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ON THE L-ORDER OF MEROMORPHIC FUNCTIONS BASED ON RELATIVE SHARING

Sanjib Kumar Datta[1 §], Sourav Kar[2]

[1]Department of Mathematics
University of North Bengal
Raja Rammohunpur, District Darjeeling
PIN Code: 734013, West Bengal, INDIA
e-mail: sanjib_kr_datta@yahoo.co.in

[2]Department of Mathematics
Siliguri Institute of Technology
District Sukna, Darjeeling, PIN Code: 734009, West Bengal, INDIA
e-mail: sourav_kar_nbu@yahoo.co.in

Abstract: In this paper we discuss the equality of L-orders (L^*-orders) of meromorphic functions based on relative sharing.

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1. Introduction, Definitions and Notations

Let f and g be two non constant meromorphic functions defined in the open complex plane $\mathbb{C}$ and let $a \in \mathbb{C} \cup \{\infty\}$. If $f - a$ and $g - a$ have the same zeros CM (counting multiplicities) and IM (ignoring multiplicities) then we say that f and g share the value a CM or IM respectively. Similarly f, g share ∞ CM or IM means that $\frac{1}{f}, \frac{1}{g}$ share 0 CM or IM respectively.

Banerjee and Dutta [1] introduced the following definition based on the idea of relative sharing of values of two meromorphic functions with respect to another meromorphic function.

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§Correspondence address: 25, School Road, Kalianibash, Barrackpore, P.O.: Nonachandanpukur, Dist.: 24 Parganas (North), P.S.: Titagarh, PIN Code: 743102, West Bengal, INDIA

Definition 1. (see Banerjee and Dutta [1]) Let f and g be two non-constant meromorphic functions and $a \in \mathbb{C} \cup \{\infty\}$. We say that f, g share a CM (IM) relatively with respect to a meromorphic function h, provided the functions F and G share a CM (IM), where $F = \frac{f}{h}$ and $G = \frac{g}{h}$.

The purpose of this definition of relative sharing of values of two meromorphic functions f and g in the paper is to study some properties of f and g by that of F and G constructed with the help of a suitably chosen meromorphic function h.

Somasundaram and Thamizharasi [4] introduced the notions of L-order and L-lower order for entire functions where $L = L(r)$ is a positive continuous function increasing slowly in the sense that $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant a. The definitions of L-order and L-lower order for meromorphic functions are as follows:

Definition 2. The L-order ρ_f^L and L-lower order λ_f^L of a meromorphic function f are defined as follows:

$$\rho_f^L = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log[rL(r)]} \quad \text{and} \quad \lambda_f^L = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log[rL(r)]}.$$

The more generalised concept of L-order and L-lower order of meromorphic functions are respectively L^*-order and L^*-lower order which are as follows:

Definition 3. The L^*-order and L^*-lower order of a meromorphic function f respectively denoted by $\rho_f^{L^*}$ and $\lambda_f^{L^*}$ are defined as follows:

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log[re^{L(r)}]} \quad \text{and} \quad \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log[re^{L(r)}]}.$$

Banerjee and Dutta [1] proved the equality of orders of two meromorphic functions based on relative sharing of values of them with respect to another meromorphic function.

In the paper we establish some results on the equality of L-orders (L-lower orders) and L^*-orders (L^*-lower orders) of two meromorphic functions based on relative sharing of values of them with respect to another meromorphic function.

We do not explain the standard notations and definitions of the theory of entire and meromorphic functions because those are available in Hayman [3]. In the sequel we use the following notation: $\exp^{[k]} x = \exp(\exp^{[k-1]} x)$ for $k = 1, 2, 3, \ldots$ and $\exp^{[0]} x = x$.

2. A Lemma

In this section we present a lemma which will be needed in the sequel.

Lemma 1. (see Gundersen [2]) *If f and g share three values IM then*

$$\frac{1}{3}T(r,g)\,(1+o(1)) \leq T(r,f) \leq 3T(r,g)\,(1+o(1))$$

as $r \to \infty$ possibly outside a set E of finite linear measure.

3. Main Theorems

In this section we present the main results of the paper.

Theorem 1. *Let f and g be two non-constant meromorphic functions. If there is a function h with $T(r,h) = o\,(T(r,f))$ and $T(r,h) = o\,(T(r,g))$ such that F, G share a_1, a_2, a_3 IM then $\rho_f^L = \rho_g^L$ and $\lambda_f^L = \lambda_g^L$, where $F = \frac{f}{h}$ and $G = \frac{g}{h}$.*

Proof. As F, G share a_1, a_2, a_3 IM, in view of Lemma 1 we get that

$$\frac{1}{3}T(r,G)\,(1+o(1)) \leq T(r,F) \leq 3T(r,G)\,(1+o(1))\,.$$

From which we have

$$\rho_F^L = \rho_G^L. \tag{1}$$

Now $F = \frac{f}{h}$ gives that

$$T(r,F) \leq T(r,f) + T(r,h) + O(1) \leq T(r,f)\,(1+o(1)) + O(1).$$

Hence $\rho_F^L \leq \rho_f^L$.

Again from $f = hF$ we obtain that $\rho_f^L \leq \rho_F^L$. So

$$\rho_f^L = \rho_F^L. \tag{2}$$

Similarly from the relation $G = \frac{g}{h}$ we get that

$$\rho_G^L = \rho_g^L. \tag{3}$$

Combining (1), (2) and (3) we obtain that $\rho_f^L = \rho_g^L$.

In the similar manner we may prove that $\lambda_f^L = \lambda_g^L$.

This proves the theorem. □

Remark 1. The condition that 'F, G share a_1, a_2, a_3' in Theorem 1 is essential which is evident from the following example.

Example 1. Let us consider the functions $f(z) = \exp z$, $g(z) = \exp^{[2]} z$, $h(z) = z$ and $L(r) = \frac{1}{p}\exp(\frac{1}{r})$ where p is any positive real number. So $F(z) = \frac{\exp z}{z}$ and $G(z) = \frac{\exp^{[2]} z}{z}$. It is clear that F, G share only ∞. Also $T(r,h) = o\,(T(r,f))$ and $T(r,h) = o\,(T(r,g))$. But

$$\rho_f^L = \lambda_f^L = 1 \text{ and } \rho_g^L = \lambda_g^L = \infty$$

$$\text{i.e., } \rho_f^L \neq \rho_g^L \text{ and } \lambda_f^L \neq \lambda_g^L.$$

Remark 2. The condition '$T(r,h) = o\,(T(r,g))$' in Theorem 1 is necessary which we see in the following example.

Example 2. Let us choose the functions $f(z) = z^2$, $g(z) = z^2 \exp z$, $h(z) = z$ and $L(r) = \frac{1}{p}\exp(\frac{1}{r})$, where p is any positive real number. So $F(z) = z$ and $G(z) = z\exp z$. It is obvious that F, G share $0, 2\pi i, \infty$. Also $T(r,h) = o\,(T(r,f))$ and $T(r,h) \neq o\,(T(r,g))$. Here also

$$\rho_f^L \neq \rho_g^L \text{ and } \lambda_f^L \neq \lambda_g^L.$$

In the line of Theorem 1 we may state the following theorem without proof.

Theorem 2. *Let f and g be two non-constant meromorphic functions. If there is a function h with $T(r,h) = o\,(T(r,f))$ and $T(r,h) = o\,(T(r,g))$ such that F, G share a_1, a_2, a_3 IM then $\rho_f^{L^*} = \rho_g^{L^*}$ and $\lambda_f^{L^*} = \lambda_g^{L^*}$ where $F = \frac{f}{h}$ and $G = \frac{g}{h}$.*

Remark 3. Let us consider the functions $f(z) = \exp z$, $g(z) = \exp^{[2]} z$, $h(z) = z$ and $L(r) = \frac{1}{p}\exp(\frac{1}{r})$ where p is any positive real number. Clearly F, G share only ∞. Also $T(r,h) = o\,(T(r,f))$ and $T(r,h) = o\,(T(r,g))$ but $\rho_f^{L^*} \neq \rho_g^{L^*}$ and $\lambda_f^{L^*} \neq \lambda_g^{L^*}$.

Remark 4. Let us choose the functions $f(z) = z^2$, $g(z) = z^2 \exp z$, $h(z) = z$ and $L(r) = \frac{1}{p}\exp(\frac{1}{r})$ where p is any positive real number. Clearly F, G share 0, $2\pi i$, ∞. Also $T(r,h) = o\,(T(r,f))$ but $T(r,h) = o\,(T(r,g))$ does not hold. Here also $\rho_f^{L^*} \neq \rho_g^{L^*}$ and $\lambda_f^{L^*} \neq \lambda_g^{L^*}$.

References

[1] D. Banerjee, R.K. Dutta, Relative sharing and order of meromorphic functions, *Indian Acad. Math.*, **29**, No. 2 (2007), 425-431.

[2] G.G. Gundersen, Meromorphic functions that share three or four values, *J. London Math. Soc.*, **20**, No. 2 (1979), 457-466.

[3] W.K. Hayman, *Meromorphic Functions*, The Clarendon Press, Oxford (1964).

[4] D. Somasundaram, R. Thamizharasi, A note on the entire functions of L-bounded index and L-type, *Indian J. Pure Appl. Math.*, **19**, No. 3 (March 1988), 284-293.