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SOME PROPERTIES OF MULTI-LINEAR OPERATORS F NUCLEAR TYPE

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Abstract: In this work we obtain two results related to multi-linear $p-$compact operators in Banach spaces. The first result establishes that an operator $\Phi \in \mathcal{L}(E_1, ..., E_n, F)$ is $p-$compact $(1 \leq p \leq \infty)$ related to $(q_1, ...q_n)$ if and only if there exist the operators $\Psi \in \mathcal{L}(E_1, ..., E_n, l_p)$ and $A \in \mathcal{L}(l_p, F)$ such that $\Phi = A \,\text{o}\, \Psi$, where A, Ψ are compact. One has $N_{(\infty,p;q_1,...,q_n)}(\Phi) = inf||A||||\Psi||$, where the infimum is taken over all possible factorizations. The second result is concerned with any multi-linear operator of finite type $\Psi \in \mathcal{L}(E_1, ..., E_n, L_p(\Omega, \mu))$ and establishes the following result:

$$N_{(\infty,p;q_1,...,q_n)}(\Phi) = N_{f,(\infty,p;q_1,...,q_n)}(\Psi) = ||\Psi||.$$

Dedicated to the memory of
Mauro R. Chumpitaz.

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1. Introduction

The authors [5], [1], [6] have made the observations that the starting point to classify the non-linear operators on Banach spaces is the study of the different

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classes of multi-linear operators. On the other hand, the investigations on analytic functions on Banach spaces of infinite dimension are intimately related to these multi-linear applications (see e.g. [1] and references therein). Moreover, it is remarkable that the more general setting of operator ideals (over the class of all Banach spaces) constitute the proper algebraic tool to study soliton equations [3]. Motivated by these ideas and developments, in this work we obtain two important results related to these operators. We obtain two results related to multi-linear $p-$compact applications on Banach spaces. The first result establishes that an operator $\Phi \in \mathcal{L}(E_1, ..., E_n, F)$ is $p-$compact $(1 \leq p \leq \infty)$ related to $(q_1, ...q_n)$ if and only if there exist the operators $\Psi \in \mathcal{L}(E_1, ..., E_n, l_p)$ and $A \in \mathcal{L}(l_p, F)$ such that $\Phi = A \,\mathrm{o}\, \Psi$, where A, Ψ are compact. Moreover, in this case one has $N_{(\infty,p;q_1,...,q_n)}(\Phi) = inf||A||||\Psi||$, where the infimum is taken over all the possible factorizations. The second result is concerned with any multi-linear operator of finite type $\Psi \in \mathcal{L}(E_1, ..., E_n, L_p(\Omega, \mu))$ and establishes the following result: $N_{(\infty,p;q_1,...,q_n)}(\Psi) = N_{f,(\infty,p;q_1,...,q_n)}(\Psi) = ||\Psi||$. The theorem presented in this paper shows Conjecture 3.1 of [4].

Let us introduce the notations used in this work. For the Banach spaces $E_1, ..., E_n$ and F over the field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) we denote by $\mathcal{L}(E_1, ..., E_n, F)$ the Banach space of all continuous multi-linear applications from $E_1 \times ... \times E_n$ into F with natural norm given by

$$\|T\| = \sup_{\substack{x_i \in B_{E_i} \\ i=1,...,n}} \|T(x_1, ..., x_n)\|, \tag{1}$$

where B_{E_i} denotes the unitary ball of E_i centered in 0. For φ belonging to the topological dual $E_k^\star$ of E_k, $k = 1, ..., n$ and $b \in F$ we denote by $\varphi_1 \times ... \times \varphi_n b$ the elements of $\mathcal{L}(E_1, ..., E_n; F)$ defined by $\varphi_1(x_1)...\varphi_n(x_n)b$ in the point $(x_1, ..., x_n)$. These applications generate the subspace $\mathcal{L}_f(E_1, ..., E_n; F)$ of multi-linear applications of finite type which is denoted by $\mathcal{L}_f(E_1, ..., E_n; F)$.

For $s \in \langle 0, +\infty \rangle$ we denote by $\ell_s(F)$ (or ℓ_s, $F = \mathbb{K}$), the vector space of all sequences $(y_j)_{j=1}^\infty$ of elements that belong to F such that

$$\|(y_j)_{j=1}^\infty\|_s = \left[\sum_{j=1}^{\infty} \|y_j\|^s\right]^{1/s} < +\infty. \tag{2}$$

For $s \geq 1$, $\| \cdot \|_s$ is a norm, and for $s < 1$, is a $s-$norm. In any case we have a complete metric vector space. We denote by $\ell_s^w(F)$ the vector space of all sequences $(y_j)_{j=1}^\infty$ of elements that belong to F such that

$$\|(y_j)_{j=1}^\infty\|_{w,s} = w_s(y_j) = \sup_{\varphi \in B_{F'}} \|(\varphi(y_j))_{j=1}^\infty\|_s < +\infty, \tag{3}$$

so, $(\ell_s^w(F), \|\cdot\|_{w,s})$ is a metric vector space. For $s = +\infty$ we consider $\ell_\infty(F) = \ell_\infty^w(F)$ as a Banach space of all sequences $(y_j)_{j=1}^\infty$ of elements of F under the norm

$$w_\infty(y_j) = \|(y_j)_{j=1}^\infty\|_\infty = \|(y_j)_{j=1}^\infty\|_{w,\infty} = \sup_{j\in\mathbb{N}} \|y_j\|. \tag{4}$$

For $r \in]0, +\infty]$, $p, q_j \in [1, \infty]$, $k = 1, ...n$, such that $n + \frac{1}{r} \geq \frac{1}{p} + \frac{1}{q_1} + ... + \frac{1}{q_n}$, $T \in \mathcal{L}(E_1, ..., E_n; F)$ is called $(r, p; q_1, ..., q_n)-$nuclear type and it takes the form

$$T = \sum_{k=1}^{\infty} \sigma_k x_{k,1}^* \times ... \times x_{k,n}^* \otimes y_k \tag{5}$$

with $(\sigma_j)_{j=1}^\infty \in \ell_r$, $(y_k)_{k=1}^\infty \in \ell_{p'}^w(F)$ and $(x_{k,j}^*)_{k=1}^\infty \in \ell_{q_j'}^w(E_j^*)$, $j = 1, ..., n$. In the case that $r = +\infty$ the condition for $(\sigma_k^j)_{k=1}^\infty$ is to be in c_0. The set of such applications satisfying such definition is a vector space and is denoted by $\mathcal{N}_{r,p;q_1,...,q_n}(E_1, ..., E_n; F)$. Considering that

$$N_{(r,p;q_1,...,q_n)}(T) = \inf \|(\sigma_k)_{k=1}^\infty\|_r \prod_{j=1}^{n} \|(x_{k,j}^*)_{k=1}^\infty\|_{w,q_j'} \|(y_k)_{k=1}^\infty\|_{w.p'} \tag{6}$$

where the infimum is taken over all possible representations of T described in (5), we obtain a $s-$norm where [4]

$$\frac{1}{s} = \frac{1}{r} + \frac{1}{p'} + \frac{1}{q_1'} + ... + \frac{1}{q_n'}.$$

2. Multi-Linear Operators of $(r, p; q_1...q_n)-$Nuclear Type

The next proposition is slightly different from the one given in [5] and its proof can be performed following the lines of this reference.

Proposition 2.1. *For any operator $T \in L_f(E_1, ..., E_n; F)$ let us define*

$$N_{f,(r,p;q_1,...,q_n)}(T) = \inf \|(\sigma_k)_{k=1}^m\|_r \prod_{j=1}^{n} \|(x_{k,j}^*)_{k=1}^m\|_{w,q_j'} \|(y_k)_{k=1}^m\|_{w.p'}, \tag{7}$$

where the infimum is taken over all possible finite representations of $T = \sum_{k=1}^m \sigma_k x_{k,1}^* \times ... \times x_{k,n}^* \otimes y_k$, we obtain a $s-$norm with

$$\frac{1}{s} = \frac{1}{r} + \frac{1}{p'} + \frac{1}{q_1'} + ... + \frac{1}{q_n'}.$$

The next multi-linear result is an extension of the linear case [6].

Lemma 2.1. *Let $\Psi : E_1 \times ... \times E_n \longrightarrow L_p(\Omega, \mu)$ be defined by*

$$\Psi(x_1, ..., x_n) = \sum_{k=1}^{m} \sigma_k x_{1,k}^*(x_1)...x_{n,k}^*(x_n) y_k, \tag{8}$$

where $\frac{1}{p} = \frac{1}{q_1'} + ... + \frac{1}{q_n'}$, then $N_{f,(\infty,p;q_1,...,q_n)}(\Psi) = N_{(\infty,p;q_1,...,q_n)}(\Psi) = \|\Psi\|$.

Proof. It is clear that for $\frac{1}{p} = \frac{1}{q_1'} + ... + \frac{1}{q_n'}$ one has from (6) and (7)

$$\|\Psi\| \leq N_{(\infty,p;q_1,...,q_n)}(\Psi) \leq N_{f,(\infty,p;q_1,..,q_n)}(\Psi). \tag{9}$$

Moreover

$$\|\Psi\| \|x_1\|...\|x_n\| \geq \left[\int_\Omega \left| \sum_{k=1}^{m} \sigma_k x_{1,k}^*(x_1)...x_{n,k}^*(x_n) y_k(t) \right|^p du(t) \right]^{1/p}. \tag{10}$$

Since $x_{i,k}^*$ is surjective there exists $\tilde{x}_i \in E_i$ such that $x_{i,k}^*(\tilde{x}_i) = M_i / 2^{k/q_i'}$ where

$$M_i = \sup_{\|x_i\|_{X_i}=1} \left(\sum_{k=1}^{m} |\langle x_{i,k}^*, x_i \rangle|^{q_i'} \right)^{1/q_i'}, \quad i = 1, ..., n. \tag{11}$$

We will show that $\|\tilde{x}_i\| \leq 1$ and $M_i < \infty$ for $i = 1, ..., n$. From the definition of M_i for a fixed i and for $\epsilon > 0$ one has

$$M_i \|\tilde{x}_i\| < (1+\epsilon) \left(\sum_{k=1}^{m} |\langle x_{i,k}^*, \widetilde{x}_i \rangle|^{q_i'} \right)^{1/q_i'} = (1+\epsilon) \left(\sum_{k=1}^{m} M_i^{q_i'} / 2^k \right)^{1/q_i'},$$

which implies that $\|\tilde{x}_i\| < (1+\epsilon)$ for all $\epsilon > 0$.

So, considering $\|\tilde{x}_i\| \leq 1$ in equation (10) one has

$$\begin{aligned} \|\Psi\| &\geq \left[\int_\Omega \left| \sum_{k=1}^{m} \sigma_k \frac{M_1}{2^{k/q_1'}} ... \frac{M_n}{2^{k/q_n'}} y_k(t) \right|^p du(t) \right]^{1/p}, \\ &\geq \left[\int_\Omega \left| \sum_{k=1}^{m} \frac{\sigma_k}{2^{k/p}} y_k(t) \right|^p du(t) \right]^{1/p} \prod_{i=1}^{m} M_i. \end{aligned} \tag{12}$$

Let $z(t) = \sum_{k=1}^{m} \sigma_k \frac{y_k(t)}{2^{k/p}}$, then for all $1 \leq p < \infty$ we have

$$|\langle \varphi, z \rangle| = |\sum_{k=1}^{m} \sigma_k \langle \varphi, \frac{y_k}{2^{k/p}} \rangle| \leq \|\varphi\| \|z\|. \tag{13}$$

In addition, let $M = span_{k \in \{1,...,n\} - \{k_0\}} \{\frac{y_k}{2^{k/p}}\}$. Moreover, as a consequence of the Hahn-Banach theorem [2] there exists φ such that $\|\varphi\| = \frac{1}{d}$, $\langle \varphi, x \rangle = 0$ for all $x \in M$ and $\langle \varphi, \frac{y_{k_0}}{2^{k_0/p}} \rangle = 1$, where $d = \inf_{x \in M} \|x - \frac{y_{k_0}}{2^{k_0/p}}\|$ and further one can choose σ_{k_0} such that

$$|\sigma_{k_0}| = \max_{k=1,\dots,n} |\sigma_k| = \ell_\infty(\sigma_k).$$

Taking into account these last relations in equation (13) one can get

$$\|z\| \geq |\sigma_{k_0}| d. \tag{14}$$

Since $x = \sum_{k=1,k\neq k_0}^m \frac{-y_k}{2^{k/p}} \in M$, then for a given $\varepsilon > 0$ one has $(1+\varepsilon)d > \| \sum_{k=1}^m \frac{y_k}{2^{k/p}} \|$. Therefore, from (14) one gets

$$(1+\varepsilon)\|z\| > \ell_\infty(\sigma_k) \| \sum_{k=1}^m \frac{y_k}{2^{k/p}} \|. \tag{15}$$

We know that $w_{p'}(y_k) = \sup_{a \in B_{\ell_p^m}} \| \sum_{k \leq m} a_k y_k \|$, and since $a = (\frac{1}{2^{k/p}})_{k=1}^m \in B_{\ell_p^m}$, for $\tilde{\varepsilon} > 0$ we have $(1+\tilde{\varepsilon}) \| \sum_{k\leq m} \frac{1}{2^{k/p}} y_k \| > w_{p'}(y_k)$. From the last relation and the equation (15) one obtains

$$(1+\varepsilon)(1+\tilde{\varepsilon})\|z\| > \ell_\infty(\sigma_k) w_{p'}(y_k), \quad \text{for all} \quad \varepsilon > 0 \;\text{ and }\; \tilde{\varepsilon} > 0. \tag{16}$$

Therefore, from the relations (12) and (16) one has

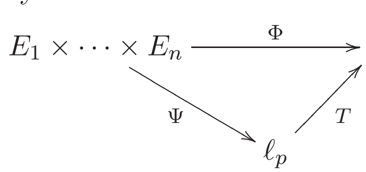
$$\begin{aligned} \|\Psi\| &\geq \ell_\infty(\sigma_k) w_{p'}(y_k) \prod_{i=1}^m M_i \\ &\geq N_{f,(\infty,p;q_1,\dots,q_n)}(\Psi). \end{aligned} \tag{17}$$

From equations (9) and (17) one has the required result. For $p = \infty$ or $q_i' = \infty$, $i = 1, \dots, n$; it can be done a similar procedure with some minor changes. □

3. Multi-Linear $p-$Compact Operators

The next theorem is an extension of the linear case presented in [6] as Theorem 18.3.2. In the present paper we deal with multi-linear operators and it will become the demonstration of Conjecture 3.1 presented in [4].

Theorem 3.1. *An operator $\Phi \in \mathcal{L}(E_1, \dots, E_n; F)$ is $p-$compact relative to $(q_1, \dots, q_n)$ if and only if there exists a commutative diagram:*



(18)

such that $\Psi \in \mathcal{L}(E_1, \dots, E_n; \ell_p)$ and $T \in \mathcal{L}(\ell_p; F)$ are compact. In this case

$$N_{(\infty,p;q_1,\dots,q_n)}(\Phi) := \inf \|T\|\|\Psi\|,$$

where the infimum is taken over all possible factorizations.

Proof. $\Rightarrow$) If $\Phi \in \mathcal{L}(E_1, ..., E_n; F)$ with Φ $p-$compact, then

$$\Phi \in \mathcal{N}_{(\infty,p;q_1,...,q_n)}(E_1, ..., E_n; F).$$

So, by the so-called factorization theorem for multi-linear operators $(r, p; q_1, ..., q_n)-$nuclear (see Appendix), we have that for $\epsilon > 0$ there exists a factorization of Φ such that

$$(1+\epsilon)N_{(\infty,p;q_1,...,q_n)}(\Phi) > \|T\|\ell_\infty(\sigma_i)\prod_{i=1}^{n}\|A_i\|, \tag{19}$$

where $\Phi = T \circ S_0 \circ (A_1, ..., A_n)$ with $A_i \in \mathcal{L}(E_i, \ell_{q_i'})$, $S_0 \in \mathcal{L}(\ell_{q_1'}, ..., \ell_{q_n'}; \ell_p)$, $T \in \mathcal{L}(\ell_p, F)$.

Let $\Psi := S_0 \circ (A_1, ..., A_n)$, so $\Psi \in \mathcal{L}(E_1, ..., E_n; \ell_p)$, and since each $A_i \in \mathcal{L}(E_i, \ell_{q_i'})$ is approximable, then A_i is compact for $i = 1, ..., n$, therefore Ψ is compact. From the construction in Appendix one has that T is compact.

Since one has

$$\|\Psi(x_1, ..., x_n)\| = \|S_0(A_1x_1, ..., A_nx_n)\| \le \|S_0\|\|A_1x_1\|\cdots\|A_nx_n\|,$$

then

$$\|\Psi\| \le \|S_0\|\prod_{i=1}^{n}\|A_i\|. \tag{20}$$

As $S_0((\varepsilon_{1,j})_{j=1}^{\infty}, ..., (\varepsilon_{n,j})_{j=1}^{\infty}) = (\sigma_j\varepsilon_{1,j}\cdots\varepsilon_{n,j})_{j=1}^{\infty}$, from this relation we have that

$$\|S_0\| \le \ell_\infty(\sigma_j). \tag{21}$$

Further, one has $\|S_0\| \ge (\sum_{j=1}^{\infty}|\sigma_j\varepsilon_{1,j}\cdots\varepsilon_{n,j}|^p)^{1/p}$ for $\|(\varepsilon_{i,j})_{j=1}^{\infty}\|_{l_{q_i'}} = 1$, $i = 1, ..., n$; since $r = \infty$, $(\sigma_j)_{j=1}^{\infty} \in c_0$. Then it is clear that there exists an $j_0 \in \mathbb{N}$ such that $\ell_\infty(\sigma_j) = |\sigma_{j_0}|$. Taken $\varepsilon_{k,j} = 1$ for $j = j_0$ and $\varepsilon_{k,j} = 0$ for $j \ne j_0$ and $k = 1, ..., n$, we have

$$\|S_0\| \ge |\sigma_{j_0}| = \ell_\infty(\sigma_j). \tag{22}$$

So, from (21) and (22) we have that $\|S_0\| = \ell_\infty(\sigma_j)$. Using this last relation in (19) we have

$$(1+\epsilon)N_{(\infty;p,q_1,...,q_n)}(\Phi) > \|T\|\|S_0\|\prod_{i=1}^{n}\|A_i\|. \tag{23}$$

Finally, substituting (20) into (23) one gets

$$(1+\epsilon)N_{(\infty;p,q_1,...,q_n)}(\Phi) > \|T\|\|\Psi\|. \tag{24}$$

$\Leftarrow$) The demonstration of this implication will require the results of the next two properties i) and ii).

i) Let $T \in \mathcal{L}(\ell_p, F)$ be any compact operator. Since $(\ell_p)^{'} = \ell_{p'}$ has the approximation property ($\frac{1}{p} + \frac{1}{p'} = 1$) (see section 10.1.4 of [6]), so, by using the results of section 18.1.3 of [6], one has the following representation for T

$$T((\varepsilon_j)_{j=1}^{\infty}) = \sum_{j=1}^{\infty} \sigma_j \varepsilon_j y_j$$

with $(\sigma_i)_{i=1}^{\infty} \in c_0$ y $T(e_j) = y_j$, where $e_j = (0,..,0,1,0,...)$ ($j-$component different from zero), and $(y_j)_{j=1}^{\infty} \in w_{p'}(F)$, $(\Pi_j)_{j=1}^{\infty} \in w_p(\ell_{p'})$; $\Pi_j((\varepsilon_m)_{m=1}^{\infty}) = \varepsilon_j$, $j \in \mathbb{N}$ and $q^{'} = p$. Then, it is clear that

$$\|T\| \leq \ell_{\infty}(\sigma_j) w_{p'}(y_j). \tag{25}$$

It is known that

$$w_{p'}(y_j) = \sup_m \sup_{a \in B_{\ell_p^m}} \| \sum_{k \leq m} a_k y_k \|. \tag{26}$$

There exists an element $|\sigma_{j_0}| = \sup_{j \in \mathbb{N}} |\sigma_j|$ because $(\sigma_k)_{k=1}^{\infty} \in c_0$. Since $a_0 = (0,...,0, \frac{\sigma_{j_0}}{|\sigma_{j_0}|}, 0,...,0) \in B_{\ell_p^m}$, for an $\varepsilon > 0$ in equation (26) we have

$$w_{p'}(y_j) < \sup_m (1+\varepsilon) \| \frac{\sigma_{j_0}}{|\sigma_{j_0}|} y_{j_0} \| = \sup_m (1+\varepsilon) \frac{\|T(e_{j_0})\|}{|\sigma_{j_0}|}.$$

From this last equation one gets $|\sigma_{j_0}| w_{p'}(y_j) < \sup_m (1+\varepsilon)\|T\|$, for all $\varepsilon > 0$ and $m \in \mathbb{N}$.

Thus

$$\ell_{\infty}(\sigma_j) w_{p'}(y_j) \leq \|T\| \tag{27}$$

and from (25) and (27) we have that

$$\|T\| = \ell_{\infty}(\sigma_j) w_{p'}(y_j).$$

ii) Let $\Psi \in \mathcal{L}(E_1,...,E_n;\ell_p)$, so one has $\Psi(x_1,...,x_n) = (\varepsilon_j(x_1,...,x_n))_{j=1}^{\infty}$ where $\varepsilon_j \in \mathcal{L}(E_1,...,E_n;\mathbb{K})$ is defined by $\varepsilon_j(x_1,....,x_n) = x_{1,j}^*(x_1)\cdots x_{n,j}^*(x_n)$ for $j \in \mathbb{N}$. Then we have that

$$\|\Psi(x_1,...,x_n)\| = (\sum_{j=1}^{\infty} |\varepsilon_j(x_1,...,x_n)^p|)^{1/p} = (\sum_{j=1}^{\infty} |x_{1,j}^*(x_1)\cdots x_{n,j}^*(x_n)|^p)^{1/p},$$

since $\frac{1}{p} = \frac{1}{q_1^{'}} + ... + \frac{1}{q_n^{'}}$, applying the Hölder's inequality we have that

$$\|\Psi\| = \sup_{\|x_i\|_{X_i}=1} \|\Psi(x_1,...,x_n)\| \leq \prod_{k=1}^{n} \|(x_{k,j}^*)_{j=1}^{\infty}\|_{w,q_k^{'}} \tag{28}$$

On the other hand, we have

$$\|\Psi\|\|x_1\|\cdots\|x_n\| \geq (\sum_{k=1}^{\infty}|x^*_{1,k}(x_1)|^p\cdots|x^*_{n,k}(x_n)|^p)^{1/p}. \tag{29}$$

Let us take the quantity M_i from equation (11) with $m=\infty$

$$\widetilde{M_i} = \sup_{\|x_i\|_{X_i}=1}\left(\sum_{k=1}^{\infty}|\langle x^*_{i,k}, x_i\rangle|^{q_i^{'}}\right)^{1/q_i^{'}}, \tag{30}$$

which can be proved to be a finite real number for $i=1,...,n$. Likewise, it was shown that there exists $x_i \in X_i$ such that $\|x_i\| \leq 1$, with $x^*_{i,k}(x_i) = \frac{\widetilde{M_i}}{2^{k/q_i'}}$. Substituting these relations into (29) one has

$$\|\Psi\|\|x_1\|\cdots\|x_n\| \geq (\sum_{k=1}^{\infty}\frac{\widetilde{M_1^p}}{2^{kp/q_1'}}\cdots\frac{\widetilde{M_n^p}}{2^{kp/q_n'}})^{1/p} = \prod_{k=1}^{n}\|(x^*_{k,j})_{j=1}^{\infty}\|_{w,q_k'}. \tag{31}$$

Finally, from the relations (28) and (31) we have that

$$\|\Psi\| = \prod_{k=1}^{n}\|(x^*_{k,j})_{j=1}^{\infty}\|_{w,q_k'}.$$

Next we make use of the previous results i) and ii). Since one has

$$\Phi(x_1,...,x_n) = (T\circ\Psi)(x_1,...,x_n) = \sum_{j=1}^{\infty}\sigma_j x^*_{1,j}(x_1)\cdots x^*_{n,j}(x_n)y_j,$$

then it is clear that $\Phi \in \mathcal{N}_{(\infty};p,q_1,...,q_n)(E_1,...,E_n;F)$ with $\frac{1}{p} = \frac{1}{q_1'}+...+\frac{1}{q_n'}$, and

$$N_{(\infty;p,q_1,...,q_n)}(\Phi) \leq \ell_\infty(\sigma_j)w_{p'}(y_k)\prod_{k=1}^{n}\|(x^*_{k,j})_{j=1}^{\infty}\|_{w,q_k'}.$$

With the relations already found this is equivalent to say that

$$N_{(\infty;p,q_1,...,q_n)}(\Phi) \leq \|T\|\|\Psi\|. \tag{32}$$

From the relations (24) and (32) we have that

$$N_{(\infty;p,q_1,...,q_n)}(\Phi) := \inf\|T\|\|\Psi\|,$$

where the infimum is taken over all possible factorizations. □

Observation 3.1. The demonstration of the Theorem for $p=\infty$ or $q_i'=\infty$, $i=1,...,n$ can be performed following similar steps with some minor changes.

In future research we will use the results presented here in the demonstrations of Conjectures 3.2 and 3.3 of [4] and it will appear elsewhere.

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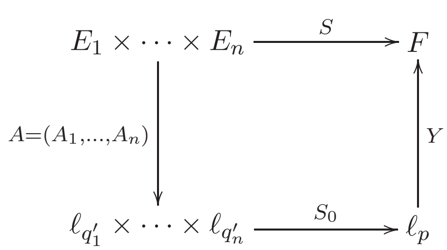
Appendix: Factorization Theorem

Theorem 3.2. (Factorization Theorem) *An operator* $S \in \mathcal{L}(E_1, \ldots, E_n; F)$ *is* $(r, p; q_1, \ldots, q_n)$*−nuclear if and only if there exists a commutative diagram:*



With $A_k \in \mathcal{L}(E_k; \ell_{q'_k})$, $Y \in \mathcal{L}(\ell_p, F)$, $S_0 \in \mathcal{L}(\ell_{q'_1}, \dots, \ell_{q'_n}; \ell_p)$, S_0 *defined as*

$$\ell_{q'_1} \times \ell_{q'_2} \times \cdots \times \ell_{q'_n} \xrightarrow{S_0} \ell_p \quad ,$$

$$S_0((\varepsilon_{1,j})_{j=1}^{\infty}, \dots, (\varepsilon_{n,j})_{j=1}^{\infty}) = (\sigma_j \varepsilon_{1,j} \cdots \varepsilon_{n,j})_{j=1}^{\infty},$$

where $(\sigma_j)_{j=1}^{\infty} \in \ell_r$ *si* $0 < r < \infty$ *and* $(\sigma_i)_{i=1}^{\infty} \in c_0$ *if* $r = \infty$.

In this case $N_{(r,p;q_1,\dots,q_n)}(S) = \inf \parallel Y \parallel \ell_r(\sigma_i) \prod_{j=1}^{n} \parallel A_j \parallel$, *where the infimum is taken for all possible factorizations, previously described.*

Proof. $\Rightarrow$) S_0 is well defined, it is multilineal and continuous, i.e.

$$\parallel S_0((\varepsilon_{1,j})_{j=1}^{\infty}, \dots, (\varepsilon_{n,j})_{j=1}^{\infty}) \parallel_{\ell_p} \leq \ell_r(\sigma_j) \prod_{k=1}^{n} \parallel (\varepsilon_{k,j})_{j=1}^{\infty} \parallel_{\ell_{q'_k}} .$$

Where $\frac{1}{q'_n} + \cdots + \frac{1}{q'_1} + \frac{1}{r} + \frac{1}{p'} \geq 1$.

Since S is a $(r, p; q_1, \dots, q_n)-$nuclear operator, so

$$S = \sum_{k=1}^{\infty} \sigma_k a_{1k} \times \cdots \times a_{nk} \otimes y_k. \tag{33}$$

With $(\sigma_k)_{k=1}^{\infty} \in \ell_r$, $(a_{jk})_{k=1}^{\infty} \in \ell_{q'_j}^{w}(E'_j)$, $\forall j = 1, \dots, n$ and $(y_k)_{k=1}^{\infty} \in \ell_{p'}^{w}(F)$. Define

$$\begin{array}{rcl} A_j : E_j & \longrightarrow & \ell_{q'_j} \\ x_j & \longmapsto & (a_{jk}(x_j))_{k=1}^{\infty} \end{array}$$

It is clear that A_j is linear and continuous since

$$\parallel A_j \parallel \leq w_{q'_j}(a_{jk}) \qquad \forall j = 1, \dots, n.$$

Now, define $Y \in \mathcal{L}(\ell_p; F)$ as

$$Y((\varepsilon_j)_{j=1}^{\infty}) = \sum_{j=1}^{\infty} \varepsilon_j y_j.$$

For $(\varepsilon_j)_{j=1}^{\infty} \in \ell_p$, and $\varphi \in F'$ we have

$$< \varphi, Y((\varepsilon_j)_{j=1}^{\infty}) >= \sum_{j=1}^{\infty} \varepsilon_j < \varphi, y_j > .$$

Then, it is clear that

$$|< \varphi, Y((\varepsilon_j)_{j=1}^{\infty}) >| \leq \sum_{j=1}^{\infty} | \varepsilon_j || < \varphi, y_j >| .$$

Applying the Hlder's inequality we get

$$|<\varphi, Y((\varepsilon_j)_{j=1}^{\infty})>|\leq (\sum_{j=1}^{\infty}|\varepsilon_j|^p)^{\frac{1}{p}}(\sum_{j=1}^{\infty}|<\varphi, y_j>|^{p'})^{\frac{1}{p'}}$$

and from this relation it is clear that

$$\sup_{\|\varphi\|\leq 1}|<\varphi, Y((\varepsilon_j)_{j=1}^{\infty})>|\leq\|(\varepsilon_j)_{j=1}^{\infty}\|_{\ell_p}\sup_{\|\varphi\|\leq 1}(\sum_{j=1}^{\infty}|<\varphi, y_j>|^{p'})^{\frac{1}{p'}}$$

and

$$\| Y((\varepsilon_j)_{j=1}^{\infty})\|\leq\|(\varepsilon_j)_{j=1}^{\infty}\|_{\ell_p} w_{p'}(y_j).$$

Therefore Y is well defined and $\| Y\|\leq w_{p'}(y_j)$.

Besides, it is clear that

$$Y\circ S_0\circ A=\sum_{k=1}^{\infty}\sigma_k a_{1k}\times\cdots\times a_{nk}\otimes y_k=S$$

In (33) we can choose a representation for S such that for $\epsilon>0$ one can get

$$\ell_r(\sigma_k)w_{p'}(y_k)\prod_{j=1}^{n}w_{q'_j}(a_{jk})<(1+\epsilon)N_{(r,p;q_1,\ldots,q_n)}(S)$$

Therefore

$$\| Y\|\prod_{k=1}^{n}\| A_k\|\ \ell_r(\sigma_k)\leq\ell_r(\sigma_k)w_{p'}(y_k)\prod_{j=1}^{n}w_{q'_j}(a_{jk})<(1+\epsilon)N_{(r,p;q_1,\ldots,q_n)}(S)$$

As it is valid for all $\epsilon>0$ we have

$$\| Y\|\prod_{j=1}^{n}\| A_j\|\ \ell_r(\sigma_k)\leq N_{(r,p;q_1,\ldots,q_n)}(S).$$

From this relation we have

$$\inf\| Y\|\prod_{j=1}^{n}\| A_j\|\ \ell_r(\sigma_k)\leq N_{(r,p;q_1,\ldots,q_n)}(S). \tag{34}$$

$\Leftarrow$) It is enough to show that S_0 is an operator $(r,p,q_1,\ldots,q_n)-$nuclear

$$S_0=\sum_{j=1}^{\infty}\sigma_j\Pi_{1j}\times\cdots\times\Pi_{nj}\otimes e_j,$$

where $\Pi_{kj}((\varepsilon_{k,m})_{m=1}^{\infty})=\varepsilon_{k,j}$, $\forall k=1,\ldots,n$ and $j\in\mathbb{N}$, and $e_j=(0,\ldots,0,1,0,\ldots)$ with 1 in the j-th component. So, let us show the following:

(i) $(\sigma_j)_{j=1}^{\infty}\in\ell_r$;

(ii) $(\Pi_{kj})_{j=1}^{\infty} \in \ell^w_{q'_k}(\ell_{q_k}) \quad \forall k = 1, 2, \ldots, n$;

(iii) $(e_j)_{j=1}^{\infty} \in \ell^w_{p'}(\ell_p)$.

Properties (i) and (iii) can be directly proved. In order to show (ii) we proceed as follows

$$\begin{aligned} w_{q'_k}(\Pi_{kj}) &= \sup_{\substack{\|a_k\|\leq 1,\\ a_k\in\ell_{q'_k}}} (\sum_{j=1}^{\infty} |< a_k, \Pi_{kj} >|^{q'_k})^{\frac{1}{q'_k}} \\ &= \sup_{\substack{\|a_k\|\leq 1,\\ a_k\in\ell_{q'_k}}} (\sum_{j=1}^{\infty} \| a_{k,j} \|^{q'_k})^{\frac{1}{q'_k}} = 1, \quad k = 1, ..., n. \end{aligned} \tag{35}$$

From the last relation one has $w_{q'_k}(\Pi_{kj}) = 1$, $\forall k = 1, \ldots, n$. Therefore

$$\begin{aligned} N_{(r,p;q_1,\ldots,q_n)}(S_0) &\leq \ell_r(\sigma_j) \prod_{j=1}^{n} w_{q'_j}(\Pi_{kj}) w_{p'}(e_j) \\ N_{(r,p;q_1,\ldots,q_n)}(S_0) &\leq \ell_r(\sigma_j), \end{aligned} \tag{36}$$

where we have used $w_{p'}(e_j) \leq 1$.

Then, $Y \circ S_0 \circ A$ is $(r, p; q_1, \ldots, q_n)-$nuclear due to the multi-linear operator ideals properties. Besides, we have that

$$N_{(r,p;q_1,\ldots,q_n)}(Y \circ S_0 \circ A) \leq \| Y \| \prod_{i=1}^{n} \| A_i \| N_{(r,p;q_1,\ldots,q_n)}(S_0). \tag{37}$$

From (36) and (37) we have that

$$N_{(r,p,q_1,\ldots,q_n)}(S) \quad \leq \quad \| Y \| \prod_{i=1}^{n} \| A_i \| \ell_r(\sigma_j), \tag{38}$$

$$N_{(r,p,q_1,\ldots,q_n)}(S) \quad \leq \quad \inf \| Y \| \prod_{i=1}^{n} \| A_i \| \ell_r(\sigma_j). \tag{39}$$

Then, from (34) and (39) we have that

$$N_{(r,p;q_1,\ldots,q_n)}(S) = \inf \| Y \| \prod_{i=1}^{n} \| A_i \| \ell_r(\sigma_j). \qquad \square$$