International Journal of Pure and Applied Mathematics
Volume 56 No. 1 2009, 7-12

Invited Lecture Delivered at
Fifth International Conference of Applied Mathematics
and Computing (Plovdiv, Bulgaria, August 12–18, 2008)

SINGULAR PERTURBATION METHOD IN ODE

A. Shayganmanesh (Golbabai)
Department of Applied Mathematics
Faculty of Mathematics
Iran University of Science and Technology
Narmak, Tehran, 16844, IRAN
e-mail: golbabai@iust.ac.ir

Abstract: In this paper, we consider in which region the perturbations are operative over very narrow region across which the dependent variables undergo very rapid changes. These narrow regions frequently adjoin the boundaries of the domain of interest, owing to the fact that the small parameter multiplies the highest derivatives, consequently, they are usually referred to as boundary layers.

The main aim of the present work is to establish fairly wide conditions under which the asymptotic matching principle [4] is correct. Two expansions are given in which one tries to recover the missing data by exploiting the fact that the two series, which are to be asymptotic expansion (in more or less adjacent regions) of the unknown function $f(x, \varepsilon)$ must somehow be related to each other. Finally we have shown that the structure of the composite expansion based on matching principle is in good agreement with exact solution of the problem.

AMS Subject Classification: 76R05
Key Words: inner layer, asymptotic expansion, singular perturbation

1. Introduction

Consider an initial or boundary value problem, for a function $f(x, \varepsilon)$ of a scalar

Received: August 13, 2008

or vector variable x and a small parameter ε, such that for $\varepsilon \to 0$ no single asymptotic expansion series of Poincare form describes f uniformly for all values of x in the domain D_x of interest. Such problems are often called, singular-perturbation problem.

In the method of matched expansions one seeks in the first instance two complementary asymptotic series which are to approximate to f in different parts of D_x. The prototype situation is that of the flow past a solid boundary of a fluid with small viscosity: the two regions are then an outer region of inviscid flow and inner boundary layer [5]. But at least one of the two problem implicitly defining these expansions is always incompletely posed, due to the loss of initial or boundary condition at points outside the region in question as we will encounter in this work. In its simplest form, the method of matched expansion proceeds as flows.

Suppose that a problem, say $P_*(f_*, x_*, \varepsilon_*)$, consisting of a differential equation and initial or boundary conditions for an unknown function f_* of the scalar independent variable x_* and the small parameter ε_*, involves a "singular perturbation" in the sense that no single asymtotic expansion of Poincare form, say

$$\sum_{n=0}^{N} \rho_n(\varepsilon_x) f_{*n}(x_*), \quad \text{where} \quad \rho_{n+1} = o(l_n) \quad \text{for} \quad \varepsilon_x \to 0,$$

describes f_* with an error which is $O(\rho_N)$ uniformly on the whole interval x_* We assume, we can find a preliminary transformation.

$$\varepsilon_* = \varepsilon_*(\varepsilon) \qquad x_* = x_*(x, \varepsilon), \quad f_* = (f, x, \varepsilon),$$

such that the transformed problem $P(f, x, \varepsilon)$ has the following properties.

(a) The variables x and ε are real; $\varepsilon \downarrow 0$ as $\varepsilon_* \to 0$; and the domain to be considered is

$$D : \varepsilon\lambda \leq x \leq l, \quad 0 < \varepsilon \leq \varepsilon_*.$$

Here λ and l are independent of ε, with $l > 0$, and ε_0 is as small as we wish (but fixed). In many problems $\lambda = 0$, or $x \leq l$ is replaced by $x < \infty$.

(b) The problem $P(f, x, \varepsilon)$ implies that under the outer limiting process (Figure 1) $\varepsilon \downarrow_0$ with x fixed and $0 < x_0 \leq x < l$, where x_0 is an arbitrarily small positive number, but is dependent of ε, the function has outer expansions to $p + 1$ terms of the form

$$f(x, \varepsilon) = \sum_{m=0}^{p} a_m(\varepsilon) f_m(x) + O\{a_{p+1}(\varepsilon); \quad x_0 \leq x \leq l\}. \tag{1.1}$$

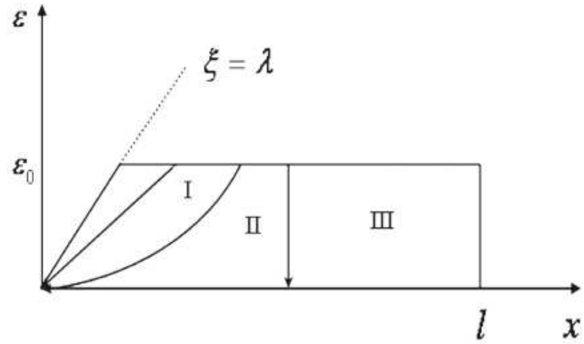

Figure 1: Notation. The paths I, II and III correspond to the inner, intermediate and outer limiting processes, respectively

Here $\{a_m\}(m = 0, 1, 2, \ldots, M)$ is some asymptotic sequence; we have $0 \le l \le M - 1$; and the qualification in the O symbol is to imply the existence of a number $A(x, p)$ independent of x and ε, such that

$$|f - \sum_{m=0}^{p} a_m f_m| \le A \mid a_{p+1} \mid \quad on \quad 0 < x_0 \le x \le l, \quad 0 < \varepsilon \le \varepsilon_0$$

(c) Under the stretching transformation $x = \varepsilon\xi$ or multiple scal, the problem becomes $Q(f, \xi, t)$, and this form of the problem implies, that under the inner limiting process

$$\varepsilon \downarrow 0 \quad \text{with} \quad \xi \quad \text{fixed and} \quad \lambda \le \xi \le \xi_0 < \infty,$$

where ξ_0 is an arbitrarily large number, but independent of ε, the function f has inner expansions to $q + 1$ terms of the form

$$f(x, \varepsilon) = \sum_{n=0}^{q} \beta_n(\varepsilon) g_n(\xi) + O\{\beta_{q+1}(\varepsilon); \quad \lambda \le \xi \le \xi_0\}. \tag{1.2}$$

Here $\{\beta_n\}$ $(n = 0, 1, 2, \ldots, N)$, is an asymptotic sequence, and we have $0 \le q \le N - 1$.

(d) In some sense, which we shall try to make more precise in an example, the inner expansion corrects the failure of the outer expansion for $x \downarrow 0$. Similarly the outer expansion corrects the failure of the inner expansion for $\xi \to \infty$. Note that ε is to be so chosen that $x = \varepsilon^n \xi$ is an appropriate stretching or contraction transformation. The following simple example can be handled in many different ways and it is treated here by the method of matched asymptotic expansions.

2. Boundary Value Problem

We consider, the simple boundary. value problem

$$\begin{aligned} &\varepsilon y'' + xy' - xy = 0, \\ &y(0) = \alpha, y(1) = \beta, \qquad 0 \leq \varepsilon \ll 1, \end{aligned} \tag{2.1}$$

where ε is small dimensionless positive number. It is assumed that the equation and the boundary conditions have been made dimensionless. Since the coefficient of y' is positive, we expect the boundary layer to be at the left end even if x vanishes there. If this is not the way, the resulting expansions cannot be matched and the results will not be mathematically consistant. Seeking an outer expansion in the form

$$y^0 = y_0(x) + \varepsilon y_1(x) + \dots,$$

we find from (2.1) that $y_0 = c_0 e^x$ since the boundary layer is assumed to be at the origin, the outer expansion must satisfy $y(1) = \beta$ or $y^0(1) = \beta$. Hence, $y_0(1) = \beta$ and $\beta = c_0 e$, then $y_0 = \beta e^{x-1}$ and

$$y^0 = \beta e^{x-1} + \dots. \tag{2.2}$$

To investigate the boundary layer at the origin, we introduce the stretching transformation.

$$\xi = \frac{x}{\varepsilon^n} \quad , \quad n > 0 \tag{2.3}$$

in (2.1) and obtain

$$\varepsilon^{1-2n} \frac{d^2 y^i}{d\xi^2} + \xi \frac{dy^i}{d\xi} - \varepsilon^n \xi y^i = 0. \tag{2.4}$$

As $\varepsilon \to 0$, the limiting form of (2.4) depends on the value of n. Only the distinguished limit is chosen. In this case, the distinguished limit is

$$\frac{d^2 y^i}{d\xi^2} + \xi \frac{dy^i}{d\xi} = 0. \tag{2.5}$$

Corresponding to $n = \frac{1}{2}$. Equation (2.4) is solvable and we have

$$y^i = a_0 \int_0^\xi e^{-(\frac{1}{2})\tau^2} d\tau + b,$$

where the lower limit in the integral was taken to be zero to facilitate satisfaction of the boundary condition. Imposing the boundary condition, $y^i(0) = \alpha$ leads to

$$y^i = a_0 \int_0^\xi e^{-(\frac{1}{2})\tau^2} d\tau + \alpha,$$

where the constant a_0 needs to be determined by matching the inner and outer expansions, see Vandyke [4]. To match the one-term outer expansion (2.2) with the one-term inner expansion (2.5), we proceed as follows:

One-term outer expansion: $y \approx \beta e^{x-1}$.

Rewritten in inner variable: $= \beta \exp(\varepsilon^{\frac{1}{2}}\xi) - 1$.

Expand for small $\varepsilon := \beta e^{-1}(1 + \varepsilon^{\frac{1}{2}}\xi + \ldots)$.

$$\beta e^{-1} \quad \text{one-term inner expansion.} \tag{2.6}$$

One-term inner expansion: $y \approx a_0 \int_0^{\xi} e^{-(\frac{1}{2})\tau^2} d\tau + \alpha$.

Rewritten in outer variable $:= a_0 \int_0^{x/\varepsilon^{\frac{1}{2}}} \exp -(\frac{1}{2}\tau^2) d\tau + \alpha$.

Expend for small $\varepsilon : a_0 \int_0^{\infty} e^{-\frac{1}{2}\tau^2 d\tau} + \alpha + \ldots$.

$$a_0\sqrt{\frac{\pi}{2}} + \alpha \quad \text{one-term after expansion.} \tag{2.7}$$

Hence, the outer and inner expansions are matchable and the results are mathematically consistent, thereby justifying our assumption that the boundary layer is at the origin. Substituting for a_0 in (2.6), we have

$$y^i = \alpha + \sqrt{\frac{\pi}{2}}(\beta e^{-1} - \alpha) \int_0^{\xi} \exp(-\frac{1}{2}\tau^2) d\tau + \ldots. \tag{2.8}$$

Finally, we determine a single composite uniform expansion by adding the outer expansion (2.2) to the inner expansion (2.9) and subtracting from the result their common part (2.6), see [4]. Thus we have,

$$y^c = \beta e^{x-1} + \alpha + \sqrt{\frac{2}{\pi}}(\beta e^{-1} - \alpha) \int_0^{\xi} e^{-(\frac{1}{2})\tau^2} d\tau - \beta e^{-1},$$

or

$$y^e = \beta e^{x-1} - (\beta e^{-1} - \alpha)\left(1 - \sqrt{\tfrac{2}{\pi}} \int_0^{\xi} e^{-(\frac{1}{2})\tau^2} d\tau \right) + \ldots. \tag{2.9}$$

Figure 2 shows that the composite expansion y^0 is very close to the exact solution y^e even for $\varepsilon = 0.2$. When $\varepsilon = 0.1$, the composite expansion is indistinguishable from the exact solution.

3. Conclusion

The method of matched asymptotic expansion has been taken into account with regard a boundary value problem in which it is shown that the boundary layer is at origin. The outer expansion and the inner expansion are found to establish the structure of the composite expansion. Finally the exact solution

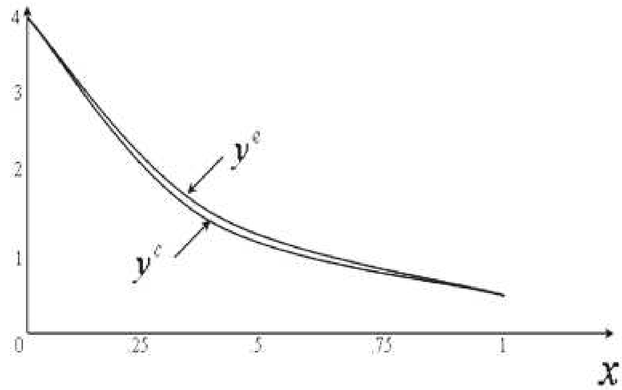

Figure 2: Comparison of the composite expansion y^c with the exact solution y^e obtained by numerically (2.1) for $\varepsilon = 0.2$, $\alpha = 4.0$, $\beta = 1.0$

of the problem is compared with the composite expansion and they are in good agreement.

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