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APPROXIMATE METHOD OF THE OPTIMAL CONTROL FOR SOME SYSTEMS WITH SMALL PARAMETER AND TIME LAG

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Abstract: The present article considers the problem of approximation of optimal control and optimal trajectory for a dynamic controlled system with small parameter and time lag from a given initial position to a given final one at quadratical criteria. The sequences of controls and trajectories approximating the optimal ones with any degree of accuracy were constructed and example for the method demonstration was given.

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1. Introduction and Previous Results

Time delays are frequently encountered in the behavior of many physical processes and very often are the main cause for poor performance and instability of control systems. The significant growth of interest in such systems is due to their various applications in control theory and automatic regulation [4], biology and ecology [2] and [3], biomedicine [1] and others. In view of this, time delay systems is a topic of great practical importance which attracted a great deal of interest for several decades; see [6]. The investigation of systems with time lag is associated with a number of specific singularities. One of them is the absence of simple expressions for the translation operator along the system trajectories. In view of this, it is difficult to extend some results of the control theory of ordinary equations to systems with time lag.

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The purpose of this paper is to continue the investigation of the system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + \epsilon f(t,\, x(t-h)),$$
$$0 \le t \le T,\;\; x(0) = a_0,\;\; x(\theta) = \phi(\theta),\;\; \theta \in [-h,\, 0), \tag{1}$$

started in [10], which transferred from the initial state a_0 to the final one a_1, so that the criterion

$$J(u) = \int_0^T |u(s)|^2\, ds \tag{2}$$

is minimized.

We know from Pontryagin's maximum principle [8, p. 233] that the optimal pair $x(t)$, $u(t)$ is a solution of the boundary value problem consisting of the system (1) and the relations

$$x(0) = a_0,\;\; x(T) = a_1,\;\; u(t) = \frac{1}{2}B'(t)\psi(t),$$
$$\dot{\psi}(t) = -A'(t)\psi(t) - \epsilon f_1(t+h,\, x(t))\psi(t+h), \tag{3}$$
$$\psi(s) \equiv 0,\;\; s > T.$$

Here $\psi(t)$ is a vector of the conjugate variables of the maximum principle, and a square matrix f_1 as i-tuple has partial derivatives of $f(t,\, x(t-h))$ with respect to $x_i(t-h)$. We remind to the reader that in [10] the convergence of the successive approximations $x_k(t)$ and $\psi_k(t)$ was established, that are solutions of the following boundary-value problem

$$\dot{x}_k(t) = A(t)x_k(t) + B(t)u_k(t) + \epsilon f(t,\, x_{k-1}(t-h)),$$
$$x_k(0) = a_0,\;\; x_k(T) = a_1,\;\; u_k(t) = \frac{1}{2}B'(t)\psi_k(t), \tag{4}$$
$$0 \le t \le T,\;\; x_k(\theta) = \phi(\theta),\;\; \theta \in [-h,\, 0),$$

and

$$\dot{\psi}_k(t) = -A'(t)\psi_k(t) - \epsilon f_1(t+h,\, x_{k-1}(t))\psi_{k-1}(t+h),$$
$$\psi_k(s) \equiv 0,\;\; s > T. \tag{5}$$

For $x_k(t)$ and $\psi_k(t)$ the exact formulas were derived in the form

$$x_k(t) = z(t,\, 0)a_0 + \int_0^t z(t,\, s)\left[\frac{1}{2}B_1(s)\psi_k(s) + \epsilon f(s,\, x_{k-1}(s-h))\right] ds,$$
$$\tag{6}$$
$$\psi_k(t) = z'(t,\, 0)^{-1}\psi_k(0) - \epsilon \int_0^t z'(t,\, s)^{-1} f_1(s+h,\, x_{k-1}(s))\psi_{k-1}(s+h)\, ds.$$

The Main Previous Result. *If $x(t)$, $\psi(t)$ is the solution to the boundary value problem (1), (3), then*

$$|x(s) - x_k(s)| + |\psi(t) - \psi_k(t)| \le \frac{1}{1-\epsilon m_6}(\epsilon m_6)^k \epsilon m_7,$$
$$-h \le s \le T, \quad 0 \le t \le T, \quad (7)$$

where the constants m_6 and m_7 can be easily estimated in terms of the parameters of (1).

2. Approximation of the Optimal Control and Optimal Trajectory with any Degree of Accuracy

The previous results show that sequence x_k, built according to (6), approximate the optimal trajectory $x(t)$ with an accuracy defined by the right-hand side of (7). In view of (3), (4), (5), the sequence $\frac{1}{2}B'(t)\psi(t)$, is also given in (6), and approximate the optimal control so, that approximation error is given by (7). From here, and boundedness of ψ, ψ_k, we obtain that the approximation error has the order ϵ^{k+1}.

Now, we start to learn an important question. Fix some k, and use approximate control $u_k(t) = \frac{1}{2}B'(t)\psi_k(t)$ in initial system (1). What is the deviation of trajectory $x(t, u_k)$ from the optimal one?

Let $x(t)$, $\psi(t)$ is a solution of the boundary value problem (1), (3), so $x(t)$ denotes the optimal trajectory. Define $y(t)$ as a solution of the following equation

$$\dot{y}(t) = A(t)y(t) + \epsilon f(t, y(t-h)) + \frac{1}{2}B_1(t)\psi_k(t), \quad (8)$$
$$y(0) = a_0, \;\; y(\theta) = \psi(\theta), \;\; \theta \in [-h, 0),$$

where $\psi_k(t)$ is defined by (6). First, we show that there exist solution of (9) in the domain $|y| \le n_0 + \epsilon\sigma$, where the constants n_0 and σ were defined easily in [10] in terms of the initial system (1). Suppose, $y_0(t)$ satisfies for $k \ge 1$ the equation

$$\dot{y}_0(t) = A(t)y_0(t) + \epsilon f(t, x_{k-1}(t-h)) + \frac{1}{2}B_1(t)\psi_k(t), \quad (9)$$
$$y_0(0) = a_0, \;\; y_0(\theta) = \psi(\theta), \;\; \theta \in [-h, 0).$$

Here $x_{k-1}(t)$ is defined by (6). For $k = 0$, the function $y_0(t)$ still is given by (10) with $\epsilon = 0$. Now, we construct successive approximations $y_i(t)$, $i = 1, 2, ...$ according to

$$\dot{y}_i(t) = A(t)y_i(t) + \epsilon f(t, y_{i-1}(t-h)) + \frac{1}{2}B_1(t)\psi_k(t), \quad (10)$$
$$y_i(0) = a_0, \;\; y_i(\theta) = \psi(\theta), \;\; \theta \in [-h, 0).$$

With the help of (10) and (6), we conclude that $y_0(t) = x_k(t)$. Then, $|y_0(t)| \leq n_0 + \epsilon\sigma$. From here and (11), and the definition of σ, it follows that for any i, $|y_i(t)| \leq n_0 + \epsilon\sigma$. From uniformly boundedness of the sequence $y_i(t)$ follows the existence of the solution of the Cauchy problem (9). Now, it is not difficult to estimate the desired difference $x(t) - y(t) = r(t)$. According to (1) and (9) we obtain

$$r(t) = \epsilon \int_0^t z(t,s) \left[f(s,\, x(s-h)) - f(s,\, y(s-h)) \right] ds$$
$$+ \frac{1}{2} \int_0^t z(t,s) B_1(s) \left[\psi(s) - \psi_k(s) \right] ds.$$

Since f satisfies the Lipshitz condition with the constant L, we obtain

$$|r(t)| \leq \epsilon L \int_0^t \|z(t,\, s)\| \cdot |r(s-h)|\, ds$$
$$+ \frac{1}{2} \int_0^t \|z(t,\, s)\| \cdot \|B_1(s)\| \cdot |\psi(s) - \psi_k(s)|\, ds. \quad (11)$$

From here and (7) with the help of Gronwall-Bellman Lemma [5], follows desired estimation for $r(t)$:

$$|r(t)| \leq m_8 (\epsilon m_6)^k \epsilon, \quad (12)$$

where the constant m_8 is easily given by the initial system coefficients. From (12) we see that the vector $y(T)$ differs from a_1, but deviation of the trajectory $y(t)$ from the final state a_1 has the order ϵ^{k+1}.

3. Example

As an illustration, we consider the dynamics of the laboratory population of the green fly *Lucilia Cuprina*. The mathematical model of this population was built in [9], [7]. We denote by $x(t)$ population density at time t, and $u(t)$ is the feed quantity that population receive at the same time. Then, according to [7]

$$\dot{x}(t) = kpu(t) - Cx(t) - \frac{1}{2} mkpx(t-h), \quad (13)$$

where C is the mortality of adult individuals, k is the effectiveness of ova lay, m is the necessary quantity of feed for existence, p is probability that each ovum will be developed to adult individual, h is the time necessary that the ovum will develops to individual.

We know initial population density $x(0) = x_0$ and desired one x_1. As a control we choose $u(t)$, that is the feed quantity that population receive at the time t. The problem is to find $u(t)$ with the minimal norm (2) so, that to move the population in final time T to state x_1. In this case, we have

$$A(t) = -C, \quad B(t) = kp, \quad \epsilon = \frac{1}{2}mp, \quad f(t, x(t-h)) = -kx(t-h).$$

Here successive approximations for the optimal trajectory and optimal control may be obtained according to formulas (6). We write the relations for null approximations:

$$x_0(t) = \exp\{-Ct\} x_0 + \frac{\exp\{CT\} x_1 - x_0}{\exp\{2CT\} - 1} \left[\exp\{Ct\} - \exp\{-Ct\}\right],$$

$$u_0(t) = \frac{2C \exp\{Ct\}}{kp} \cdot \frac{\exp\{CT\} x_1 - x_0}{\exp\{2CT\} - 1}.$$

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