell,\ell,\ell}(\Delta^2_\ell u(k)\Delta^{n-2}_\ell v(k+2\ell)) + \dots$$
$$+nC_n\Delta^n_{\ell,\ell,\ell}(\Delta^n_\ell u(k)v(k+n\ell)).$$

Proof. The proof follows by the generalized Leibnitz Theorem ([4], Theorem 2.5) and (7). □

Using Stirling numbers of first kind s^n_r , the following can be easily obtained.

Lemma 3.2. *If $m \in \mathbb{N}$, then $\Delta_{\ell,\ell,\ell,\ell}k^n_{\ell,\ell,\ell,\ell} = (n\ell)^{(4)}_\ell k^{(n-4)}_{\ell,\ell,\ell,\ell}$ and hence*

$$\Delta^m_{\ell,\ell,\ell,\ell}k^n_{\ell,\ell,\ell,\ell} = \begin{cases} n!\, 4\ell^n, & \text{if} \quad n = 4m+3; \\ 0, & \text{if} \quad n < 4m+3. \end{cases}$$

Proof. The proof follows by (7) and induction on m. □

The following theorem is the generalized version of Newton's formula with reference to $\Delta_{\ell,\ell,\ell,\ell}$ and $k^{(n)}_{\ell,\ell,\ell,\ell}$.

Theorem 3.3. *Let $f(k)$ be a polynomial of degree $(3n+2)$ in k. Then $f(k)$ can be expressed as*

$$f(k) = f(0) + \frac{\Delta_{\ell,\ell,\ell,\ell}f(0)}{7!4\ell^7}k^{(7)}_{\ell,\ell,\ell,\ell} + \frac{\Delta^2_{\ell,\ell,\ell,\ell}f(0)}{11!4\ell^{11}}k^{(11)}_{\ell,\ell,\ell,\ell} + \cdots$$
$$+ \frac{\Delta^n_{\ell,\ell,\ell,\ell}f(0)}{(4n+3)!4\ell^{(4n+3)}}k^{(4n+3)}_{\ell,\ell,\ell,\ell}. \tag{12}$$

Proof. Assume that

$$f(k) = a_0 + a_1k^{(7)}_{\ell,\ell,\ell,\ell} + a_2k^{(11)}_{\ell,\ell,\ell,\ell} + \cdots + a_nk^{(4n+3)}_{\ell,\ell,\ell,\ell}. \tag{13}$$

Clearly $f(0) = a_0$. The coefficients $a_i, i = 1,2,3,\dots,n$ are determined from the relation

$$\Delta^r_{\ell,\ell,\ell,\ell}f(0) = (4r+3)!\ 4\ell^{4r+3}\ a_r, \quad r > 0. \tag{14}$$

Then the proof follows from (13) and (14). □

Corollary 3.4. *Let $f(k)$ be a polynomial of degree $(4n+3)$ in k. Then*

$f(k-t)$ can be expressed as

$$f(k-t) = f(t) + \frac{\Delta_{\ell,\ell,\ell,\ell} f(t)}{7!4\ell^7}(k-t)^{(7)}_{\ell,\ell,\ell,\ell} + \frac{\Delta^2_{\ell,\ell,\ell,\ell} f(t)}{11!4\ell^{11}}(k-t)^{(11)}_{\ell,\ell,\ell,\ell} + \cdots + \frac{\Delta^n_{\ell,\ell,\ell,\ell} f(t)}{(4n+3)!4\ell^{(4n+3)}}(k-t)^{(4n+3)}_{\ell,\ell,\ell,\ell}.$$

Proof. The proof follows from replacing k by $(k-t)$ and 0 by t in (12). □

Lemma 3.5. *There exist constants $c_{\ell_1}, c_{\ell_2}, c_{\ell_3}$ and c such that*

$$\Delta^{-1}_{\ell_1,\ell_2,\ell_3,\ell_4}\big(k^{(n)}_{\ell_1,\ell_2,\ell_3,\ell_4}\big) = \frac{k^{(n+1)}_{\ell_1}}{\ell_1(n+1)} + \frac{k^{(n+1)}_{\ell_2}}{\ell_2(n+1)} + \frac{k^{(n+1)}_{\ell_3}}{\ell_3(n+1)} + \frac{k^{(n+1)}_{\ell_4}}{\ell_4(n+1)} + c_{3j}\left(\frac{k^{(3)}_{\ell_3}}{6\ell_3^3}\right) + c_{2j}\left(\frac{k^{(2)}_{\ell_2}}{2\ell_2^2}\right) + c_{1j}\left(\frac{k^{(1)}_{\ell_1}}{\ell_1}\right) + c_{0j} \quad (15)$$

and

$$\Delta^{-1}_{\ell,\ell,\ell,\ell} k^{(n)}_{\ell,\ell,\ell,\ell} = 4\,\frac{k^{(n+1)}_{\ell}}{\ell(n+1)} + c. \quad (16)$$

Proof. The proof follows from (9) and $\Delta_{\ell,\ell,\ell,\ell}\Big(\frac{4\ k^{(n+1)}_{\ell}}{\ell(n+1)} + c\Big) = k^{(n)}_{\ell,\ell,\ell,\ell}$. □

4. Applications

The following theorem gives a general rule to find the sum of the third partial sums of the products of n consecutive terms of an arithmetic progression.

Theorem 4.1. *If $j \geq (n-1)\ell$, then*

$$\sum_{t=2}^{n_t^*}\sum_{s=1}^{n_s^*}\sum_{r=1}^{n_r^*}\sum_{q=0}^{n_q^*}(k-t\ell-s\ell-r\ell-q\ell)^{(n)}_{\ell} = \frac{k^{(n+4)}_{\ell}}{\ell^4\prod_{i=1}^{4}(n+i)} - \left\{c_{3j}\left(\frac{k^{(3)}_{\ell}}{6\ell^3}\right) + c_{2j}\left(\frac{k^{(2)}_{\ell}}{2\ell^2}\right) + c_{1j}\left(\frac{k}{\ell}\right) + c_{0j}\right\}, \quad (17)$$

where $n_t^*, n_s^*, n_r^*, n_q^*$ are taken as in Theorem 2.5 and the constants c_{0j}, c_{1j}, c_{2j}, c_{3j}'s are obtained by solving the four simultaneous equations which are obtained by substituting $k = (n+a)\ell + j$, for $a = 3, 4, 5, 6$ in (17).

Proof. We have

$$\Delta^{-1}_{\ell,\ell,\ell,\ell}k^{(n)}_{\ell} = \sum_{t=2}^{n^*_t}\sum_{s=1}^{n^*_s}\sum_{r=1}^{n^*_r}\sum_{q=0}^{n^*_q}(k-t\ell-s\ell-r\ell-q\ell)^{(n)}_{\ell} + c_{3j}\left(\frac{k^{(3)}_{\ell}}{6\ell^3}\right)$$
$$+c_{2j}\left(\frac{k^{(2)}_{\ell}}{2\ell^2}\right)+c_{1j}\left(\frac{k}{\ell}\right)+c_{0j} = \frac{k^{(n+4)}_{\ell}}{(n+1)(n+2)(n+3)(n+4)\ \ell^4}. \quad (18)$$

Now, the theorem follows by substituting $k=(n+a)\ell+j$, for $a=3,4,5,6$ in (18) and solving the system of four equations. □

The following example is to the illustrate of the Theorem 4.1.

Example 4.2. For
$C_n = (3)(6)(9)(12) + (6)(9)(12)(15) + (9)(12)(15)(18) + \cdots + (39)(42)(45)(48)$, we shall find PC_n, P^2C_n and P^3C_n.

Solution. We present the procedure for finding P^3C_n, since P^2C_n, PC_n and C_n can be determined using $\Delta_{\ell,\ell,\ell}, \Delta_{\ell,\ell}$ and Δ_ℓ respectively which are discussed in [4], [9], [11]. We have

$$P^3C_n = \big[(3)(6)(9)(12)\big] + \big[\overline{(3)(6)(9)(12) + \big((3)(6)(9)(12)}$$
$$+\overline{\overline{(3)(6)(9)(12) + (6)(9)(12)(15)}}\big)\big] + ... + \big[\overline{(3)(6)(9)(12) + (3)(6)(9)(12)}$$
$$\overline{+\overline{(3)(6)(9)(12) + (6)(9)(12)(15)}} + \cdots + \big[\overline{\big((3)(6)(9)(12)}$$
$$\overline{\overline{((3)(6)(9)(12) + (6)(9)(12)(15)) + \cdots +}}$$
$$\overline{\big(\overline{(3)(6)(9)(12) + (6)(9)(12)(15) + \cdots + (39)(42)(45)(48)}\big)\big)\big]}.$$

Here $n=4, \ell=3, i=12, k=60$

$$P^3C_n = (60)^{(8)}_3 - (33)^{(8)}_3 + \sum_{t=2}^{n^*}\sum_{s=1}^{n^*}\sum_{r=1}^{n^*}\sum_{q=0}^{n^*}(33-3t-3s-3r-3q)^{(4)}_3$$
$$\big(7+\frac{12-60}{3}\big)\Big\{\frac{(36)^{(8)}_3-(33)^{(8)}_3}{3^{(4)}\ 1680} - \sum_{t=2}^{n^*}\sum_{s=1}^{n^*}\sum_{r=1}^{n^*}(36-3t-3s-3r)^{(4)}_3\Big\}$$
$$+\frac{1}{2(3)^2}\left[(33)^{(2)}_3 + \left(7+\frac{12-60}{3}\right)[(33)^{(2)}_3 - (36)^{(2)}_3] - (60)^{(2)}_3\right]$$
$$\Big\{\frac{(39)^{(8)}_3-(33)^{(8)}_3}{3^{(4)}\ 1680} + (-2)\left(\frac{(36)^{(8)}_3-(33)^{(8)}_3}{3^{(4)}\ 1680}\right)$$
$$-\sum_{t=2}^{n^*}\sum_{s=1}^{n^*}(36-3t-3s)^{(4)}_3\Big\} + \frac{1}{6(3)^3}\Big\{(33)^{(3)}_3 + \left(7+\frac{12-60}{3}\right)$$

$$[(33)_3^{(3)}-(36)_3^{(3)}]+\frac{1}{2(3)^2}\Big\{\Big[(33)_3^{(2)}+\Big(7+\frac{12-60}{3}\Big)$$

$$[(33)_3^{(2)}-(36)_3^{(2)}]-(60)_3^{(2)}\Big][(33)_3^{(3)}-2[(33)_3^{(3)}-(36)_3^{(3)}]$$

$$-(39)_3^{(3)}]\Big\}-(60)_3^{(3)}\Big\}\Big\{\frac{(42)_3^{(8)}-(33)_3^{(8)}}{3^{(4)}\ 1680}+(-3)\Big(\frac{(36)_3^{(8)}-(33)_3^{(8)}}{3^{(4)}\ 1680}\Big)$$

$$+(-3)\Big\{\frac{(39)_3^{(8)}-(33)_3^{(8)}}{3^{(4)}\ 1680}+(-2)\Big(\frac{(36)_3^{(8)}-(33)_3^{(8)}}{3^{(4)}\ 1680}\Big)\Big\}-\sum_{t=2}^{n^*}(36-3t)_3^{(4)}\Big\}$$

$= 244885680.$

Lemma 4.3. *If a is a nonzero number, then*

$$\Delta^{-1}_{\ell,\ell,\ell,\ell}\left(ka^k\right)=\frac{a^k}{(a^\ell-1)^4}\Big[k-\frac{4\ell a^\ell}{a^\ell-1}\Big]+\frac{c_{3j}}{6\ell^3}k_\ell^{(3)}+\frac{c_{2j}}{2\ell^2}k_\ell^{(2)}+\frac{c_{1j}}{\ell}k+c_{0j}. \quad (19)$$

Proof. The proof follows from

$$\Delta_{\ell,\ell,\ell,\ell}\Big\{\frac{a^k}{(a^\ell-1)^4}\big[k-\frac{4\ell a^\ell}{(a^\ell-1)}\big]+\frac{c_{3j}}{6\ell^3}k_\ell^{(3)}+\frac{c_{2j}}{2\ell^2}k_\ell^{(2)}+\frac{c_{1j}}{\ell}k+c_{0j}\Big\}=ka^k. \quad \square$$

The following theorem is to find the formula for the sum of the third partial sums of an arithmetic-geometric progression.

Theorem 4.4. *If $k>4\ell$ and $k-n^*\ell>0$, then*

$$\sum_{t=2}^{n_t^*}\sum_{s=1}^{n_s^*}\sum_{r=1}^{n_r^*}\sum_{q=0}^{n_q^*}(k-t\ell-s\ell-r\ell-q\ell)a^{(k-t\ell-s\ell-r\ell-q\ell)}$$
$$=\frac{a^k}{(a^\ell-1)^4}\left(k-\frac{4\ell a^\ell}{(a^\ell-1)}\right)-\Big\{\frac{c_{3j}}{3!\ell^3}k_\ell^{(3)}+\frac{c_{2j}}{2\ell^2}k_\ell^{(2)}+\frac{c_{1j}}{\ell}k+c_{0j}\Big\}, \quad (20)$$

where $n_t^*, n_s^*, n_r^*, n_q^*$ are taken as in Theorem 2.5 and the constants c_{0j}, c_{1j}, c_{2j}, c_{3j}'s are obtained by solving the four simultaneous equations which are obtained by substituting $k=(n+a)\ell+j$, for $a=3,4,5,6$ in (20).

Proof. From (7) and (19), we get

$$\sum_{t=2}^{n_t^*}\sum_{s=1}^{n_s^*}\sum_{r=1}^{n_r^*}\sum_{q=0}^{n_q^*}(k-t\ell-s\ell-r\ell-q\ell)a^{(k-t\ell-s\ell-r\ell-q\ell)}+\frac{c_{3j}}{3!\ell^3}k_\ell^{(3)}$$
$$+\frac{c_{2j}}{2\ell^2}k_\ell^{(2)}+\frac{c_{1j}}{\ell}k+c_{0j}=\frac{a^k}{(a^\ell-1)^4}\left(k-\frac{4\ell a^\ell}{(a^\ell-1)}\right). \quad (21)$$

Now, the theorem follows by substituting $k=(n+a)\ell+j$, for $a=3,4,5,6$ in

(21) and solving the system of four equations. □

The following example is to illustrate the result of Theorem 4.4.

Example 4.5. For

$$nS^n = (1)2^1 + (4)2^4 + \cdots + (31)2^{31} + (34)2^{34},$$

we shall find the values of PnS^n, P^2nS^n and P^3nS^n.

Solution. As in Example 4.2, we present the procedure for finding P^3nS^n, since nS^n, PnS^n and P^2nS^n can be determined by referring [4], [9], [10], [11]. We have

$$P^3nS^n = (1)2^1 + \Big((1)2^1 + [(1)2^1 + \overline{(1)2^1 + (4)2^4}]\Big) + \cdots$$
$$+ \Big((1)2^1 + [(1)2^1 + \overline{(1)2^1 + (4)2^4}] + \cdots + [(1)2^1 + \overline{(1)2^1 + (4)2^4} + \cdots$$
$$+ \overline{(1)2^1 + (4)2^4 + \cdots + (34)2^{34}}]\Big).$$

Here $i = 1, a = 2, \ell = 3, k = 46$.

$$P^3nS^n = \frac{1}{(2^3-1)^4}\left[(46)(2)^{46} - (7)2^7\right] - \frac{(4)(3)2^3}{(2^3-1)^5}\left[2^{46} - 2^7\right]$$
$$+(2 + \frac{1-46}{3})\Big\{\frac{1}{(2^3-1)^4}\left[(10)2^{10} - (7)2^7\right] - \frac{(4)(3)2^3}{(2^3-1)^5}\left[2^{10} - 2^7\right]\Big\}$$
$$+\frac{1}{2!(3)^2}\left[(7)_3^{(2)} + (-13)[(7)_3^{(2)} - (10)_3^{(2)}] - (46)_3^{(2)}\right]$$
$$\Big\{\frac{1}{(2^3-1)^4}\left[(13)2^{13} - (7)2^7\right] - \frac{(4)(3)2^3}{(2^3-1)^5}\left[2^{13} - 2^7\right]$$
$$+(-2)\left(\frac{1}{(2^3-1)^4}\left[(10)2^{10} - (7)2^7\right] - \frac{(4)(3)2^3}{(2^3-1)^5}\left[2^{10} - 2^7\right]\right) - 2\Big\}$$
$$+\frac{1}{3!(3)^3}\Big\{(7)_3^{(3)} + (-13)[(7)_3^{(3)} - (10)_3^{(3)}] +$$
$$\frac{1}{2!(3)^2}[(7)_3^{(2)} + (-13)[(7)_3^{(2)} - (10)_3^{(2)}] - (46)_3^{(2)}]$$
$$([(7)_3^{(3)} - (13)_3^{(3)}] + (-2)[(7)_3^{(3)} - (10)_3^{(3)}]) - (46)_3^{(3)}\Big\}$$
$$\Big\{\frac{1}{(2^3-1)^4}\left[(16)2^{16} - (7)2^7\right] - \frac{(4)(3)2^3}{(2^3-1)^5}\left[2^{16} - 2^7\right] + (-3)$$

$$\{\frac{1}{(2^3-1)^4}\left[(10)2^{10}-(7)2^7\right]-\frac{(4)(3)2^3}{(2^3-1)^5}\left[2^{10}-2^7\right]\}+(-3)$$

$$\Big\{\frac{1}{(2^3-1)^4}\left[(13)2^{13}-(7)2^7\right]-\frac{(12)2^3}{(2^3-1)^5}\left[2^{13}-2^7\right]+(-2)(\frac{1}{(2^3-1)^4}$$

$$\left[(10)2^{10}-(7)2^7\right]-\frac{(4)(3)2^3}{(2^3-1)^5}\left[2^{10}-2^7\right])\Big\}-\left[(4)2^4+(1)2^1\right]\Big\}$$

$$P^3nS^n = 946232890300.$$

Acknowledgments

This research is supported by University Grants Commission, New Delhi.

References

[1] R.P. Agarwal, *Difference Equations and Inequalities*, Marcel Dekker, New York (2000).

[2] Saber N. Elaydi, *An Introduction to Difference Equations*, Second Edition, Springer (1999).

[3] Walter G. Kelley, Allan C. Peterson, *Difference Equations, An Introduction with Applications*, Academic Press (1991).

[4] M. Maria Susai Manuel, G. Britto Antony Xavier, E. Thandapani, Theory of generalized difference operator and its applications, *Far East Journal of Mathematical Sciences*, **20**, No. 2 (2006), 163-171.

[5] M. Maria Susai Manuel, G. Britto Antony Xavier, E. Thandapani, Qualitative properties of solutions of certain class of difference equations, *Far East Journal of Mathematical Sciences*, **23**, No. 3 (2006), 295-304.

[6] M. Maria Susai Manuel, G. Britto Antony Xavier, E. Thandapani, Generalized Bernoulli polynomials through weighted Pochhammer symbols, *Far East Journal of Applied Mathematics*, **26**, No. 3 (2007), 321-333.

[7] M. Maria Susai Manuel, A. George Maria Selvam, G. Britto Antony Xavier, Rotatory and boundedness of solutions of certain class of difference equations, *International Journal of Pure and Applied Mathematics*, **33**, No. 3 (2006), 333-343.

[8] M. Maria Susai Manuel, G. Britto Antony Xavier, Recessive, dominant and spiral behaviours of solutions of certain class of generalized difference equations, *International Journal of Differential Equations and Applications*, **10**, No. 4 (2007), 423-433.

[9] M. Maria Susai Manuel, G. Britto Antony Xavier, V. Chandrasekar, Generalized difference operator of the second kind and its application to number theory, *International Journal of Pure and Applied Mathematics*, **47**, No. 1 (2008), 127-140.

[10] M. Maria Susai Manuel, G. Britto Antony Xavier, V. Chandrasekar, R. Pugalarasu, S. Elizabeth, On generalized difference operator of third kind and its applications in number theory, *International Journal of Pure and Applied Mathematics*, **53**, No. 1 (2009), 69-81.

[11] M. Maria Susai Manuel, G.B. Antony Xavier, V. Chandrasekar, R. Pugalarasu, S. Elizabeth, Generalized difference operator of third kind and on second partial sums of products of consecutive terms of arithmetic and arithmetic-geometric progressions, *International Journal of Pure and Applied Mathematics*, **53**, No. 1 (2009), 109-120.

[12] Ronald E. Mickens, *Difference Equations*, Van Nostrand Reinhold Company, New York (1990).