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NUMERICAL ANALYSIS FOR ESTIMATING PERMEABILITY IN REAL TISSUE

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Abstract: The purpose of this paper is to present efficient algorithm for estimation of parameter in real tissue. Some equation of liver-perfusion image are proposed. C. Narazawa et al [5] derive mathematical modeling from compartment model and present the perfusion pattern well. They [5] also studied some algorithms for estimating the permeability in real tissue by using the optimization. In this paper, we will analysis the equation of estimation in real tissue and present algebraic algorithms for estimation of parameter in real tissue. The numerical tests justifying the results are presented.

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Key Words: permeability, liver perfusion, compartmental model, mathematical modeling, numerical methods, Newton-Raphson method, inverse problem

1. Introduction

For the last year, many people present formulate of blood flow in liver to analyze liver perfusion. CT system is greatly improved and Hepatic perfusion parameters measured with CT were significantly altered. In 1987, S.W. Atlas et al proposed a method for determine tissue perfusion using CT data. The methods is based on the use of a dual-input single -compartment model of liver circulation. K. Miles et al [3] proposed mathematical modeling (ODE) of blood flows between hepatic vessels by calculating inflow at arterial and portal veins. R. Materne et al [2] study compartmental model where the arterial and portal component are separated and present mathematical modeling given by:

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$$\frac{dC_L(t)}{dt}(t) = k_{1a}C_a(t) + k_{1p}C_p(t) - k_2C_L(t),$$

where $C_a(t), C_p(t)$ and $C_L(t)$ are the concentration from the aorta, portal vein and liver compartments respectively, k_{ia} is the aortic inflow rate constant, k_{1p} the portal venous inflow rate constant, k_2 the outflow rate constant. Y. Nazawa et al [5] study permeability of liver by minimizing the difference between values obtained from measurements and values obtained as the solution of the expression. They [5] present mathematical modeling given by

$$\frac{dC(t)}{dt}(t) + \frac{Pm_2}{(1-rbv)}\,C(t) = rbv\,\frac{dC_{cap}(t)}{dt} + (Pm_1 + \frac{rbv}{1-rbv}Pm_2)\,C_{cap}(t). \quad (1.1)$$

P_{m1} is permeability from inside to outside of blood vessel, $P_{m2}(t)$ from outside to inside, $C_{cap}(t)$ is the concentration of radio contrast inside, $C_{ecf}(t)$ is the concentration of radio contrast outside, rv is the rate of inside and outside blood vessel.

We study the mathematical modeling given by

$$\frac{dC(t)}{dt}(t) + \frac{Pm_2}{(1-rbv)}\,C(t) = rbv\,\frac{dC_{cap}(t-\tau)}{dt} + (Pm_1 + \frac{rbv}{1-rbv}Pm_2)\,C_{cap}(t-\tau), \quad (1.2)$$

where τ is the transit time from the aorta to the liver.

We set

$$C(t) = x(t), \quad C_{cap}(t) = g(t), \quad a = -\frac{Pm_2}{(1-rbv)}, \quad b = rbv,$$

$$c = (Pm_1 + \frac{rbv}{1-rbv}Pm_2). \quad (1.3)$$

Then, from (1.2), we have

$$\frac{dx}{dt}(t) - a\,x(t) = b\,\frac{dg}{dt}(t-\tau) + c\,g(t-\tau). \quad (1.4)$$

2. Solution of Differential Equation

In this section, we solve ODE (1.4) by Laplace transform. Set Laplace transform of the function $f(t)$ by

$$L[f(t)] = F(s). \quad (2.1)$$

We have some formula;

$$L[\frac{dx}{dt}(t)] = \int_0^\infty e^{-st}\frac{dx}{dt}(t) = [e^{st}x(t)]_0^\infty + \int_0^\infty se^{-st}x(t)dt$$

$$= -x(0) + sX(s)L[\frac{d}{dt}g(t-\tau)] = \int_0^\infty e^{-st}(\frac{d}{dt}g(t-\tau))dt. \quad (2.2)$$

Taking $T = t - \tau$, we have

$$\int_0^\infty e^{-st}(g_t(t-\tau))dt = e^{-s\tau}\{\int_{-\tau}^0 e^{-sT}(g_t(T))dT + \int_0^\infty e^{-sT}(g_t(T))dT\}$$

$$= e^{-s\tau}\{G_t(s) + h_1(\tau)\}, \quad (2.3)$$

where

$$g_t(t) = \frac{d}{dt}g(t), G_t(s) = L[\frac{d}{dt}g(t)], h_1(\tau) = \int_{-\tau}^0 e^{-st}g_v(t)dt. \quad (2.4)$$

Then we have

$$L[g_v(t-\tau)] = e^{-s\tau}\{G_v(s) + h_1(\tau)\} \quad (2.5)$$

and similarly to (2.5), we have

$$L[g(t-\tau)] = e^{-s\tau}\{G(s) + h_2(\tau)\}, \quad (2.6)$$

where

$$h_2(t) = \int_{-\tau}^0 e^{-st}g(t)dt. \quad (2.7)$$

We take Laplace transform of (1.4), from (2.5), (2.6), we have

$$X(s) = \frac{1}{s-a}[be^{-s\tau}\{G(s) + h_1(\tau)\} + ce^{-s\tau}\{G(s) + h_2(\tau)\}$$

$$-bg(-\tau) + x(0)]. \quad (2.8)$$

We define Laplace inverse as follows;

$$L^{-1}[L[f(t)]] = f(t).$$

We take Laplace inverse of (2.8) and have the solution of (1.4);

$$x(t) = be^{a(t-\tau)}\int_0^{t-\tau} e^{-a\sigma}g_v(\sigma)d\sigma + ce^{a(t-\tau)}\int_0^{t-\tau} e^{-a\sigma}g(\sigma)d\sigma$$

$$+(bh_1(\tau) + ch_2(\tau))e^{a(t-\tau)} + \{x(0) - bg(-\tau)\}e^{at},$$

where $h_1(\tau), h_2(\tau)$ are defined by (2.4) and (2.7), respectively, or

$$x(t) = be^{a(t-\tau)}\int_{-\tau}^{t-\tau} e^{-a\sigma}g_v(\sigma)d\sigma + ce^{a(t-\tau)}\int_{-\tau}^{t-\tau} e^{-a\sigma}g(\sigma)d\sigma$$

$$+\{x(0) - bg(-\tau)\}e^{at}. \quad (2.9)$$

We set the derivative of $x(t)$ by

$$v(t) = \frac{d}{dt}x(t). \tag{2.10}$$

We have transform of $V(s) = L[v(t)]$;

$$V(s) = -x(0) + sX(s). \tag{2.11}$$

From (2.8) and (2.11), we have

$$V(s) = a\{\frac{1}{s}(V(s) + x(0))\} + be^{-s\tau}\{G_v(s) + h_1(\tau)\} + ce^{-s\tau}\{G(s) + h_2(\tau)\} - bg(-\tau),$$

or

$$V = b\{1 + \frac{a}{s-a}\}e^{-s\tau}G_v(s) + c\{1 + \frac{a}{s-a}\}e^{-s\tau}G(s) + b\{1 + \frac{a}{s-a}\}e^{-s\tau}h_1(\tau) + c\{1 + \frac{a}{s-a}\}e^{-s\tau}h_2(\tau) + \frac{a}{s-a}x(0) - \{1 + \frac{a}{s-a}\}bg(-\tau), \tag{2.12}$$

where $h_1(\tau), h_2(\tau)$ are defined by (2.4) and (2.7).

Taking Laplace inverse of (2.12), we have

$$v(t) = b\{g_v(t-\tau) + ae^{a(t-\tau)}\int_0^{t-\tau} e^{-a\sigma}g_v(\sigma)d\sigma\} + c\{g(t-\tau) + ae^{a(t-\tau)} \times \int_0^{t-\tau} e^{-a\sigma}g(\sigma)d\sigma\} + a(bh_1(\tau) + ch_2(\tau))e^{a(t-\tau)} + av(0)e^{at} - b(\delta(t) + ae^{at})g(-\tau)$$

or

$$v(t) = b\{g_v(t-\tau) + ae^{a(t-\tau)}\int_{-\tau}^{t-\tau} e^{-a\sigma}g_v(\sigma)d\sigma\} + c\{g(t-\tau) + ae^{a(t-\tau)}\int_{-\tau}^{t-\tau} e^{-a\sigma}g(\sigma)d\sigma\} + a\{v(0) - b(\delta(t) + g(-\tau))\}e^{at}. \tag{2.13}$$

3. Numerical Method

We will present the numerical example of inverse problem. We assume that the solutions $x(t)$ of (1.4) and $v(t)$ of (2.13) are given. We set $c = c(t, a)$ in (1.3), Then, from (2.9) and (2.13), we have

$$c(t, a) = \frac{(v(t) - bg_v(t) - ax(t))}{g(t-\tau)}. \tag{3.1}$$

We approximate the integration (2.9) by the trapezoidal rule;

$$\int_{t_1}^{t_2} f(t)dt \cong \sum_{k=0}^{n} hf(t_1 + kh), \tag{3.2}$$

with $h = (t_2 - t_1)/n$.

Applying (3.2) to (2.9), we have numerical approximation $x_n(t)$ to $x(t)$:

$$x_n(t) = be^{a(t-\tau))} \sum_{k=0}^{n} e^{-akh} g_v(kh) + c(t,a)e^{a(t-\tau))} \sum_{k=0}^{n} e^{-akh} g(kh) + \{x(0) - bg(-\tau)\}e^{at}, \tag{3.3}$$

where $c(t,a)$ is given by (3.1) and

$$h = \frac{(t-\tau)}{n},$$

$$\tilde{g}(t) = \begin{cases} g(t) & (g(t) \neq 0), \\ k & (g(t) = 0,\ k \neq 0,\ \text{constant.}) \end{cases} \tag{3.4}$$

We study the scheme (3.3) with the example;

$$g(t) = \begin{cases} \alpha \sin(\beta t + \gamma) & (\frac{\pi}{15} \leq t \leq \frac{\pi}{2}), \\ 0. & (\text{else}). \end{cases} \tag{3.5}$$

If we set constant $\alpha, \beta, \gamma, rbv, \tau, Pm_1, Pm_2$ in (1.2) as $\alpha = 1.5, \beta = 1, \gamma = \pi/6, rbv = 0.01,\ \tau = 0.5$. $Pm_1 = 0.01,\ Pm_2 = 1.8$. Then, from (1.3), we have

$$a = -1.81818,\ b = 0.01,\ c = 0.0281818. \tag{3.6}$$

From (2.9), we have

$$x(t) = be^{a(t-\tau)}(f_1(t-\tau) - f_1(-\tau)) + ce^{a(t-\tau)}(f_2(t-\tau) - f_2(-\tau)) + \{x(0) - bg(-\tau)\}e^{at}, \tag{3.7}$$

with

$$f_1(t) = (\frac{\beta}{\alpha^2 + \beta^2})e^{-\alpha t}\{\frac{-\alpha}{\beta} \sin(\beta t + \gamma) - \cos(\beta t + \gamma)\},$$

$$f_2(t) = (\frac{\beta}{\alpha^2 + \beta^2})e^{-\alpha t}\{\sin(\beta t + \gamma) - \frac{\alpha}{\beta} \cos(\beta t + \gamma))\}.$$

The function $g(t)$ and $x(t)$ are given in Figure 1 and Figure 2, respectively.

We set parameter a in (3.3) as variable and put

$$x(t) = x(t,a), x_n(t) = x_n(t,a). \tag{3.8}$$

We set the function,

$$h_n(t,p) = x(t,a) - x_n(t,p). \tag{3.9}$$

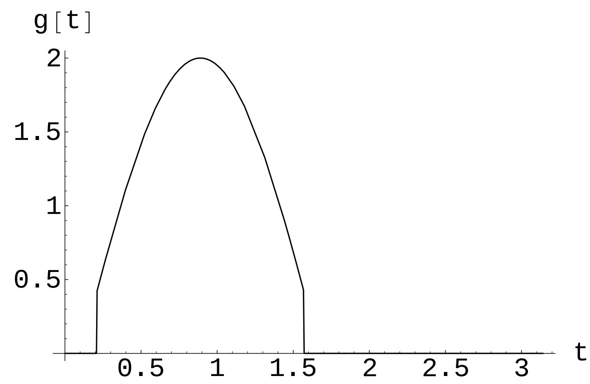

Figure 1:

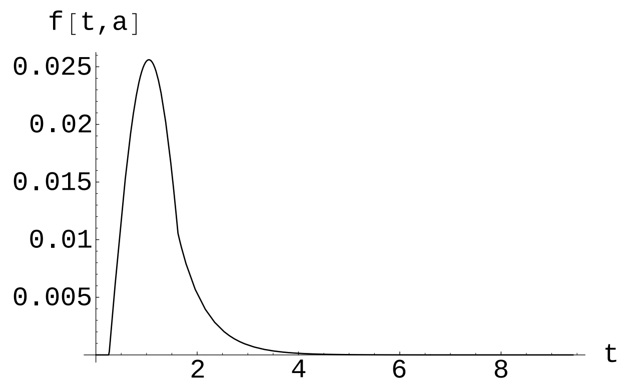

Figure 2:

From the numerical integration of (3.2), we have

$$h_n(t,a) = x(t,a) - x_n(t,a) = O(h^2).$$

The solution p of equation

$$h_n(t_1, p) = 0 \tag{3.10}$$

is approximation to a at $t = t_1$ in (2.9).

We consider the example $n = 2, 4, 10, t = \frac{\pi}{6}$ in the integration (3.3) and we have

$$h_2(\frac{\pi}{6}, x) = 0.01284 - 0.00596e^{(-0.457+\pi/6)x} - 0.00593e^{(-0.325+\pi/6)x}$$
$$+0.00143xe^{(-0.457+\pi/6)x} + 0.00084xe^{(-0.325+\pi/6)x} \tag{3.11}$$

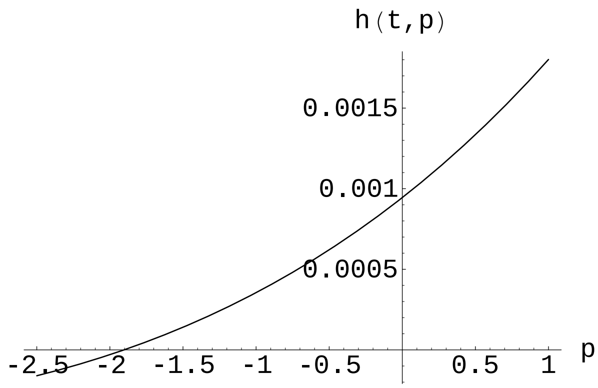

Figure 3:

which is shown in Figure 3.

We solve equation $h_2(t,x)=0$ by Newton-Raphson method

$$x_{l+1}(t)=x_l(t)-\frac{h_2(t,x_l)}{h_2'(t,x_l)} \qquad (l=1,2,3,...), \tag{3.12}$$

where $h'(t)$ is the derivative of $h(t)$. We set the solution of (3.10) by $x_l(t)=\alpha$ if

$$|x_{l+1}(t)-x_l(t)|\le e, \tag{3.13}$$

where e is preassigned.

If we set $x_1(t)=1, e=10^{-3}$ in (3.13), we have

$$\alpha=-1.903, \quad l=4. \tag{3.14}$$

The case $t=\pi/6, n=4$, we have

$$h_4(\pi/6,x)=0.0128-0.0029e^{(-0.490+\pi/6)x}-0.0030e^{(-0.424+\pi/6)x}$$
$$-0.0029e^{(-0.358+\pi/6)x}-0.0029e^{(-0.292+\pi/6)x}+0.0007xe^{(-0.490+\pi/6)x}$$
$$+0.0006xe^{(-0.424+\pi/6)x}+0.0005xe^{(-0.358+\pi/6)x}+0.0003xe^{(-0.292+\pi/6)x} \tag{3.15}$$

which is shown in Figure 4.

In the same way as in (3.14), we have the algebraic solution of (3.15) with $e=10^{-3}$,

$$\alpha=-1.836, \quad l=4, \tag{3.16}$$

and for $n=10$, we have

$$h_{10}(\frac{\pi}{6},x)=0.012849-0.00117e^{(-0.5103+\pi/6)x}-0.00118e^{(-0.4839+\pi/6)x}$$

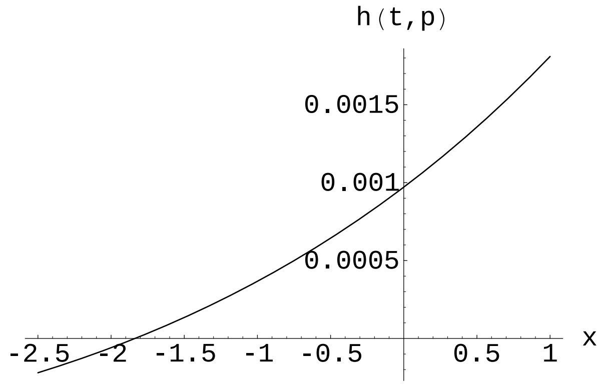

Figure 4:

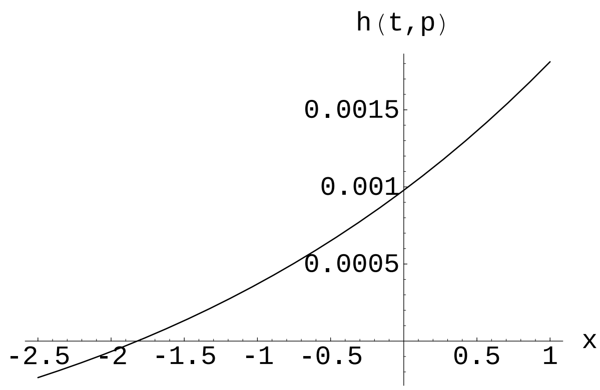

Figure 5:

$$
\begin{aligned}
&-0.00119e^{(-0.4575+\pi/6)x} - 0.00119e^{(-0.4311\pi/6)x} - 0.00120e^{(-0.4047+\pi/6)x} \\
&-0.00119e^{(-0.3783\pi/6)x} - 0.00119e^{(-0.3518\pi/6)x} - 0.00118e^{(-0.3254\pi/6)x} \\
&-0.00117e^{(-0.2990\pi/6)x} - 0.00116e^{(-0.2726\pi/6)x} + 0.0003xe^{(-0.5103\pi/6)x} \\
&+0.00030xe^{(-0.4839\pi/6)x} + 0.00028xe^{(-0.4575\pi/6)x} + 0.00026xe^{(-0.4311\pi/6)x} \\
&+0.00024xe^{(-0.4047\pi/6)x} + 0.00021xe^{(-0.3783\pi/6)x} + 0.00019xe^{(-0.3518\pi/6)x} \\
&+0.00016xe^{(-0.3254\pi/6)x} + 0.00014xe^{(-0.2990\pi/6)x} + 0.00011xe^{(-0.2726\pi/6)x} \quad (3.17)
\end{aligned}
$$

which is shown in Figure 5.

In the same way as in (3.14), we have the algebraic solution of (3.17) with

$e = 10^{-3}$,

$$\alpha = -1.8206, \quad l = 4.$$

We have the approximation results of $h_n(t,x) = 0$ with $es = 10^{-3}$ for the time, $t = (1/6)\pi, (7/30)\pi, (11/30)\pi, (1/3)\pi$ in Table 1.

	$h_n(t,a)$	$h_2(t,x)$	$h_4(t,x)$	$h_{10}(t,x)$
t	$(1/6)\pi$	$-1.903(l=4)$	$-1.836(l=4)$	$-1.820(l=4)$
t	$(7/30)\pi$	$-1.925(l=5)$	$-1.833(l=5)$	$-1.819(l=4)$
t	$(11/30)\pi$	$-1.280(l=5)$	$-1.679(l=5)$	$-1.795(l=5)$
t	$(1/2)\pi$	$-1.390(l=6)$	$-1.675(l=6)$	$-1.792(l=6)$

Table 1: l is the iteration number in (3.12)

We know that, if we take n in the integration to large in (3.12), the solution of (3.9) converge to the exact solution $p = a$ for each time. The coefficient $Pm_1(x,t), Pm_2(x,t)$ are given in the following results. Comparing the data of

x	-1.792	-1.795	-1.819	-1.820
Pm_1	$1.02E-02$	$1.02E-02$	$0.9E-02$	$9.9E-03$
Pm_2	$1.774E+00$	$1.777E+00$	$1.800E+00$	$1.801E+00$

Table 2:

exact solution (3.6) with that of Table 2, it is shown that the numerical solution of $h_n(t,x) = 0$ is enough to the estimation of the permeability.

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