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ZEROS OF THE LAPLACE TRANSFORM

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Abstract: In this paper we discus the zeros of analytic functions that admit a Laplace transform represention. In particular, we complement a classical result of Yu-Cheng Shen from 1947 by showing that for all sequences $\lambda_n \in \mathbb{C}$ with $Re\lambda_n \geq \gamma > 0$ and $|\arg(\lambda_n)| \leq \theta < \frac{\pi}{2}$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} 1 - \frac{|\lambda_n - 1|}{|\lambda_n + 1|} < \infty$, there exists $0 \neq f \in L^1_{loc}[0, \infty)$ such that the Laplace transform $\hat{f}$ exists for $Re\lambda > \gamma$ and satisfies $\hat{f}(\lambda_n) = 0$ for all $n \in \mathbb{N}$.

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1. Müntz Sequences and Uniqueness Sequences

A sequence of positive real numbers $(\beta_n)_{n\in\mathbb{N}}$ is a *Müntz sequence* provided that, for all $n \in \mathbb{N}$,

$$\beta_{n+1} - \beta_n \geq 1 \text{ and } \sum_{n=1}^{\infty} \frac{1}{\beta_n} = \infty. \tag{1.1}$$

A sequence of distinct complex numbers $(\lambda_n)_{n\in\mathbb{N}}$ with no accumulation point and with the property that $Re\lambda_n \geq \gamma > 0$ is called a *uniqueness sequence* if

$$\sum_{n=1}^{\infty} 1 - \frac{|\lambda_n - 1|}{|\lambda_n + 1|} = \infty. \tag{1.2}$$

It is easy to see that every Müntz sequence is a uniqueness sequence. Uniqueness sequences play an important role in Laplace transform theory. A classical result of Yu-Cheng Shen [5] is as follows. If two Laplace transforms $\hat{f}_i : \lambda \to \int_0^\infty e^{-\lambda t} f_i(t)\, dt$ $(i = 1, 2)$ of functions $f_i \in L^1_{loc}[0, \infty)$ satisfy $\hat{f}_1(\lambda_n) = \hat{f}_2(\lambda_n)$

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on a uniqueness sequence $(\lambda_n)_{n\in\mathbb{N}}$, then $f_1 = f_2$ almost everywhere. In this note we discuss this result in some detail and prove the reverse implication for sectorial sequences; i.e., we show that for complex sequences $(\lambda_n)_{n\in\mathbb{N}}$ with $Re\lambda_n \geq \gamma > 0$ and $|\arg(\lambda_n)| \leq \theta < \frac{\pi}{2}$ for all $n \in \mathbb{N}$ that are not uniqueness sequences, there exist $f_1, f_2 \in L^1_{loc}[0,\infty)$ with $f_1 \neq f_2$ such that $\hat{f}_1(\lambda_n) = \hat{f}_2(\lambda_n)$ for all $n \in \mathbb{N}$. This result was mentioned, without proof and without the assumption that the sequence is sectorial, in Theorem 1.11.1 in [1]. It seems to be not known if this statement holds without assuming that the sequence is sectorial (i.e., $|\arg(\lambda_n)| \leq \theta < \frac{\pi}{2}$ for all $n \in \mathbb{N}$).

Examples of sequences satisfying (1.1) are the equidistant sequences $\lambda_n = a + n^\alpha b$ $(a, b > 0, 0 < \alpha \leq 1)$. Examples of sequences not satisfying (1.2) are given by $\lambda_n = n^\alpha$ for $\alpha > 1$ and $\lambda_n = 1 + in$. Next we will show that there are sequences satisfying the condition in the previous proposition, on any vertical line $x = \gamma > 0$ and on any smooth curve $(t, \gamma(t))$ with $\gamma(t) > 0$ and $\gamma'(t) \geq 0$ for all $t > 0$.

Let $\lambda = (x, y) \in \mathbb{C}$ with $Re\lambda > 0$. Then the distance from λ to the point $(1, 0)$ is smaller than the distance to $(-1, 0)$. Therefore,

$$\frac{|\lambda - 1|}{|\lambda + 1|} < 1.$$

Now fix $0 < \epsilon < 1$. Then it is easy to show that the set of $\lambda \in \mathbb{C}$ with

$$\frac{|\lambda - 1|}{|\lambda + 1|} = \epsilon$$

is a circle in the right complex half plane centered at $(x_\epsilon, 0)$ with $x_\epsilon = \frac{1+\epsilon^2}{1-\epsilon^2}$ and with radius $r_\epsilon = \frac{2\epsilon}{1-\epsilon^2} < x_\epsilon$. We should note that as $\epsilon \to 1^-$ both the radius r_ϵ and the x-coordinate x_ϵ of the center are going to infinity whereas the left intersection of the circle with the x-axis approaches the origin; i.e., $0 < x_\epsilon - r_\epsilon \to 0$. Define $\epsilon_n = \frac{n-1}{n}$. Then there is a circle C_n with center at $(\frac{2n^2-2n+1}{2n-1}, 0)$ and radius $\frac{2n^2-2n}{2n-1}$ such that for any choice of $\lambda_n \in C_n$ we have that

$$\sum_{n=1}^{N} 1 - \frac{|\lambda_n - 1|}{|\lambda_n + 1|} = \sum_{n=1}^{N} 1 - \epsilon_n = \sum_{n=1}^{N} \frac{1}{n} \to \infty$$

as $N \to \infty$. At this point we can show now that there are uniqueness sequences on any vertical line $x = \gamma$ with $\gamma > 0$. For this, let $\gamma > 0$. Then C_n intersects the line $x = \gamma$ at the points (γ, y_n) where

$$y_n = \pm\sqrt{(2n - 1 - \gamma)(\gamma - \frac{1}{2n-1})} \to \infty \text{ as } n \to \infty.$$

Therefore, there exists a uniqueness sequence on any vertical line $\gamma + i\mathbb{R}$ with $\gamma > 0$. Let Γ be a smooth curve $(t, \gamma(t)), t > 0$, with $\gamma(t) > 0$ and $\gamma'(t) \geq 0$ for all $t > 0$. Then the circles C_n intersect Γ at points λ_n with $|\lambda_n| \to \infty$ as $n \to \infty$. Therefore there are uniqueness sequences on any such curve Γ.

Let X be a complex Banach space. We denote by $\mathrm{Lip}_0(\mathbb{R}_+, X)$ the Banach space of all functions $F : [0, \infty) \to X$ with $F(0) = 0$ and $\|F\|_{\mathrm{Lip}} := \sup_{t,s\geq 0} \frac{\|F(t)-F(s)\|}{|t-s|} < \infty$. If $F \in \mathrm{Lip}_0(\mathbb{R}_+, X)$, then the Laplace-Stieltjes integral

$$\widehat{dF}(\lambda) := \int_0^\infty e^{-\lambda t}\, dF(t) := \lim_{T\to\infty} \int_0^T e^{-\lambda t} dF(t), \tag{1.3}$$

exists for all $\lambda \in \mathbb{C}$ with $Re\lambda > 0$. If $F' = f$ almost everywhere, then $f \in L^\infty_{loc}(\mathbb{R}_+, X)$, and $\widehat{dF}(\lambda) = \hat{f}(\lambda)$. By the fundamental theorem of calculus, a Lipschitz-continuous function F is differentiable a.e. if and only if X has the Radon-Nikodym property (see [1], Chapter 1, for a discussion of the Radon-Nicodym property). We remark that reflexive Banach spaces have the Radon-Nikodym property. The following is one of the key results in Laplace transform theory (for a proof see [1], Section 2.1). We denote by $\mathcal{L}(L^1(0,\infty), X)$ the Banach space of bounded linear operators from $L^1(0, \infty)$ into X.

Theorem 1.1. (Riesz-Stieltjes Representation) *There exists an isometric isomorphism* $\mathcal{R} : Lip_0([0,\infty), X) \to \mathcal{L}(L^1(0,\infty), X)$ *given by* $\mathcal{R}(F) := T$ *where* $Tg := \int_0^\infty g(s)\, dF(s)$ *for all continuous functions* $g \in L^1(0,\infty)$ *and* $T\chi_{[0,t]} = F(t)$ *for all* $t \geq 0$.

The Riesz-Stieltjes Representation Theorem allows us to see how properties of F and its transform $\widehat{dF}$ relate to each-other. Since

$$\widehat{dF}(\lambda) = T_F e^{-\lambda\cdot} \quad \text{and} \quad F(t) = T_F \chi_{[0,t]}$$

for all $\lambda, t > 0$ it follows that the function F determines the operator T_F on the set of characteristic functions, which is total in $L^1(0, \infty)$ (i.e., its linear span is dense). Therefore T_F, and in particular $T_F e^{-\lambda\cdot} = \widehat{dF}(\lambda)$, is completely determined. On the other hand, any information on $\widehat{dF}(\lambda)$ for $\lambda > 0$ translates into information on T_F on the set of exponential functions which is also total in $L^1(0, \infty)$ (see Proposition 1.4). Therefore, $\widehat{dF}$ determines the properties of T_F and, in particular, of $T_F \chi_{[0,t]} = F(t)$, $(t \geq 0)$. The following results form complex analysis will be used, their proofs can be found in [4], pp. 300-311.

Proposition 1.2. *Suppose that* h *is a bounded analytic function on the* *unit disc not identically zero, and* $\alpha_1, \alpha_2, \dots$ *are the zeros of* h*. Then*

$$\sum_{n=1}^{\infty}(1-|\alpha_n|) < \infty.$$

Proposition 1.3. *Suppose that* r_n *is analytic on the region* Ω *for each* $n \in \mathbb{N}$*, that no* r_n *is identically zero there, and that*

$$\sum_{n=1}^{\infty} |1 - r_n(\lambda)| \tag{1.4}$$

converges uniformly on compact subsets of Ω*. Then the product*

$$r(\lambda) = \prod_{n=1}^{\infty} r_n(\lambda) \tag{1.5}$$

converges uniformly on compact subsets of Ω*. Hence* r *is analytic on* Ω*. Moreover we have that*

$$m(r,\lambda) = \sum_{n=1}^{\infty} m(r_n,\lambda), \quad \lambda \in \Omega, \tag{1.6}$$

where $m(r,\lambda)$ *is defined to be the multiplicity of the zero of* r *at* λ *(if* $r(\lambda) \neq 0$*, then* $m(r,\lambda) = 0$*).*

The following proposition is an essential result for our purposes; for a proof, see Lemma 1.2 in [2].

Proposition 1.4. *Let* $(\lambda_n)_{n\in\mathbb{N}}$ *be a uniqueness sequence. Then the exponential functions* $t \mapsto e^{-\lambda_n t}$ *are total in* $L^1(0,\infty)$*.*

We say that sequence of complex numbers $(\lambda_n)_{n\in\mathbb{N}}$ with $Re\lambda_n \geq \gamma > 0$ is called a Laplace uniqueness sequence if given a Laplace transformable function $f \in L^1_{loc}([0,\infty), X)$ with the property that $\hat{f}(\lambda_n) = 0$ for each n, implies $f \equiv 0$. In the following we will see how the concept of Laplace uniqueness sequence and uniqueness sequences as defined in (1.2) are related. We need the following preliminary result for Lipschitz continuous functions, which is an extension of Theorem 1.7.3 in [1].

Proposition 1.5. (Uniqueness) *Let* $(\lambda_n)_{n\in\mathbb{N}}$ *be a sequence of complex numbers with no accumulation point and* $Re\lambda_n \geq \gamma > 0$ *for all* $n \in \mathbb{N}$*. Consider the following statements.*

(i) $(\lambda_n)_{n\in\mathbb{N}}$ *is a uniqueness sequence.*

(ii) For all $F \in Lip_0([0,\infty), X)$ *we have that* $\widehat{dF}(\lambda_n) = 0$ *for all* $n \in \mathbb{N}$ *if and only if* $F = 0$*.*

Then (i) implies (ii). Moreover, if $|arg(\lambda_n)| \leq \theta < \frac{\pi}{2}$ *for all* $n \in \mathbb{N}$, *then (ii) implies (i).*

Proof. To prove that (i) implies (ii) we have to show that if $\widehat{dF}(\lambda_n) = 0$ for any $n \in \mathbb{N}$ then $F \equiv 0$. By Proposition 1.4, if (i) holds, then the family of exponentials $\{e^{-\lambda_n \cdot}\}$ is dense in $L^1(0,\infty)$. Thus, the operator $T_F := T_F g := \int_0^\infty g(t)\, dF(t)$ (see Theorem 1.1) has the property that

$$T_F(e^{-\lambda_n \cdot}) = \int_0^\infty e^{-\lambda_n t}\, dF(t) = \widehat{dF}(\lambda_n) = 0.$$

Since the induced operator is zero on a total set it follows that it is identically zero. Therefore the function F is identically zero which implies that $(\lambda_n)_{n\in\mathbb{N}}$ is a Laplace uniqueness sequence.

To prove that (ii) implies (i) with the additional assumption that $|arg(\lambda_n)| \leq \theta$ for some $\theta < \frac{\pi}{2}$ and all $n \in \mathbb{N}$, we have to show that if $(\lambda_n)_{n\in\mathbb{N}}$ is a Laplace uniqueness sequence then $\sum_{n=1}^{N} 1 - \frac{|\lambda_n - 1|}{|\lambda_n + 1|} \to \infty$. We will prove this by contradiction. Suppose $\sum_{n=1}^{\infty} 1 - \frac{|\lambda_n - 1|}{|\lambda_n + 1|} < \infty$. For $\lambda = de^{i\alpha}$ we have that

$$\frac{|\lambda + 1| - |\lambda - 1|}{\cos(\alpha)} = \frac{4\,|\lambda|}{|\lambda + 1| + |\lambda - 1|} \to 2 \ \text{ as } \ n \to \infty.$$

Thus, $\frac{|\lambda_n + 1| - |\lambda_n - 1|}{\cos(\alpha_n)} \geq 1$ for sufficiently large n. In particular, $|\lambda_n + 1| - |\lambda_n - 1| \geq \cos(\alpha_n) \geq \cos(\theta)$ for sufficiently large n. Thus,

$$1 - \frac{|\lambda_n - 1|}{|\lambda_n + 1|} = \frac{|\lambda_n + 1| - |\lambda_n - 1|}{|\lambda_n + 1|} > \frac{\cos(\theta)}{|\lambda_n + 1|}.$$

Therefore, $\sum_{n=1}^{\infty} \frac{1}{|\lambda_n + 1|} < \infty$ and for any $\varepsilon > 0$ we have that $\sum_{n=1}^{\infty} \frac{1}{|\lambda_n + \varepsilon + 1|} < \sum_{n=1}^{\infty} \frac{1}{|\lambda_n + 1|} < \infty$. Moreover,

$$\begin{aligned} \sum_{n=1}^{\infty} 1 - \frac{|\lambda_n + \varepsilon - 1|}{|\lambda_n + \varepsilon + 1|} &= \sum_{n=1}^{\infty} \frac{|\lambda_n + \varepsilon + 1| - |\lambda_n + \varepsilon - 1|}{|\lambda_n + \varepsilon + 1|} \\ &\leq \sum_{n=1}^{\infty} \frac{2}{|\lambda_n + \varepsilon + 1|}. \end{aligned}$$

Hence, $\sum_{n=1}^{\infty} 1 - \frac{|\lambda_n + \varepsilon - 1|}{|\lambda_n + \varepsilon + 1|} < \infty$.

For each $n \in \mathbb{N}$ define $\mu_n = \lambda_n + \varepsilon$ and $r_n(\lambda) := \frac{\mu_n - \lambda}{2 + \mu_n + \lambda}$. Then r_n is analytic on the region $Re\lambda > 0$ and it has a zero at $\lambda = \mu_n$. Now we will check that the conditions in Proposition 1.3 are satisfied. First we show that

$$\sum_{n=1}^{\infty} |1 - r_n(\lambda)| = \sum_{n=1}^{\infty} \left| \frac{2\lambda + 2}{2 + \mu_n + \lambda} \right| < \infty. \tag{1.7}$$

Let U be a compact subset of the region $Re\lambda > 0$ and let

$$a_n := \left|\frac{2\lambda+2}{2+\mu_n+\lambda}\right| \text{and } b_n := \frac{1}{|\mu_n+1|}. \tag{1.8}$$

Then

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{|\mu_n+1|\,|2\lambda+2|}{|1+\mu_n+1+\lambda|} = \lim_{n\to\infty}\frac{2\,|\lambda+1|}{\left|1+\frac{1+\lambda}{\mu_n+1}\right|}, \tag{1.9}$$

which exists for any $\lambda \in U$. Therefore $\sum_{n=1}^{\infty}|1-r_n(\lambda)|$ converges uniformly on U. By Proposition 1.3 it follows that the product $r(\lambda) = \prod_{n=1}^{\infty} r_n(\lambda)$ converges uniformly on compact subsets of the open right half plane $Re\lambda > 0$. Hence $r(\lambda)$ is analytic on $Re\lambda > 0$. Moreover, since only a finite number of factors $r_n(\lambda)$ have $|r_n(\lambda)| > 1$ it follows that $0 \leq |r(\lambda)| < 1$. By Proposition 1.3, $r(\lambda) = 0$ if and only if $\lambda = \mu_n$ for some $n \in \mathbb{N}$. Now let $q(\lambda) := \frac{r(\lambda)}{\lambda}$. Then $|\lambda q(\lambda)| = |r(\lambda)| \leq 1$ for $Re\lambda > 0$. By [1], Theorem 2.5.1, it follows that there exists $f \in C_0(\mathbb{R}_+, X)$ such that

$$\sup_{t>0}\left\|\frac{f(t)}{t}\right\| < \infty \quad \text{and} \quad q(\lambda) = \lambda\hat{f}(\lambda)$$

for $Re\lambda > 0$. Thus,

$$\hat{f}(\lambda) = \frac{q(\lambda)}{\lambda} = \frac{1}{\lambda}\prod_{n=1}^{\infty}\frac{\mu_n-\lambda}{2+\mu_n+\lambda}.$$

Since $\sup_{t>0}\left\|\frac{f(t)}{t}\right\| < \infty$ it follows that $\left\|f(t)e^{-\varepsilon t}\right\| \leq M$ for any $\varepsilon > 0$. Therefore, $h(t) := f(t)e^{-\varepsilon t}$ has the Laplace transform $\hat{h}(\lambda) = \int_0^{\infty} e^{-(\lambda+\varepsilon)t} f(t)\,dt = \hat{f}(\lambda+\varepsilon)$. Since h is continuous and bounded it follows that $H(t) := \int_0^t h(s)\,ds$ belongs to $Lip_0([0,\infty), X)$ and we have that

$$\widehat{dH}(\lambda) = \hat{h}(\lambda) = \hat{f}(\lambda+\varepsilon).$$

Thus, $\widehat{dH}(\lambda_n) = \hat{f}(\lambda_n+\varepsilon) = \hat{f}(\mu_n) = 0$, for all $n \in \mathbb{N}$, which is a contradiction. This finishes the proof of the proposition. □

In 1947, Yu-Cheng Shen [5] showed that in the following theorem the statement (i) implies statement (ii). In Theorem 1.11.1 in [1] it is stated without proof that statement (ii) always implies statement (*i*). We can show this with the additional assumption that $|arg(\lambda_n)| \leq \theta < \frac{\pi}{2}$ for all $n \in \mathbb{N}$. It remains open if the implication holds without this assumption.

Theorem 1.6. *Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers with $Re\lambda_n \geq \gamma > 0$ for all $n \in \mathbb{N}$. Consider the following statements.*

(i) $(\lambda_n)_{n\in\mathbb{N}}$ is a uniqueness sequence.

(ii) For all $f \in L^1_{loc}([0,\infty), X)$*with* $abs(f) < \gamma$ *we have that* $\hat{f}(\lambda_n) = 0$ *for all* $n \in \mathbb{N}$ *if and only if* $f = 0$.

Then (i) *implies* (ii)*. Moreover, if* $|arg(\lambda_n)| \leq \theta < \frac{\pi}{2}$ *for all* $n \in \mathbb{N}$*, then* (ii) *implies* (i).

Proof. The proof of this statement is a consequence of the previous proposition and the fact that there is an isometric isomorphism between $Lip_0([0,\infty), X)$ and $Lip_\omega([0,\infty), X)$ given by

$$I_\omega(F)(t) := \int_0^t e^{-\omega s} dF(s),$$

where $Lip_\omega([0,\infty), X) := \{F : \mathbb{R}_+ \to X : F(0) = 0, \|F\|_{Lip_\omega} < \infty\}$ and where $\|F\|_{Lip_\omega} := \sup_{t>s\geq 0} \frac{\|F(t)-F(s)\|}{\int_s^t e^{\omega r} dr}$. For a proof of this fact see [1], Section 2.4. Now, if $f \in L^1_{loc}(0,\infty)$ is Laplace transformable, then $F(t) := \int_0^t f(s)ds$ is absolutely continuous and exponentially bounded. Thus, $G(t) := \int_0^t F(s)\, ds \in Lip_\omega[0,\infty)$. Since

$$\hat{f}(\lambda) = \int_0^\infty e^{-\lambda t} f(t)\, dt = \lambda \int_0^\infty e^{-\lambda t} F(t)\, dt = \lambda^2 \int_0^\infty e^{-\lambda t} G(t)\, dt$$

it follows that $(\lambda_n)_{n\in\mathbb{N}}$ is a Laplace uniqueness sequence for f if and only if is a uniqueness sequence for G. Therefore $G(t) = 0$ implies $f(t) = 0$, which completes the proof. □

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