International Journal of Pure and Applied Mathematics

Volume 56 No. 1 2009, 21-36

FUZZY, ROUGH AND ROUGH FUZZY IDEALS IN TERNARY SEMIGROUPS

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Abstract: The formal definition of a ternary semigroup was given by Lehmer in 1932, but earlier such structures was studied by Kasner and Prüfer. Any semigroup can be reduced to a ternary semigroup but a ternary semigroup does not necessarily reduce to a semigroup. The notion of fuzzy sets was introduced by Zadeh in 1965 and that of rough sets by Pawlak in 1982. In this paper, we study fuzzy, rough and rough fuzzy ternary subsemigroups (left ideals, right ideals, lateral ideals, ideals) of ternary semigroups.

AMS Subject Classification: 20N10, 08A72, 03E72, 20M12
Key Words: fuzzy sets, rough sets, ternary semigroups, ternary subsemigroups, left ideals, right ideals, lateral ideals, ideals

1. Introduction and Preliminaries

The formal definition of a ternary semigroup was given by Lehmer [7] in 1932, but earlier such structures was studied by Kasner [4] and Prüfer [11]. Lehmer [7] gave the definition of ternary semigroups in 1932. A nonempty set T is called a *ternary semigroup* [7] if there exists a ternary operation $T \times T \times T \to T$, written as $(x_1, x_2, x_3) \mapsto x_1x_2x_3$ satisfying the following identity for any $x_1, x_2, x_3, x_4, x_5 \in T, [[x_1x_2x_3]x_4x_5] = [x_1[x_2x_3x_4]x_5] = [x_1x_2[x_3x_4x_5]]$.

We can see that any semigroup can be reduced to a ternary semigroup.

Received: August 27, 2008

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Banach showed that a ternary semigroup does not necessarily reduce to a semigroup by this example.

Example 1.1. (see [7]) $T = \{-i, 0, i\}$ is a ternary semigroup while T is not a semigroup under the multiplication over complex numbers.

Example 1.2. $T = \mathbb{Z}^-$ is a ternary semigroup while T is not a semigroup under the multiplication over integers.

However, Los [8] proved that every ternary semigroup can be embedded in a semigroup.

Let T be a ternary semigroup. For nonempty subsets A, B and C of T, let $ABC := \{abc \mid a \in A, b \in B \text{ and } c \in C\}$. A nonempty subset S of T is called a *ternary subsemigroup* [1] if $SSS \subseteq S$. Sioson [9] studied ideal theory in ternary semigroups. Now we give the definition of ideals of ternary semigroups. A nonempty subset A of a ternary semigroup T is called a *left ideal* of T if $TTA \subseteq A$, a *right ideal* of T if $ATT \subseteq A$ and a *lateral ideal* of T if $TAT \subseteq A$. If A is a left, right and lateral ideal of T, A is called an *ideal* of T.

The notion of fuzzy sets was introduced by Zadeh [13] and that of rough sets by Pawlak [10]. Several researchs were conducted on the generalizations of the notion of fuzzy sets and rough sets. Let T be a ternary semigroup. A function f from T to the unit interval $[0, 1]$ is called a *fuzzy subset* of T. The ternary semigroup T itself is a fuzzy subset of T such that $T(x) = 1$ for all $x \in T$, denoted also by T. If $A \subseteq T$, the *characteristic function* f_A of A is a fuzzy subset of T defined as follows:

$$f_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

In this paper, we study fuzzy, rough and rough fuzzy ternary subsemigroups (left ideals, right ideals, lateral ideals, ideals) of ternary semigroups.

2. Fuzzy Ideals of Ternary Semigroups

In this section, we study fuzzy ternary subsemigroups, fuzzy left ideals, fuzzy right ideals, fuzzy lateral ideals and fuzzy ideals of ternary semigroups analogous to fuzzy subsemigroups, fuzzy left ideals, fuzzy right ideals and fuzzy ideals of semigroups. Theorems in this section are consequence of the transfer principle described in [5].

Now we define fuzzy ternary subsemigroups, fuzzy left ideals, fuzzy right ideals, fuzzy lateral ideals and fuzzy ideals of ternary semigroups. Let T be a ternary semigroup. A fuzzy subset f of T is called

a *fuzzy ternary subsemigroup* of T if $f(xyz) \geq \min\{f(x), f(y), f(z)\}$ for all $x, y, z \in T$,

a *fuzzy left ideal* of T if $f(xyz) \geq f(z)$ for all $x, y, z \in T$,

a *fuzzy right ideal* of T if $f(xyz) \geq f(x)$ for all $x, y, z \in T$,

a *fuzzy lateral ideal* of T if $f(xyz) \geq f(y)$ for all $x, y, z \in T$ and

a *fuzzy ideal* of T if $f(xyz) \geq \max\{f(x), f(y), f(z)\}$ for all $x, y, z \in T$.

Theorem 2.1. *Let T be a ternary semigroup and A a nonempty subset of T. The following statements are true.*

(1) A is a ternary subsemigroup of T if and only if f_A is a fuzzy ternary subsemigroup of T.

(2) A is a left ideal (right ideal, lateral ideal, ideal) of T if and only if f_A is a fuzzy left ideal (fuzzy right ideal, fuzzy lateral ideal, fuzzy ideal) of T.

Proof. (1) Assume that A is a ternary subsemigroup of T. Let $x, y, z \in T$.

Case 1. $x, y, z \in A$. Since A is a ternary subsemigroup of T, $xyz \in A$. Then $f_A(xyz) = 1 \geq \min\{f(x), f(y), f(z)\}$.

Case 2. $x \notin A$ or $y \notin A$ or $z \notin A$. Thus $f_A(x) = 0$ or $f_A(y) = 0$ or $f_A(z) = 0$. Therefore $\min\{f_A(x), f_A(y), f_A(z)\} = 0 \leq f_A(xyz)$.

Conversely, let $x, y, z \in A$. So $f_A(x) = f_A(y) = f_A(z) = 1$. Since f_A is a fuzzy ternary subsemigroup of T, $f_A(xyz) \geq \min\{f_A(x), f_A(y), f_A(z)\} = 1$. Hence $xyz \in A$.

(2) Assume that A is a left ideal of T. Let $x, y, z \in T$.

Case 1. $z \in A$. Since A is a left ideal of T, $xyz \in A$. Then $f_A(xyz) = 1$. Therefore $f_A(xyz) \geq f_A(z)$.

Case 2. $z \notin A$. So $f_A(z) = 0$. Hence $f_A(xyz) \geq f_A(z)$.

Conversely, let $x, y \in T$ and $z \in A$. Since f_A is a fuzzy left ideal of T and $z \in A$, $f_A(xyz) \geq f_A(z) = 1$. Thus $xyz \in A$.

The other parts can be seen in similarly way. □

Let T be a ternary semigroup. A nonempty subset S of T is called a *prime subset* of T if for all $x, y, z \in T, xyz \in S$ implies $x \in S$ or $y \in S$ or $z \in S$. A ternary subsemigroup S of T is called a *prime ternary subsemigroup* of T if S is a prime subset of T. *Prime left ideals, prime right ideals, prime lateral ideals* and *prime ideals* of T are defined analogously. A fuzzy subset f of T is called a *prime fuzzy subset* of T if $f(xyz) \leq \max\{f(x), f(y), f(z)\}$ for all $x, y, z \in T$. A fuzzy ternary subsemigroup f of T is called a *prime fuzzy ternary subsemigroup* of T if f is a prime fuzzy subset of T. *Prime fuzzy left ideals, prime fuzzy right ideals, prime fuzzy lateral ideals* and *prime fuzzy ideals* of T are defined analogously.

Theorem 2.2. *Let T be a ternary semigroup and A a nonempty subset of T. The following statements are true.*

(1) A is a prime subset of T if and only if f_A is a prime fuzzy subset of T.

(2) A is a prime ternary subsemigroup (prime left ideal, prime right ideal, prime lateral ideal, prime ideal) of T if and only if f_A is a prime fuzzy ternary subsemigroup (prime fuzzy left ideal, prime fuzzy right ideal, prime fuzzy lateral ideal, prime fuzzy ideal) of T.

Proof. (1) Assume that A is a prime subset of T. Let $x, y, z \in T$.

Case 1. $xyz \in A$. Since A is prime, $x \in A$ or $y \in A$ or $z \in A$. Thus $\max\{f_A(x), f_A(y), f_A(z)\} = 1 \geq f_A(xyz)$.

Case 2. $xyz \notin A$. Thus $f_A(xyz) = 0 \leq \max\{f_A(x), f_A(y), f_A(z)\}$.

Conversely, let $x, y, z \in T$ such that $xyz \in A$. Thus $f_A(xyz) = 1$. Since f_A is prime, $\max\{f_A(x), f_A(y), f_A(z)\} = 1$. This implies $f_A(x) = 1$ or $f_A(y) = 1$ or $f_A(z) = 1$. Hence $x \in A$ or $y \in A$ or $z \in A$.

(2) follows from (1) and Theorem 1. □

Let f be a fuzzy subset of a set (a ternary semigroup) T. For any $t \in [0, 1]$, the set

$$f_t = \{x \in T \mid f(x) \geq t\} \text{ and } f_t^s = \{x \in T \mid f(x) > t\}$$

are called a *t-levelset* and a *t-strong levelset* of f, respectively [12].

Theorem 2.3. *Let f be a fuzzy subset of a ternary semigroup T. The following statements are true.*

(1) f is a fuzzy ternary subsemigroup of T if and only if for all $t \in [0, 1]$, if $f_t \neq \emptyset$, then f_t is a ternary subsemigroup of T.

(2) f is a fuzzy left ideal (fuzzy right ideal, fuzzy lateral ideal, fuzzy ideal) of T if and only if for all $t \in [0, 1]$, if $f_t \neq \emptyset$, then f_t is a left ideal (right ideal, lateral ideal, ideal) of T.

Proof. (1) Assume that f is a fuzzy ternary subsemigroup of T. Let $t \in [0, 1]$ such that $f_t \neq \emptyset$. Let $x, y, z \in f_t$. So $f(x), f(y), f(z) \geq t$. Thus $\min\{f(x), f(y), f(z)\} \geq t$. Since f is a fuzzy ternary subsemigroup of T, $f(xyz) \geq t$. Hence $xyz \in f_t$.

Conversely, let $x, y, z \in T$. Choose $t = \min\{f(x), f(y), f(z)\}$. Then $f(x), f(y),\ f(z) \geq t$. Thus $x, y, z \in f_t$. Since f_t is a ternary subsemigroup of T, $xyz \in f_t$. Thus $f(xyz) \geq t = \min\{f(x), f(y), f(z)\}$.

(2) Assume that f is a fuzzy left ideal of T. Let $t \in [0,1]$. Suppose that $f_t \neq \emptyset$. Let $x, y \in T$ and $z \in f_t$. Thus $f(xyz) \geq f(z) \geq t$. Therefore $xyz \in f_t$.

Conversely, let $x, y, z \in T$. Choose $t = f(z)$. Thus $z \in f_t$, this implies $f_t \neq \emptyset$. By assumption, we have f_t is a left ideal of T. So $xyz \in f_t$. Therefore $f(xyz) \geq t$. Thus $f(xyz) \geq f(z)$.

The other parts can be proved in a similar way. □

Theorem 2.4. *Let f be a fuzzy subset of a ternary semigroup T. The following statements are true.*

(1) f is a prime fuzzy subset of T if and only if for all $t \in [0,1]$, if $f_t \neq \emptyset$, then f_t is a prime subset of T.

(2) f is a prime fuzzy ternary subsemigroup (prime fuzzy left ideal, prime fuzzy right ideal, prime fuzzy lateral ideal, prime fuzzy ideal) of T if and only if for all $t \in [0,1]$, if $f_t \neq \emptyset$, then f_t is a prime ternary subsemigroup (prime left ideal, prime right ideal, prime lateral ideal, prime ideal) of T.

Proof. (1) Assume that f is a prime fuzzy subset of T. Let $t \in [0,1]$. Suppose that $f_t \neq \emptyset$. Let $x, y, z \in T$ such that $xyz \in f_t$. Thus $f(xyz) \geq t$. Since f is prime, $f(x) \geq t$ or $f(y) \geq t$ or $f(z) \geq t$. This implies $x \in f_t$ or $y \in f_t$ or $z \in f_t$.

Conversely, let $x, y, z \in T$. Choose $t = f(xyz)$. Then $xyz \in f_t$. Since f_t is prime, $x \in f_t$ or $y \in f_t$ or $z \in f_t$. Then $f(x) \geq t$ or $f(y) \geq t$ or $f(z) \geq t$. Hence $\max\{f(x), f(y), f(z)\} \geq t = f(xyz)$.

(2) follows from (1) and Theorem 2.3. □

Theorem 2.5. *Let f be a fuzzy subset of a ternary semigroup T. Then f is a fuzzy ternary subsemigroup (fuzzy left ideal, fuzzy right ideal, fuzzy lateral ideal, fuzzy ideal) of T if and only if for all $t \in [0,1]$, if $f_t^s \neq \emptyset$, then f_t^s is a ternary subsemigroup (left ideal, right ideal, lateral ideal, ideal) of T.*

Proof. The proof of this theorem is similar to the proof of Theorem 2.3. □

Theorem 2.6. *Let f be a fuzzy subset of a ternary semigroup T. Then f is a prime fuzzy subset (prime fuzzy ternary subsemigroup, prime fuzzy left ideal, prime fuzzy right ideal, prime fuzzy lateral ideal, prime fuzzy ideal) of T if and only if for all $t \in [0,1]$, if $f_t^s \neq \emptyset$, then f_t^s is a prime subset (prime ternary subsemigroup, prime left ideal, prime right ideal, prime lateral ideal,* *prime ideal) of T.*

Proof. The proof of this theorem is similar to the proof of Theorem 2.4. □

3. Rough Ideals of Ternary Semigroups

In this section, we study rough ternary subsemigroups, rough left ideals, rough right ideals, rough lateral ideals and rough ideals of ternary semigroups analogous to rough subsemigroups, rough left ideals, rough right ideals and rough ideals of semigroups considered by Kuroki [6].

Kar and Maity [3] studied congruences on ternary semigroups. Let T be a ternary semigroup. A *congruence* ρ on T is an equivalence relation on T such that for all $a, b, x, y \in T$, $(a, b) \in \rho$ implies $(xya, xyb) \in \rho, (xay, xby) \in \rho$ and $(axy, bxy) \in \rho$. For $a \in T$, the ρ-congruence class containing a denoted by $[a]_\rho$. A congruence ρ of T is called *complete* if $[a]_\rho[b]_\rho[c]_\rho = [abc]_\rho$ for all $a, b, c \in T$. Let ρ be a congruence on T and A a nonempty subset of T. The sets

$$\rho_-(A) = \{x \in T \mid [x]_\rho \subseteq A\} \text{ and } \rho^-(A) = \{x \in T \mid [x]_\rho \cap A \neq \emptyset\}$$

are called the ρ-*lower* and ρ-*upper approximations* of A, respectively.

Example 3.1. Define a relation ρ on a ternary semigroup $\mathbb{Z}^-$ under the usual multiplication by

$$x\rho y \leftrightarrow 2 \mid x - y \text{ for all } x, y \in \mathbb{Z}^-.$$

Then ρ is a congruence on $\mathbb{Z}^-$ and $\mathbb{Z}^-/\rho = \{[-1]_\rho, [-2]_\rho\}$. Let $A = \{-2, -4\}$. We have that $\rho_-(A) = \emptyset$ and $\rho^-(A) = [-2]_\rho$.

Proposition 3.1. *Let ρ and λ be congruences on a ternary semigroup T and A and B nonempty subsets of T. The following statements are true.*

(1) $\rho_-(A) \subseteq A \subseteq \rho^-(A)$.

(2) $\rho^-(A \cup B) = \rho^-(A) \cup \rho^-(B)$.

(3) $\rho_-(A \cap B) = \rho_-(A) \cap \rho_-(B)$.

(4) $A \subseteq B$ *implies* $\rho_-(A) \subseteq \rho_-(B)$.

(5) $A \subseteq B$ *implies* $\rho^-(A) \subseteq \rho^-(B)$.

(6) $\rho_-(A) \cup \rho_-(B) \subseteq \rho_-(A \cup B)$.

(7) $\rho^-(A \cap B) \subseteq \rho^-(A) \cap \rho^-(B)$.

(8) $\rho \subseteq \lambda$ *implies* $\lambda_-(A) \subseteq \rho_-(A)$.

(9) $\rho \subseteq \lambda$ *implies* $\rho^-(A) \subseteq \lambda^-(A)$.

Proof. The proof of this theorem is similar to the proof of Theorem 2.1 in Kuroki [6]. □

Theorem 3.2. *Let* ρ *be a complete congruence on a ternary semigroup* T *and* A, B *and* C *nonempty subsets of* T. *Then (1)* $\rho^-(A)\rho^-(B)\rho^-(C) \subseteq \rho^-(ABC)$ *and (2)* $\rho_-(A)\rho_-(B)\rho_-(C) \subseteq \rho_-(ABC)$.

Proof. (1) Let $t \in \rho^-(A)\rho^-(B)\rho^-(C)$. Then $t = abc$ for some $a \in \rho^-(A), b \in \rho^-(B)$ and $c \in \rho^-(C)$. Thus $x, y, z \in T$ such that $x \in [a]_\rho \cap A, y \in [b]_\rho \cap B$ and $z \in [c]_\rho \cap C$. Since ρ is complete, $xyz \in [a]_\rho[b]_\rho[c]_\rho = [abc]_\rho$. Since $x \in A, y \in B$ and $z \in C$, $xyz \in ABC$. Hence $xyz \in [abc]_\rho \cap ABC$. Then $t = abc \in \rho^-(ABC)$.

(2) Let $t \in \rho_-(A)\rho_-(B)\rho_-(C)$. Then $t = abc$ for some $a \in \rho_-(A), b \in \rho_-(B)$ and $c \in \rho_-(C)$. Thus $[a]_\rho \subseteq A, [b]_\rho \subseteq B$ and $[c]_\rho \subseteq C$. Since ρ is complete, $[abc]_\rho = [a]_\rho[b]_\rho[c]_\rho \subseteq ABC$. So $abc \in \rho_-(ABC)$. □

A nonempty subset A of a ternary semigroup T is called a *ρ-upper rough ternary subsemigroup* of T if $\rho^-(A)$ is a ternary subsemigroup of T and A is called a *ρ-lower rough ternary subsemigroup* of T if $\rho_-(A)$ is a ternary subsemigroup of T. *ρ-upper* and *ρ-lower left ideals (right ideals, lateral ideals, ideals)* of T are defined analogously.

Theorem 3.3. *Let* ρ *be a congruence on a ternary semigroup* T *and* A *a nonempty subset of* T. *The following statements are true.*

(1) If A *is a ternary subsemigroup of* T, *then* A *is a* ρ*-upper rough ternary subsemigroup of* T.

(2) If A *is a left ideal (right ideal, lateral ideal, ideal) of* T, *then* A *is a* ρ*-upper rough left ideal (*ρ*-upper rough right ideal,* ρ*-upper rough lateral ideal,* ρ*-upper rough ideal) of* T.

Proof. (1) Assume A is a ternary subsemigroup of T. By Proposition 3.1(1), $\rho^-(A) \neq \emptyset$. By Theorem 3.2 and Proposition 3.1(5), we have $\rho^-(A)\rho^-(A)\rho^-(A) \subseteq \rho^-(AAA) \subseteq \rho^-(A)$.

(2) Assume A is a left ideal of T. Then $\rho^-(A) \neq \emptyset$. By Theorem 3.2 and Proposition 3.1(5), we have $TT\rho^-(A) = \rho^-(T)\rho^-(T)\rho^-(A) \subseteq \rho^-(TTA) \subseteq \rho^-(A)$.

The other parts can be proved in a similar way. □

However, the converse of Theorem 3.3 is not true in general. For example, we can see in Example 3.1, $\rho^-(A)$ is a ternary subsemigroup (left ideal, right ideal, lateral ideal, ideal) of $\mathbb{Z}^-$ but A is not.

Theorem 3.4. *Let ρ be a complete congruence on a ternary semigroup T and A a nonempty subset of T such that $\rho_-(A) \neq \emptyset$. If A is a ternary subsemigroup (left ideal, right ideal, lateral ideal, ideal) of T, then A is a ρ-lower rough ternary subsemigroup (ρ-lower rough left ideal, ρ-lower rough right ideal, ρ-lower rough lateral ideal, ρ-lower rough ideal) of T.*

Proof. The proof of this theorem is similar to Theorem 3.3. □

Note that the converse of Theorem 3.4 is not true in general.

Theorem 3.5. *Let ρ be a congruence on a ternary semigroup T. If A, B and C are a right ideal, a lateral ideal and a left ideal of T, respectively, then*

(1) $\rho^-(ABC) \subseteq \rho^-(A) \cap \rho^-(B) \cap \rho^-(C)$ and

(2) $\rho_-(ABC) \subseteq \rho_-(A) \cap \rho_-(B) \cap \rho_-(C)$.

Proof. By assumption, $ABC \subseteq ATT \cap TBT \cap TTC \subseteq A \cap B \cap C$.

(1) By Proposition 3.1(5) and (7), we have

$$\rho^-(ABC) \subseteq \rho^-(A \cap B \cap C) \subseteq \rho^-(A) \cap \rho^-(B) \cap \rho^-(C).$$

(2) By Proposition 3.1(3) and (4), we have

$$\rho_-(ABC) \subseteq \rho_-(A \cap B \cap C) = \rho_-(A) \cap \rho_-(B) \cap \rho_-(C).$$ □

4. Rough Sets with Respect to the Ternary Idempotent Congruences

In this section, we study rough sets with respect to the ternary idempotent congruence of ternary semigroups analogous to rough sets with respect to the idempotent congruence of semigroups considered by Kuroki [6].

Proposition 4.1. *Let T be a ternary semigroup and ρ a complete congruence on T. Then $T/\rho = \{[a]_\rho \mid a \in T\}$ is a ternary semigroup under the ternary operation by $[a]_\rho[b]_\rho[c]_\rho = [abc]_\rho$ for all $a, b, c \in T$.*

Let ρ be a complete congruence on a ternary semigroup T. The ternary semigroup T/ρ is called a *quotient ternary semigroup of T by a congruence ρ*.

Example 4.1. Let I be an ideal of a ternary semigroup T. Define the relation ρ_I on T by

$$a\rho_I b \Leftrightarrow a, b \in I \text{ or } a = b \text{ for all } a, b \in T.$$

Then ρ_I is a congruence on T. The congruence ρ_I is called a *Rees congruence on T determined by an ideal I*. We have that $T/\rho_I = \{I\} \cup \{\{a\} \mid a \notin I\}$. This quotient ternary semigroup T/ρ_I is called a *Rees quotient ternary semigroup of T by an ideal I*.

A ternary semigroup T is called a *ternary idempotent semigroup* if $a^3 = a$ for all $a \in T$. A congruence ρ on T is called a *ternary idempotent congruence* on T if the quotient ternary semigroup T/ρ is a ternary idempotent semigroup. A nonempty subset P of a ternary semigroup T is called *semiprime* if for $a \in T$, $a^3 \in P$ implies $a \in P$.

Theorem 4.2. *Let ρ be a complete ternary idempotent congruence on a ternary semigroup T and A a nonempty subset of T. Then $\rho^-(A)$ is semiprime.*

Proof. Let $a \in T$ such that $a^3 \in \rho^-(A)$. Since ρ is a ternary idempotent congruence on T,

$$[a]_\rho \cap A = [a]_\rho[a]_\rho[a]_\rho \cap A = [a^3]_\rho \cap A \neq \emptyset.$$

Thus $a \in \rho^-(A)$. Hence $\rho^-(A)$ is semiprime. □

Theorem 4.3. *Let ρ be a ternary idempotent on a ternary semigroup T and A, B and C nonempty subset of T. The following statements are true.*

(1) $\rho^-(A) \cap \rho^-(B) \cap \rho^-(C) \subseteq \rho^-(ABC)$.

(2) $\rho_-(A) \cap \rho_-(B) \cap \rho_-(C) \subseteq \rho_-(ABC)$.

Proof. (1) Let $x \in \rho^-(A) \cap \rho^-(B) \cap \rho^-(C)$. Then there exist $a, b, c \in T$ such that $a, b, c \in [x]_\rho, a \in A, b \in B$ and $c \in C$. Since ρ is a ternary idempotent congruence of T, $abc \in [x]_\rho[x]_\rho[x]_\rho = [x]_\rho$. And since $abc \in ABC$, $abc \in [x]_\rho \cap ABC$. This implies that $c \in \rho^-(ABC)$.

(2) Let $x \in \rho_-(A) \cap \rho_-(B) \cap \rho_-(C)$. Thus $[x]_\rho \subseteq A, [x]_\rho \subseteq B$ and $[x]_\rho \subseteq C$. Since ρ is a ternary idempotent congruence of T, $[x]_\rho = [x]_\rho[x]_\rho[x]_\rho \subseteq ABC$. Hence $x \in \rho_-(ABC)$. □

Corollary 4.4. *Let ρ be a ternary idempotent congruence on a ternary semigroup T. If A, B and C are a right ideal, a lateral ideal and a left ideal of T, respectively, then:*

(1) $\rho^-(ABC) = \rho^-(A) \cap \rho^-(B) \cap \rho^-(C)$ and

(2) $\rho_-(ABC) = \rho_-(A) \cap \rho_-(B) \cap \rho_-(C)$.

Proof. This follows from Theorem 3.5 and Theorem 4.3. □

5. Rough Sets in a Quotient Ternary Semigroup

In this section, we study rough sets in a quotient ternary semigroup analogous to rough sets in a quotient semigroup considered by Kuroki [6].

Let ρ be a congruence on a ternary semigroup T. The ρ-lower and ρ-upper approximations can be presented in an equivalent form as shown below:

$$\rho_{-}(A)/\rho = \{[x]_\rho \in T/\rho \mid [x]_\rho \subseteq A\} \text{ and}$$

$$\rho^{-}(A)/\rho = \{[x]_\rho \in T/\rho \mid [x]_\rho \cap A \neq \emptyset\},$$

respectively. Now we discuss these sets as subsets of a quotient ternary semigroup T/ρ.

Theorem 5.1. *Let ρ be a complete congruence on a ternary semigroup T. The following statements are true.*

(1) If A is a ternary subsemigroup of T, then $\rho^{-}(A)/\rho$ is a ternary subsemigroup of T/ρ.

(2) If A is a left ideal (right ideal, lateral ideal, ideal) of T, then $\rho^{-}(A)/\rho$ is a left ideal (right ideal, lateral ideal, ideal) of T/ρ.

Proof. (1) Let $[a]_\rho, [b]_\rho, [c]_\rho \in \rho^{-}(A)/\rho$. Then there exist $x \in [a]_\rho \cap A, y \in [b]_\rho \cap A$ and $z \in [c]_\rho \cap A$. Since ρ is complete, $xyz \in [a]_\rho[b]_\rho[c]_\rho = [abc]_\rho$. Since A is a ternary subsemigroup of T, $xyz \in A$. Then $[abc]_\rho \cap A \neq \emptyset$. Hence $[a]_\rho[b]_\rho[c]_\rho \in \rho^{-}(A)/\rho$.

(2) is similar to (1) □

Theorem 5.2. *Let ρ be a complete congruence on a ternary semigroup T and A a nonempty subset of T such that $\rho_{-}(A)/\rho \neq \emptyset$. The following statements are true.*

(1) If A is a ternary subsemigroup of T, then $\rho_{-}(A)/\rho$ is a ternary subsemigroup of T/ρ.

(2) If A is a left ideal (right ideal, lateral ideal, ideal) of T, then $\rho_{-}(A)/\rho$ is a left ideal (right ideal, lateral ideal, ideal) of T/ρ.

Proof. (1) Let $[a]_\rho, [b]_\rho, [c]_\rho \in \rho_{-}(A)/\rho$. Then $[a]_\rho \subseteq A, [b]_\rho \subseteq A$ and $[c]_\rho \subseteq A$. Since A is a ternary subsemigroup of T, $[a]_\rho[b]_\rho[c]_\rho \subseteq A$. Hence $[a]_\rho[b]_\rho[c]_\rho \in \rho_{-}(A)/\rho$.

(2) is similar to (1). □

6. Rough Fuzzy Ideals of Ternary Semigroups

In this section, we study rough fuzzy ternary subsemigroups, left ideals, right ideals, lateral ideals and ideals of ternary semigroups analogous to rough fuzzy ideals in a semigroup considered by Xiao and Zhang [12].

Let f be a fuzzy subset of a ternary semigroup. Then the sets

$$\rho^-(f)(x) = \sup_{a\in[x]_\rho} f(a) \text{ and } \rho_-(f)(x) = \inf_{a\in[x]_\rho} f(a)$$

are called the *ρ-upper* and *ρ-lower approximations* of a fuzzy set f, respectively [2].

Example 6.1. Define a relation ρ on a ternary semigroup $\mathbb{Z}^-$ under the usual multiplication by

$$x\rho y \leftrightarrow 2 \mid x - y \text{ for all } a, b \in \mathbb{Z}^-.$$

It is easy to see that ρ is a complete congruence on $\mathbb{Z}^-$. Let $f(x) = \dfrac{1}{-x}$ for all $x \in \mathbb{Z}^-$. Then

$$\rho_-(f)(x) = 0 \text{ for all } x \in \mathbb{Z}^-$$

and

$$\rho^-(f)(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Z}^- \text{ is odd,} \\ \dfrac{1}{2} & \text{if } x \in \mathbb{Z}^- \text{ is even.} \end{cases}$$

Lemma 6.1. *Let ρ be a congruence on a ternary semigroup T, f a fuzzy subset of T and $t \in [0,1]$, then (1) $(\rho_-(f))_t = \rho_-(f_t)$ and (2) $(\rho^-(f))^s_t = \rho^-(f^s_t)$.*

Proof. (1) Let $x \in (\rho_-(f))_t$. Then $\rho_-(f)(x) \geq t$. So $\inf_{a\in[x]_\rho} f(a) \geq t$. Therefore $f(a) \geq t$ for all $a \in [x]_\rho$. This implies $[x]_\rho \subseteq f_t$. Therefore $x \in \rho_-(f_t)$.

Conversely, assume $x \in \rho_-(f_t)$. Thus $[x]_\rho \subseteq f_t$. Then $f(a) \geq t$ for all $a \in [x]_\rho$. This implies $\inf_{a\in[x]_\rho} f(a) \geq t$. Thus $\rho_-(f)(x) \geq t$. Hence $x \in (\rho_-(f))_t$.

(2) Let $x \in (\rho^-(f))^s_t$. Then $\rho^-(f)(x) > t$. So $\sup_{a\in[x]_\rho} f(a) > t$. Therefore $f(a) > t$ for some $a \in [x]_\rho$. This implies $[x]_\rho \cap f^s_t \neq \emptyset$. Therefore $x \in \rho^-(f^s_t)$.

Conversely, assume $x \in \rho^-(f^s_t)$. Thus $[x]_\rho \cap f^s_t \neq \emptyset$. Then $f(a) > t$ for some $a \in [x]_\rho$. This implies $\inf_{a\in[x]_\rho} f(a) > t$. Thus $\rho^-(f)(x) > t$. Hence $x \in (\rho^-(f))^s_t$. □

Theorem 6.2. *Let ρ be a complete congruence on a ternary semigroup T. If f is a fuzzy ternary subsemigroup (fuzzy left ideal, fuzzy right ideal, fuzzy lateral ideal, fuzzy ideal) of T, then $\rho^-(f)$ and $\rho_-(f)$ are fuzzy ternary subsemigroups (fuzzy left ideals, fuzzy right ideals, fuzzy lateral ideals, fuzzy* *ideals) of T.*

Proof. By Theorem 2.3, Theorem 2.5, Theorem 3.3, Theorem 3.4 and Lemma 6.1, we can obtain the conclusion easily. □

The converse of Theorem 6.2 is not true. For example, we can see in Example 6.1, $\rho_-(f)$ is a fuzzy ternary subsemigroup (fuzzy left ideal, fuzzy right ideal, fuzzy later ideal, fuzzy ideal) of T but f is not.

7. Problems of Homomorphisms

Let T_1 and T_2 be ternary semigroups. A mapping φ from T_1 to T_2 is called a *homomorphism* from T_1 to T_2 if $\varphi(abc) = \varphi(a)\varphi(b)\varphi(c)$ for all $a, b, c \in T_1$. Then the set $\kappa = \{(a, b) \in T_1 \times T_1 \mid \varphi(a) = \varphi(b)\}$ is called the *kernel* of φ. We have that κ is a congruence on T_1.

Theorem 7.1. *Let T_1 and T_2 be two ternary semigroups and φ a homomorphism from T_1 to T_2. If A is a nonempty subset of T_1, then $\varphi(\kappa^-(A)) = \varphi(A)$.*

Proof. Since $A \subseteq \kappa^-(A), \varphi(A) \subseteq \varphi(\kappa^-(A))$. To prove the converse, let $y \in \varphi(\kappa^-(A))$. Then there exists $x \in \kappa^-(A)$ such that $\varphi(x) = y$. Thus there exists $a \in T_1$ such that $a \in [x]_\kappa \cap A$. Then $a \in [x]_\kappa$ and $a \in A$. So $\varphi(a) = \varphi(x)$. Therefore $y = \varphi(x) = \varphi(a) \in \varphi(A)$. □

Lemma 7.2. *Let φ be an onto homomorphism from a ternary semigroup T_1 to a ternary semigroup T_2, ρ_2 a congruence on T_2, $\rho_1 = \{(x, y) \in T_1 \times T_1 \mid (\varphi(x), \varphi(y)) \in \rho_2\}$ and A a nonempty subset of T_1. The following statements are true.*

(1) ρ_1 is a congruence on T_1.

(2) If ρ_2 is complete and φ is 1-1, then ρ_1 is complete.

(3) $\varphi(\rho_1^-(A)) = \rho_2^-(\varphi(A))$.

(4) $\varphi(\rho_{1-}(A)) \subseteq \rho_{2-}(\varphi(A))$.

(5) If φ is 1-1, then $\varphi(\rho_{1-}(A)) = \rho_{2-}(\varphi(A))$.

Proof. (1) Clearly.

(2) Let $x_1, x_2, x_3 \in T_1$. Then $[x_1]_{\rho_1}[x_2]_{\rho_1}[x_3]_{\rho_1} \subseteq [x_1x_2x_3]_{\rho_1}$. To prove the other hand, let $a \in [x_1x_2x_3]_{\rho_1}$. Since ρ_2 is complete, $\varphi(a) \in \varphi([x_1x_2x_3]_{\rho_1}) = [\varphi(x_1x_2x_3)]_{\rho_2} = [\varphi(x_1)]_{\rho_2}[\varphi(x_2)]_{\rho_2}[\varphi(x_3)]_{\rho_2}$. Since φ is an onto homomorphism, there exist $b_1, b_2, b_3 \in T_1$ such that $\varphi(b_1) \in [\varphi(x_1)]_{\rho_2}$, $\varphi(b_2) \in [\varphi(x_2)]_{\rho_2}$,

$\varphi(b_3) \in [\varphi(x_3)]_{\rho_2}$ and $\varphi(a) = \varphi(b_1)\varphi(b_2)\varphi(b_3) = \varphi(b_1 b_2 b_3)$. Since φ is 1-1, $b_1 \in [x_1]_{\rho_1}, b_2 \in [x_2]_{\rho_1}, b_3 \in [x_3]_{\rho_1}$ and $a = b_1 b_2 b_3$. Thus $a \in [x_1]_{\rho_1}[x_2]_{\rho_1}[x_3]_{\rho_1}$. Therefore ρ_1 is complete.

(3) Let $y \in \varphi(\rho_1^-(A))$. Then there exists $x \in \rho_1^-(A)$ such that $\varphi(x) = y$. So $[x]_{\rho_1} \cap A \neq \emptyset$. Let $z \in [x]_{\rho_1} \cap A$. Then $\varphi(z) \in \varphi(A)$. By the definition of ρ_1, we have $\varphi(z) \in [\varphi(x)]_{\rho_2}$. So $[\varphi(x)]_{\rho_2} \cap \varphi(A) \neq \emptyset$, Thus $y = \varphi(x) \in \rho_2^-(\varphi(A))$.

Conversely, let $y \in \rho_2^-(\varphi(A))$. Then there exists $x \in T_1$ such that $\varphi(x) = y$. Thus $[\varphi(x)]_{\rho_2} \cap \varphi(A) \neq \emptyset$. So there exists $z \in A$ such that $\varphi(z) \in \varphi(A)$ and $\varphi(z) \in [\varphi(x)]_{\rho_2}$. By the definition of ρ_1, we have $z \in [x]_{\rho_1}$. Thus $[x]_{\rho_1} \cap A \neq \emptyset$. Then $x \in \rho_1^-(A)$. So $y = \varphi(x) \in \varphi(\rho_1^-(A))$.

(4) Let $y \in \varphi(\rho_{1-}(A))$. Then there exists $x \in \rho_{1-}(A)$ such that $\varphi(x) = y$. Thus $[x]_{\rho_1} \subseteq A$. Let $z \in [y]_{\rho_2}$. Then there exists $w \in T_1$ such that $\varphi(w) = z$ and $\varphi(w) \in [\varphi(x)]_{\rho_2}$. Hence $w \in [x]_{\rho_1}$. This implies $w \in A$. So $z = \varphi(w) \in \varphi(A)$. Thus $[y]_{\rho_2} \subseteq \varphi(A)$. Therefore $y \in \rho_{2-}(\varphi(A))$.

(5) Assume φ is 1-1. Let $y \in \rho_{2-}(\varphi(A))$. Then there exists $x \in T_1$ such that $\varphi(x) = y$ and $[\varphi(x)]_{\rho_2} \subseteq \varphi(A)$. Let $z \in [x]_{\rho_1}$. Then $\varphi(z) \in [\varphi(x)]_{\rho_2} \subseteq \varphi(A)$. So $z \in A$. Thus $[x]_{\rho_1} \subseteq A$. Then $x \in \rho_{1-}(A)$. Therefore $y = \varphi(x) \in \varphi(\rho_{1-}(A))$. □

Theorem 7.3. *Let φ be an onto homomorphism from a ternary semigroup T_1 to a ternary semigroup T_2, ρ_2 a congruence on T_2, $\rho_1 = \{(x, y) \in T_1 \times T_1 \mid (\varphi(x), \varphi(y)) \in \rho_2\}$ and A a nonempty subset of T_1. The following statements are true.*

(1) $\rho_1^-(A)$ is a ternary subsemigroup of T_1 if and only if $\rho_2^-(\varphi(A))$ is a ternary subsemigroup of T_2.

(2) $\rho_1^-(A)$ is a left ideal (right ideal, later ideal, ideal) of T_1 if and only if $\rho_2^-(\varphi(A))$ is a left ideal (right ideal, later ideal, ideal) of T_2.

Proof. (1) Assume that $\rho_1^-(A)$ is a ternary subsemigroup of T_1. So

$$\varphi(\rho_1^-(A))\varphi(\rho_1^-(A))\varphi(\rho_1^-(A))) = \varphi(\rho_1^-(A)\rho_1^-(A)\rho_1^-(A)) \subseteq \varphi(\rho_1^-(A)).$$

By Lemma 7.2(3), $\rho_2^-(\varphi(A))\rho_2^-(\varphi(A))\rho_2^-(\varphi(A)) \subseteq \rho_2^-(\varphi(A))$. So $\rho_2^-(\varphi(A))$ is a ternary subsemigroup of T_2.

Conversely, assume that $\rho_2^-(\varphi(A))$ is a ternary subsemigroup of T_2. By Lemma 7.2(3), $\varphi(\rho_1^-(A))\varphi(\rho_1^-(A))\varphi(\rho_1^-(A))) = \rho_2^-(\varphi(A))\rho_2^-(\varphi(A))\rho_2^-(\varphi(A)) \subseteq \rho_2^-(\varphi(A)) = \varphi(\rho_1^-(A))$. Therefore $\rho_1^-(A)$ is a ternary subsemigroup of T_1.

(2) is similar to (1). □

Theorem 7.4. *Let φ be an isomorphism from a ternary semigroup T_1*

to a ternary semigroup T_2, ρ_2 a congruence on T_2, $\rho_1 = \{(x,y) \in T_1 \times T_1 \mid (\varphi(x), \varphi(y)) \in \rho_2\}$ and A a nonempty subset of T_1. The following statements are true.

(1) $\rho_1^-(A)$ is a prime subset of T_1 if and only if $\rho_2^-(\varphi(A))$ is a prime subset of T_2.

(2) $\rho_1^-(A)$ is a prime ternary subsemigroup (prime left ideal, prime right ideal, prime lateral ideal, prime ideal) of T_1 if and only if $\rho_2^-(\varphi(A))$ is a prime ternary subsemigroup (prime left ideal, prime right ideal, prime lateral ideal, prime ideal) of T_2.

Proof. (1) Assume that $\rho_1^-(A)$ is a prime subset of T_1. Let $x, y, z \in T_2$ such that $xyz \in \rho_2^-(\varphi(A))$. Since φ is onto, there exist $a, b, c \in T_1$ such that $\varphi(a) = x, \varphi(b) = y$ and $\varphi(c) = z$. Then $\varphi(abc) \in \rho_2^-(\varphi(A))$. By Lemma 7.2(3), we have $\varphi(abc) \in \varphi(\rho_1^-(A))$. Since φ is one-to-one, $abc \in \rho_1^-(A)$. Since $\rho_1^-(A)$ is prime, $a \in \rho_1^-(A)$ or $b \in \rho_1^-(A)$ or $c \in \rho_1^-(A)$. Then $x = \varphi(a) \in \varphi(\rho_1^-(A)) = \rho_2^-(\varphi(A))$ or $y = \varphi(b) \in \varphi(\rho_1^-(A)) = \rho_2^-(\varphi(A))$ or $z = \varphi(c) \in \varphi(\rho_1^-(A)) = \rho_2^-(\varphi(A))$. Similarly, the other hand holds.

(2) follows from (1) and Theorem 7.3. □

Theorem 7.5. *Let φ be an isomorphism from a ternary semigroup T_1 to a ternary semigroup T_2, ρ_2 a congruence on T_2, $\rho_1 = \{(x,y) \in T_1 \times T_1 \mid (\varphi(x), \varphi(y)) \in \rho_2\}$ and A a nonempty subset of T_1. The following statements are true.*

(1) $\rho_{1-}(A)$ is a ternary subsemigroup of T_1 if and only if $\rho_{2-}(\varphi(A))$ is a ternary subsemigroup of T_2.

(2) $\rho_{1-}(A)$ is a left ideal (right ideal, later ideal, ideal) of T_1 if and only if $\rho_{2-}(\varphi(A))$ is a left ideal (right ideal, later ideal, ideal) of T_2.

Proof. By Lemma 7.2(5), we have $\varphi(\rho_{1-}(A) = \rho_{2-}(\varphi(A))$. The proof of this theorem is similar to the proof of Theorem 7.3. □

Theorem 7.6. *Let φ be an isomorphism from a ternary semigroup T_1 to a ternary semigroup T_2, ρ_2 a congruence on T_2, $\rho_1 = \{(x,y) \in T_1 \times T_1 \mid (\varphi(x), \varphi(y)) \in \rho_2\}$ and A a nonempty subset of T_1. The following statements are true.*

(1) $\rho_{1-}(A)$ is a prime subset of T_1 if and only if $\rho_{2-}(\varphi(A))$ is a prime subset of T_2.

(2) $\rho_{1-}(A)$ is a prime ternary subsemigroup (prime left ideal, prime right ideal, prime lateral ideal, prime ideal) of T_1 if and only if $\rho_{2-}(\varphi(A))$ is a prime

ternary subsemigroup (prime left ideal, prime right ideal, prime lateral ideal, prime ideal) of T_2*.*

Proof. By Lemma 7.2(5), we have $\varphi(\rho_{1-}(A) = \rho_{2-}(\varphi(A))$. The proof of this theorem is similar to the proof of Theorem 7.4. □

We can obtain the following conclusion easily in a quotient ternary semigroup.

Corollary 7.7. *Let* φ *be an isomorphism from a ternary semigroup* T_1 *to a ternary semigroup* T_2*,* ρ_2 *a complete congruence on* T_2*,* $\rho_1 = \{(x, y) \in T_1 \times T_1 \mid (\varphi(x), \varphi(y)) \in \rho_2\}$ *and* A *a nonempty subset of* T_1*. The following statements are true.*

(1) $\rho_{1-}(A)/\rho_1$ *is a ternary subsemigroup (left ideal, right ideal, lateral ideal, ideal) of* T_1 *if and only if* $\rho_{2-}(\varphi(A))/\rho_2$ *is a ternary subsemigroup (left ideal, right ideal, lateral ideal, ideal) of* T_2*.*

(2) $\rho_1^-(A)/\rho_1$ *is a ternary subsemigroup (left ideal, right ideal, lateral ideal, ideal) of* T_1 *if and only if* $\rho_2^-(\varphi(A))/\rho_2$ *is a ternary subsemigroup (left ideal, right ideal, lateral ideal, ideal) of* T_2*.*

(3) $\rho_{1-}(A)/\rho_1$ *is a prime ternary subsemigroup (prime left ideal, prime right ideal, prime lateral ideal, prime ideal) of* T_1 *if and only if* $\rho_{2-}(\varphi(A))/\rho_2$ *is a prime ternary subsemigroup (prime left ideal, prime right ideal, prime lateral ideal, prime ideal) of* T_2*.*

(4) $\rho_1^-(A)/\rho_1$ *is a prime ternary subsemigroup (prime left ideal, prime lateral ideal, prime ideal) of* T_1 *if and only if* $\rho_2^-(\varphi(A))/\rho_2$ *is a prime ternary subsemigroup (prime left ideal, prime right ideal, prime lateral ideal, prime ideal) of* T_2*.*

Acknowledgments

Most of the work in this paper is a part of the research written by Miss Pattarawan Petchkhaew under the supervision of Assistant Professor Dr.Ronnason Chinram. Miss Pattarawan Petchkhaew would like to thank him for helpful suggestions and would like to thank DSPT for supports through the education grant.

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