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MIXED T-, T*-CONVEX FUNCTIONS AND THEIR CORRESPONDING MIXED HESSIAN MATRICES

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Abstract: Many minimization problems include functions with integer variables or a combination of integer and real variables.

L and $L^{\natural}$-convex functions are integer functions with the property that any local minimum is a global minimum. In this paper, the definition of these functions is extended to include real variables in the domain. Mixed Hessian matrices are defined for T and T*-convex functions with properties similar to those of the Hessian matrix for real variables.

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1. Introduction

There are many definitions of convexity for functions defined on a discrete set: M-convex and L-convex functions by Murota [6] and [7], $M^{\natural}$-convex functions by Murota and Shioura [8], $L^{\natural}$-convex functions by Fujishige and Murota [2], discretely convex functions by Miller [4], strong discrete convex functions by Yuceer [10], integrally convex functions by Favati and Tardella [1], and D-convex functions by Ui [9]. Ui also shows that integrally, discretely, M, L, $M^{\natural}$, and $L^{\natural}$-convex functions are special cases of D-convex functions. One result is

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that these unified forms of convex functions have the property that every local minimum is a global minimum and vice versa. Integer or discretely convex functions can also be characterized by properties of their Hessian matrices. Hessian matrices are defined for strong discrete convex functions by Yuceer [10], for L-convex functions by Moriguchi and Murota [5], and for M-convex functions by Hirai and Murota [3]. This paper extends the definition of $L^{\natural}$ and L convex functions to include continuous variables and defines a Hessian matrix that gives similar results.

The discrete Hessian matrix is symmetric in a local neighborhood and vanishes when the function is affine. It is also known that the Hessian matrix of a discrete function corresponds to the coefficient matrix of the function that it represents if the function is a second degree polynomial with respect to its integer variables [5].

Let $\mathbb{Z}^n$ denote n-dimensional integer space, where $n \in \mathbb{N}$ ($\mathbb{N}$ denote the natural numbers) and $\mathbb{R}^m$ denote m-dimensional Euclidean space.

A function $\Psi : \mathbb{R}^m \to \mathbb{R}$ is convex if

$$\Psi(\lambda x + (1-\lambda)y) \leq \lambda\Psi(x) + (1-\lambda)\Psi(y)$$

for $\forall\ x, y \in \mathbb{R}^m$ and $0 \leq \lambda \leq 1$. One way of verifying the convexity of a C^2 function is to show that the corresponding Hessian matrix at each point is positive definite.

Moriguchi and Murota [5] constructed matrices that represent second degree L-convex and $L^{\natural}$-convex polynomial functions. In this paper, we construct Hessian matrices to represent functions that are defined on a combination of integer and real spaces which satisfy the L-convexity and $L^{\natural}$-convexity properties of second degree polynomials with respect to the integer variables and convexity with respect to the continuous variables.

The first difference of a discrete function $\Phi : \mathbb{Z}^n \to \mathbb{R}$ is defined to be

$$\nabla_j(\Phi(\psi,\xi)) = \Phi(\psi + e_j, \xi) - \Phi(\psi, \xi),$$

where e_j is a unit vector and the second difference is defined to be

$$\nabla_{ij}(\Phi(\psi,\xi)) = \Phi(\psi + e_i + e_j, \xi) - \Phi(\psi + e_i, \xi) - \Phi(\psi + e_j, \xi) + \Phi(\psi, \xi).$$

All definitions and applications below are based on the local neighborhood of a point.

2. Mixed T-, T*- convex functions

Definition 2.1. A function $\Xi : \mathbb{Z}^n \to \mathbb{R} \cup \{\infty\}$ is called L-convex [4] if for any $\zeta, \eta \in \mathbb{Z}^n$,

$$Ave(\ \Xi(\zeta),\ \Xi(\eta)\ ) \geq Ave(\ \Xi(M(\zeta,\eta)),\ (\Xi(m(\zeta,\eta))\ )$$

such that the equality holds when $\Xi(\zeta)$ or $\Xi(\eta)$ is ∞, and

$$\exists\ \alpha \in \mathbb{R} \text{ such that } \Xi(\zeta + 1_n) - \alpha - \Xi(\zeta) = 0,$$

where Ave represents the arithmetic average, M represents the componentwise maximum, m represents the componentwise minimum and $1_n \in \mathbb{Z}^n$ represents the vector that has 1 at each component.

Definition 2.2. A function $\Phi : \mathbb{Z}^n \times \mathbb{R}^m \to \mathbb{R}$ is called mixed T-convex if Φ is L-convex with respect to the integer variables in $\mathbb{Z}^n$ and strictly convex with respect to the real variables in $\mathbb{R}^m$ which is differentiable.

Denote by Γ_1 the set of integer convex functions, by Γ_2 the set of L-convex functions in Γ_1, by Ω_1 the collection of Hessian matrices that represents the discrete convex functions in Γ_1, and by Ω_2 the collection of Hessian matrices that represents the L-convex functions in Γ_2. We thus have,

$$\Omega_1 = \left\{ \begin{array}{c} A \in \mathbb{Z}^{n\times n}:\ \text{A is the Hessian matrix of} \\ \text{some } f \in \Gamma_1 \text{ in a neighborhood } U \end{array} \right\}, \tag{2.1}$$

$$\Gamma_1 = \left\{ \begin{array}{c} \Phi : \mathbb{Z}^n \to \mathbb{R}:\ \text{second differences} \\ \text{of the function } \Phi \text{ are positive in } U \end{array} \right\},$$

$$= \left\{ \begin{array}{c} \Phi : \mathbb{Z}^n \to \mathbb{R}:\ \Phi(\varphi) = \frac{1}{2}\varphi^T A \varphi \\ \text{such that } A \in \Omega_1 \text{ in } U \end{array} \right\}, \tag{2.2}$$

$$\Gamma_2 = \left\{ \begin{array}{c} \Phi : \mathbb{Z}^n \to \mathbb{R} : \Phi \in \Gamma_1 \\ \text{and } \Phi \text{ satisfies Definition 2.1} \end{array} \right\}, \tag{2.3}$$

$$\Omega_2 = \left\{ \begin{array}{c} B \in \mathbb{Z}^{n\times n}:\ \text{B is the Hessian} \\ \text{matrix of } f \in \Gamma_2 \text{ in } U \end{array} \right\}. \tag{2.4}$$

According to Moriguchi and Murota [5] the Hessian matrix of L-convex functions satisfies Definition 2.1.

Throughout this paper Φ will be defined as $\Phi : \mathbb{Z}^n \times \mathbb{R}^m \to \mathbb{R}$ unless stated otherwise. In this paper, $\Phi|_{\mathbb{Z}^n}$ is considered to be an element of Γ_2 with its matrix form in Ω_2.

Let S be a $(n+m) \times (n+m)$ mixed Hessian matrix. For mixed T-convex functions, Φ, the $n \times n$ upper left discrete submatrix portion of S will be assumed to be in Ω_2. The lower $m \times m$ submatrix of S is the $m \times m$ Hessian matrix of Φ associated with the real variables of Φ. The upper right and lower left submatrices consist of the first differences of first differential and the first differential of first difference of the given mixed function. Thus S has the following form:

$$S = \begin{bmatrix} \left[\nabla_{ij}\Phi(\psi,\xi)\right]_{n\times n} & \left[\frac{\partial}{\partial\xi_k}(\nabla_j(\Phi(\psi,\xi)))\right]_{n\times m} \\ \left[\nabla_j(\frac{\partial}{\partial\xi_k}\Phi(\psi,\xi))\right]_{m\times n} & \left[\frac{\partial^2}{\partial\xi_k\partial\xi_t}\Phi(\psi,\xi)\right]_{m\times m} \end{bmatrix} \tag{2.5}$$

$$= \begin{bmatrix} X_{n\times n} & Y_{n\times m} \\ Z_{m\times n} & W_{m\times m} \end{bmatrix}, \tag{2.6}$$

where $\psi \in \mathbb{Z}^n, \xi \in \mathbb{R}^m, 1 \le i,\ j \le n$ and $1 \le k,\ t \le m$. In the matrix S, ∇_i and ∇_{ij} denote the first and second differences with respect to the discrete variables ψ_k of Φ and, $\frac{\partial}{\partial\xi_k}$ and $\frac{\partial^2}{\partial\xi_k\partial\xi_t}$ denote the first and second differentials of Φ with respect to the continuous variables ξ_i, respectively.

Many cases are possible, depending on the conditions placed on the upper right and lower left submatrices of S in a neighborhood of a point. In the simplest simple case we have the following lemma:

Lemma 2.1. *Let Φ be a mixed T-convex function that is linear for both the discrete and the real variables. Then the mixed Hessian matrix S above is the zero matrix if the degree of the polynomial is one. If the degree of the polynomial is not one then the corresponding matrix has the form of* (2.6) *with at least one of X, W, Y or Z*

Proof. If the degree of the given polynomial is one, the first differences of Φ are constants hence the second differences and first differentials of the first differences of Φ are zero. Similarly, the first differentials of Φ are constant hence the second differentials and first differences of the first differentials of Φ are zero. This proves that all the entries of the matrix S are zero. In the simplest case, if Φ has only integer variables then X is constant and $Y = Z = W = 0$, if Φ has only real variables then W is non-zero and $Y = Z = X = 0$. In other possible cases, it can be easily shown that one of Y, Z, W or X is non-zero. □

As a special case of Lemma 2.1, when Φ has only integer variables, X is constant, $Y = Z = W = 0$ and we have a quadratic L-convex function which satisfies all of the properties that are stated in [8].

Note that the lemma above is independent of the definition of mixed T-convex functions since we assume that the mixed T-convex function is strictly convex. In Lemma 2.1, the definition of mixed T-convexity refers to the linear case of the function and the corresponding matrix representation. In general, we have the following theorem:

Theorem 2.1. *Φ is a mixed T-convex function if and only if* $\Phi(\varphi,\xi) = (\Phi|_{\mathbb{Z}^n}(\varphi), \Phi|_{\mathbb{R}^m}(\xi))$ *with* $A = [a_{ij}]_{n\times n}$*, where A is symmetric such that* $a_{st} \leq 0$, $(s,t \in \mathbb{Z}^+, s \neq t)$, $\sum_{s\in\mathbb{Z}^+} a_{st} = 0$*, and* $\Phi(\varphi, \lambda\xi^1 + (1-\lambda)\xi^2) < \lambda\Phi(\varphi,\xi^1) + (1-\lambda)\Phi(\varphi,\xi^2)$ *for all* $\xi^1, \xi^2 \in \mathbb{R}^m$ *at each component of* $\mathbb{R}^m$ *with* $0 \leq \lambda \leq 1$.

Proof. The proof follows from [4] and the strict convexity of Φ with respect to the real variable.

Lemma 2.2. *Let Φ be a mixed T-convex function defined on* $\mathbb{Z}^n \times \mathbb{R}^m$*, then there exists a unique global minimum value of Φ in $\mathbb{R}$.*

Proof. Denote the minimum of Φ in $K \subset \mathbb{Z}^n \times \mathbb{R}^m$ to be $\min\left\{\Phi_{|_K}\right\}$. According to [8] every local minimum of an L-convex function is global. The restriction of Φ to m-dimensional Euclidean space has one global minimum value. Hence in the entire domain of the function,

$$\min\left\{(\min\left\{\Phi_{|_{\mathbb{Z}^n}}\right\}, \min\left\{\Phi_{|_{\mathbb{R}^m}}\right\})\right\}$$

gives a global minimum value in $\mathbb{R}$.

Lemma 2.2 shows the existence of a point in $(n+m)$-dimensional real Euclidean space with a minimum value; however, this point is not necessarily unique. If we consider T-convex functions defined on $\mathbb{Z}^n \times \mathbb{R}^m$, we have the following general result:

Theorem 2.2. *If* $\Phi : \mathbb{Z}^n \times \mathbb{R}^m \to \mathbb{R}$ *is a mixed T-convex function then the minimization of Φ in* $\mathbb{Z}^n \times \mathbb{R}^m$ *depends on the local minimum points of* $\Phi|_{\mathbb{Z}^n}$ *and* $\Phi|_{\mathbb{R}^m}$*. In this case, the mixed variables of the function Φ define a new local minimum point set. The elements of this set are also global minimum points of* Φ.

Proof. Suppose that Φ is a mixed T-convex function. Considering the restriction of Φ in the domain $\mathbb{Z}^n$, $\Phi|_{\mathbb{Z}^n}$, we have local minimum points that are also global minimum points for each fixed real variable. Extension of the domain to $\mathbb{R}^m$ with respect to the real variables will form another discrete set of minimum points that are not necessarily in $\mathbb{Z}^{n+m}$. One can think of the new set of minimum points as the extension of the minimum set that exists on the restricted function $\Phi|_{\mathbb{Z}^n}$. The extension of the domain of Φ by $\mathbb{R}^m$ gives a set of distinct continuous functions in $\mathbb{Z}^n \times \mathbb{R}^m$ and by finding the minimums of such distinct continuous functions we can identify the new set of discrete minimums. The discrete set of minimums is isomorphic to $\mathbb{Z}^{n+m}$. Since every local minimum is a global minimum in $\mathbb{Z}^{n+m}$, the new set also forms a set of global minimums.

One can visualize the mixed T-convex functions by considering L-convex discrete functions that are elements of Γ_2, and at each point there is a continuous part of the mixed T-convex function attached to it from $\mathbb{R}^m$.

Definition 2.3. The mixed T*-convex function $\Phi_1 : \mathbb{Z}^n \times \mathbb{R}^m \to \mathbb{R}$ is defined to be the same as a mixed T-convex function except the discrete variable part of the function Φ_1 consists of L$^\natural$-convex functions.

Theorems 2.1 and 2.2, and Lemmas 2.1 and 2.2 can be restated and proven with respect to mixed T*-convex functions easily with justification as follows:

Theorem 2.3. *Φ_1 is a mixed T*-convex function if and only if $\Phi_1(\varphi,\xi) = (\Phi_1|_{\mathbb{Z}^n}(\varphi^T A\varphi), \Phi_1|_{\mathbb{R}^m}(\xi))$ with $A = [a_{ij}]_{n\times n}$, where A is symmetric such that $a_{st} \le 0$, $(s,t \in \mathbb{Z}^+, s \ne t)$, $\sum\limits_{s\in\mathbb{Z}^+} a_{st} \ge 0$ and $\Phi_1(\varphi, \lambda\xi^1 + (1-\lambda)\xi^2) < \lambda\Phi_1(\varphi,\xi^1) + (1-\lambda)\Phi_1(\varphi,\xi^2)$ for all $\xi^1, \xi^2 \in \mathbb{R}^m$ at each component of $\mathbb{R}^m$ with $0 \le \lambda \le 1$.*

Lemma 2.3. *Let Φ_1 be a mixed T*-convex function, then there exists a unique global minimum value in $\mathbb{R}$.*

Proof. Denote the minimum of Φ_1 in space K to be $\min\left\{\Phi_{1|_K}\right\}$. We know that every local minimum is also a global minimum for L-convex functions and vice versa. The restriction of Φ_1 to m-dimensional real space has one global minimum value. Hence, in the entire domain of the function, $\min\left\{(\min\left\{\Phi_{1|_{\mathbb{Z}^n}}\right\}, \min\left\{\Phi_{1|_{\mathbb{R}^m}}\right\})\right\}$ gives a unique global minimum value in $\mathbb{R}$.

The minimum value that is found in Lemma 2.3 shows that there exists a minimum point in $\mathbb{Z}^n \times \mathbb{R}^m$ for mixed T*- convex functions defined on $\mathbb{Z}^n \times \mathbb{R}^m$. Considering the points in the domain $\mathbb{Z}^n \times \mathbb{R}^m$ of a mixed T*-convex function Φ_1 that corresponds to the global minimum value in $\mathbb{R}$, we have the following general result:

Theorem 2.4. *If $\Phi_1 : \mathbb{Z}^n \times \mathbb{R}^m \to \mathbb{R}$ is a mixed T*-convex function then the minimization of Φ_1 in $\mathbb{Z}^n \times \mathbb{R}^m$ depends on the local minimums of $\Phi_1|_{\mathbb{Z}^n}$ and $\Phi_1|_{\mathbb{R}^m}$. In this case, the mixed variables of the function Φ_1 define a new local minimum point set. The elements of this set are also global minimum points of Φ_1.*

Proof. The proof follows similarly from Theorem 2.2. □

The results above are based on the neighborhoods of integer and real variables. Let U be a local neighborhood for the integer variable and V be the local neighborhood for the continuous variable. Let the local neighborhood of a point $(\varphi,\xi) \in \mathbb{Z}^n \times \mathbb{R}^m$ be (U,V),

$$(U,V)(\varphi,\xi) = (U(\varphi), V(\xi)) = \left\{ \begin{array}{c} (\varphi + e_a + e_b, t) \mid a,b \in \{0\} \cup \mathbb{N}, \\ a \ne b, -\epsilon < t < \epsilon \end{array} \right\},$$

where the properties of the mixed T-convex function will be characterized for a fixed point (φ_0, ξ_0) in the neighborhood $(U, V)(\varphi_0, \xi_0)$. The Hessian matrix for the mixed T-convex functions has the form of (2.6) where we use the following equations.

$$\nabla_a(\varphi) = \Phi(\varphi + e_a, \xi) - \Phi(\varphi, \xi),$$
$$\nabla_{ab}(\varphi) = \Phi(\varphi + e_a + e_b, \xi) + \Phi(\varphi, \xi) - [\Phi(\varphi + e_a, \xi) + \Phi(\varphi + e_b, \xi)],$$
$$\nabla_i(\frac{\partial}{\partial \xi_k}\Phi(\varphi, \xi)) = \nabla_i(\lim_{t\to 0}\frac{\Phi(\varphi, \xi_k + tv) - \Phi(\varphi, \xi_k)}{t}),$$

where v is the directional derivative at that point,

$$= \lim_{t\to 0}(\frac{\Phi(\varphi + e_i, \xi_k + tv) - \Phi(\varphi, \xi_k + tv) - [\Phi(\varphi + e_i, \xi_k) - \Phi(\varphi, \xi_k)]}{t}) \quad (2.7)$$

$$\frac{\partial}{\partial \xi_k}(\nabla_i \Phi(\varphi, \xi)) = \frac{\partial}{\partial \xi_k}(\Phi(\varphi + e_i, \xi) + \Phi(\varphi, \xi))$$

$$= \lim_{t\to 0}(\frac{\Phi(\varphi + e_i, \xi_k + tv) - \Phi(\varphi + e_i, \xi_k) - \Phi(\varphi, \xi_k + tv) + \Phi(\varphi, \xi_k)}{t}). \quad (2.8)$$

We thus have from (2.7) and (2.8) that

$$\frac{\partial}{\partial \xi_k}(\nabla_i \Phi(\psi, \xi)) = \nabla_i(\frac{\partial}{\partial \xi_k}\Phi(\psi, \xi)).$$

Hence,

Lemma 2.4. *Under the assumptions above, the mixed Hessian matrix of a mixed T^*- or T-convex function is symmetric with respect to the diagonal if it has a square matrix form.*

Proof. Since it is shown that $\frac{\partial}{\partial \xi_k}(\nabla_i \Phi(\psi, \xi) = \nabla_i(\frac{\partial}{\partial \xi_k}\Phi(\psi, \xi)$, the square matrix form of S is symmetric. □

Lemma 2.5. (2.6) *is linear if the mixed T-convex function is affine.*

Proof. This simply follows from the fact that both L-convex and the real variable convex portions of the Hessian matrix (2.6) are linear. □

Theorem 2.5. *A function $\Phi : \mathbb{Z}^n \times \mathbb{R}^m \to \mathbb{R}$ is a mixed T-convex function with Y and Z equivalent to zero in* (2.6) *at a neighborhood of a point in $\mathbb{Z}^n \times \mathbb{R}^m$ if and only if the determinant of W in* (2.6) *is positive definite and for each point of $\mathbb{Z}^n$, X satifies $-\nabla_{kl}(\Phi)$ is negative or zero with $\sum_{k=1}^{n}\nabla_{kl}(\Phi)$ positive or zero.*

Proof. The proof follows from (2.6), [5] and real variable convexity. □

Theorem 2.6. *A function $\Phi_1 : \mathbb{Z}^n \times \mathbb{R}^m \to \mathbb{R}$ is a mixed T^*- convex function with Y and Z equivalent to zero in* (2.6) *at a neighborhood of a point in $\mathbb{Z}^n \times \mathbb{R}^m$ if and only if determinant of W in* (2.6) *is positive definite and for each point of $\mathbb{Z}^n$, X satifies $-\nabla_{kl}(\Phi_1)$ is negative or zero when k and l are not the same with $\sum_{k=1}^{n} \nabla_{kl}(\Phi_1)$ that is positive or zero.*

Proof. The proof follows from (2.6), [5] and real variable convexity. □

Let $\bar{\Phi}_1$ denote the function that is obtained by the extension of the domain of $\Phi|_{\mathbb{Z}^n}$ to $\mathbb{R}^n$ and $\bar{\Phi}_2$denote similarly for $\Phi|_{\mathbb{R}^m}$.

Theorem 2.7. *The existence of a unique global minimum point of a mixed T-convex function $\Phi : \mathbb{Z}^n \times \mathbb{R}^m \to \mathbb{R}$ depends on the functions $\bar{\Phi}_1$ and $\bar{\Phi}_2$ defined as above such that: If the global minimum point of $\bar{\Phi}$ (that is obtained from the global minimums of $\bar{\Phi}_1$ and $\tilde{\Phi}_2$) is the same as the global minimum of Φ then the global minimum point of Φ is unique; If the global minimum point of $\bar{\Phi}$ is not in the domain of Φ then the uniqueness of the global minimum point depends on the integer points in the neigborhood of the integer component of $\bar{\Phi}|_{\mathbb{Z}^n \times \mathbb{R}^m}$. In this case, the lowest value in the range gives the unique global minimum if it is unique.*

Proof. The uniqueness of the global minimum point of a mixed T-convex function $\Phi : \mathbb{Z}^n \times \mathbb{R}^m \to \mathbb{R}$ follows from the global minimum point properties of the extension of Φ, that is $\bar{\Phi} : \mathbb{R}^{n+m} \to \mathbb{R}$. By continuous convexity, there exists a unique global minimum point of $\bar{\Phi}$. If the global minimum of real extension of $\bar{\Phi} : \mathbb{R}^{n+m} \to \mathbb{R}$ exists in the domain of Φ then the global minimum point of Φ is unique. If the global minimum point of $\bar{\Phi}$ is not in the domain of Φ then one can look at the integer points in the integer component neighborhood (componentwise for each integer component) of $\bar{\Phi}|_{\mathbb{Z}^n \times \mathbb{R}^m}$ to find the points that are possible global minimum points in the domain of Φ. By comparison, one can observe the existence of a unique global minimum if there exists a unique point with the lowest value in the range of Φ. The uniqueness of the minimum point value of Φ is based on the closest integer values of $\bar{\Phi}|_{\mathbb{Z}}$ for all the components in the domain. This can be done by induction, that is by finding the minimum with respect to each component in the domain and checking the values based on the function that is given. □

Theorem 2.7 can be also stated and proved similarly for mixed T*-convex functions:

Theorem 2.8. *The existence of a unique global minimum point of a mixed T^*-convex $\Phi_1 : \mathbb{Z}^n \times \mathbb{R}^m \to \mathbb{R}$ depends on the functions $\bar{\Phi}_1$ and $\bar{\Phi}_2$ defined as* *above such that: If the global minimum point of* $\bar{\Phi}$ *(that is obtained from the global minimums of* $\bar{\Phi}_1$ *and* $\tilde{\Phi}_2$*) is the same as the global minimum of* Φ *then the global minimum point of* Φ *is unique; If the global minimum point of* $\bar{\Phi}$ *is not in the domain of* Φ *then the uniqueness of the global minimum point depends on the integer points in the neigborhood of the integer component of* $\bar{\Phi}|_{\mathbb{Z}^n\times\mathbb{R}^m}$. *In this case, the lowest value in the range gives the unique global minimum if it is unique.*

Proof. Follows similar to the proof of Theorem 2.7. □

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