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ON S-STRONG JORDAN IDEALS

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Abstract: S-strong Jordan ideals of a semi-prime ring R with char $R \neq 2$ and $2R = R$ have been defined and studied. Necessary conditions for an S-strong Jordan ideal of R have been obtained. It has been proved that if J is an S-strong Jordon ideal of R, S is the set of symmetric elements of R and B_J is the set associated with J, then $SB_J \subseteq B_J$ and for all $u \in J$, $u^2 \in B_J$. Finally, as an application, we prove that if ϕ is a non-zero additive mapping of R into an associative ring A such that $\phi(ab + b^*a^*) = \phi(a)\phi(b) + \phi(b^*)\phi(a^*)$, $a, b \in R$, then $\operatorname{Ker}\phi \cap S = (0)$ and for all $x \in R$, $k \in K$, $(\phi(x^2) - (\phi(x))^2)\phi(k) = \phi(k)(\phi(x^{*^2}) - (\phi(x^*))^2)$.

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1. Introduction

Throughout the paper, we assume that R is a non-commutative ring, the symbol J denotes the Jordan ideal of R. A ring R is said to be *prime* if for $a, b \in R$, $aRb = (0)$ implies $a = 0$ or $b = 0$. An additive subgroup J of R is said to be a Jordan ideal of R if $ur + ru \in J$ for all $u \in J$, $r \in R$. For $x, y \in R$, by $[x, y]$, we mean $xy - yx$.

Throughout the paper, we assume that R is a semi-prime ring with char $R \neq 2$ and $2R = R$.

By an involution $*$ on a ring R we mean a mapping of R into itself satisfying:

(i) $a^{**} = a$, for all $a \in R$;

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(ii) $(a+b)^* = a^* + b^*$, for all $a, b \in R$;

(iii) $(ab)^* = b^* a^*$, for all $a, b \in R$.

Let $K = \{a \in R \,|\, a^* = -a\}$ and $S = \{a \in R \,|\, a^* = a\}$ be the set of skew-symmetric and symmetric elements of R, respectively.

2. Main Results

Definition 2.1. Let R be a ring having an involution $*$. A Jordan ideal J of S (set of symmetric elements of R) is called S-strong if for all $s \in S$ and$u \in J$ we have $sus \in J$.

Theorem 2.2. *Let J be an S-strong Jordan ideal of S. If $u, v \in J$ and $s, t \in S$, then:*

(i) $sut + tus \in J$;

(ii) $tsu + ust \in J$;

(iii) $utu \in J$.

Proof. (i) Follows in view of the definition.

(ii) Since J is Jordan ideal of $S, us + su \in J$.

Again, since J is a Jordan ideal of $S, t(us+su)+(us+su)t \in J$. Therefore, by part(i), $tsu + ust \in J$.

(iii) Note that $tu + ut \in J$. Therefore $u(tu+ut) + (tu+ut)u \in J$.

Thus, $2utu + u^2t + tu^2 \in J$.

In part (ii) if we take $s = u$, then $u^2t + tu^2 \in J$.

Hence $2utu \in J$. Since $2R = R, utu \in J$. □

Regarding the set B_J =$\{b \in R : ba + a^*b^* \in J$, for all $a \in R\}$, associated with S-strong Jordan ideal J, one may observe that B_J is a right ideal of R. Also, we have the following result.

Theorem 2.3. *Let R be a ring with unity. If J is an S-strong Jordan ideal of S and S is the set of symmetric elements of R, then:*

(i) $SB_J \subseteq B_J$

(ii) $u^2 \in B_J$, for all $u \in J$.

Proof. (i) Let $s \in S, b \in B_J$. Clearly by definition $ba + a^*b^* \in J$.

Since J is S-strong Jordan ideal of S

$$sbas + sa^*b^*s \in J, \quad \text{for all } s \in S. \tag{2.1}$$

Thus

$$\begin{aligned} & sba - sba + a^*(sb)^* - a^*(sb)^* + sbas + sa^*b^*s \in J, \\ \text{or,} \quad & sba + sba(s-1) + a^*(sb)^* - a^*b^*s + sa^*b^*s \in J, \\ \text{or,} \quad & sba + a^*b^*s + sba(s-1) + (s-1)a^*b^*s \in J. \end{aligned}$$

Since $(s-1)^* = s^* - 1^* = s - 1$, $s - 1 \in S$.

Therefore

$$sba + a^*b^*s + sbat + ta^*b^*s \in J, \quad \text{where } \ t = s - 1 \in S,$$

using (2.1), we get

$$sba + a^*(sb)^* \in J.$$

Hence $sb \in B_J$.

(ii) Note that

$$\begin{aligned} u^2a + a^*(u^2)^* &= u^2a + a^*(u^*)^2 \\ &= u^2a + a^*u^2. \end{aligned}$$

Therefore

$$\begin{aligned} u^2a + a^*u^2 &= u(ua + a^*u) + (ua + a^*u)u - uau - ua^*u \\ &= u(ua + a^*u) + (ua + a^*u)u - (u(a + a^*)u). \end{aligned}$$

Now

$$\begin{aligned} ua + (ua)^* &= ua + a^*u^* \\ &= ua + a^*u \in S. \end{aligned}$$

Since J is S-strong Jordan ideal of S, we have

$$u(ua + a^*u) + (ua + a^*u)u \in J.$$

Again, since $a+a^* \in S$, $u \in J$ and J is S-strong, by Theorem 1 (iii), $u(a+a^*)u \in J$. Hence $u^2 \in B_J$. □

Let R be a simple ring with involution '$*$' and characteristic not 2 such that $Z = (0)$ or the dimension of R over Z is greater than 4. We observe that under these conditions R cannot be commutative. Let ϕ be a non-zero additive mapping from R into an associative ring A. Assume $R' = \overline{\phi(R)}$, the subring of A generated by $\{\phi(r) : r \in R\}$, is a non-commutative prime ring such that $2R' = R'$ and R' is 2-torsion free.

Let ϕ satisfy the property

$\phi(ab + b^*a^*) = \phi(a)\phi(b) + \phi(b^*)\phi(a^*)$, for all $a, b \in R$.

As an application of S-strong Jordan ideals, we prove the following results.

Result I. $\ker\phi \cap S = (0)$.

Proof. First we show that $\ker\phi \cap S$ is an S-strong Jordan ideal. Let $p \in \ker\phi \cap S$ and $s \in S$. Then

$$\begin{aligned}\phi(sp+ps) &= \phi(s)\phi(p) + \phi(s^*)\phi(p^*)\\ &= 0 \qquad\qquad (\text{because } p = p^*,\ s = s^*).\end{aligned}$$

This gives that $sp + ps \in \ker\phi$. Now $sp + ps \in \ker\phi, s \in S$ and $\ker\phi \cap S$ is a Jordan ideal of S. Therefore

$$s(ps+sp) + (ps+sp)s \in \ker\phi \cap S.$$

This gives

$$\begin{aligned}&\phi(sps + s^2p + ps^2 + sps) = (0),\\ \text{or,}\quad &\phi(s^2p + ps^2) + 2\phi(sps) = (0).\end{aligned}$$

Note that

$$\phi(s^2p + ps^2) - \phi(s^2)\phi(p) + \phi((s^2)^*)\phi(p^*) = 0.$$

So $2\phi(sps) = (0)$. Since R is 2-torsion free ring, $\phi(sps) = (0)$. Hence $\ker\phi \cap S$ is a S-strong Jordan ideal of R. Therefore, by Theorem 2.6 in [5], $\ker\phi \cap S = (0)$ or $\ker\phi \cap S = S$.

Let $\ker\phi \cap S = S$. Then any $x \in R$ can be written as $x = s + k$, where $s \in S, k \in K$. Now $s \in S = \ker\phi \cap S$. So $\phi(s) = 0$, for all $s \in S$.

Also, by [5], $sk - ks \in S$, for all $s \in S$ and $k \in K$ this gives $\phi([s,k]) = 0$. Therefore, we have

$$\begin{aligned}[\phi(x),\phi(y)] &= \phi(s+k)\phi(y) - \phi(y)\phi(s+k)\\ &= 0,\ \text{for all}\ \ x,y \in R,\\ \text{i.e.}\qquad \phi(x)\phi(y) &= \phi(y)\phi(x),\ \text{for all}\ \ x,y \in R.\end{aligned}$$

Thus, R' is commutative. This is a contradiction. ☐

Result II. *For all* $x \in R$ *and* $k \in K$.

$$(\phi(x^2) - (\phi(x))^2)\phi(k) = \phi(k)(\phi(x^{*^2}) - (\phi(x^*))^2).$$

Proof. Let $x, y \in R$. Then

$$\begin{aligned}\phi((xy + y^*x^*)x^* + x(xy + y^*x^*)^*) &= \{\phi(x)\phi(y) + \phi(y^*)\phi(x^*)\}\phi(x^*)\\ &\quad + \phi(x)\{\phi(y^*)\phi(x^*) + \phi(x)\phi(y)\}.\end{aligned}$$

If $y = k \in K$, then we have

$$\phi((xk + k^*x^*)x^* + x(xk + k^*x^*)) = \phi(x^2k - kx^{*2})$$

$$= \phi(x^2)\phi(k) - \phi(k)\phi(x^{*2}), \qquad (2.2)$$

also, for all $x \in R$ and $k \in K$, we have

$$\begin{aligned} &\{\phi(x)\phi(y) + \phi(y^*)\phi(x^*)\}\phi(x^*) + \phi(x)\{\phi(y^*)\phi(x^*) + \phi(x)\phi(y)\} \\ &= \{\phi(x)\phi(k) + \phi(k^*)\phi(x^*)\}\phi(x^*) + \phi(x)\{\phi(k^*)\phi(x^*) + \phi(x)\phi(k)\} \\ &= (\phi(x))^2\phi(k) - \phi(k)(\phi(x^*))^2 . \qquad (2.3) \end{aligned}$$

Combing (2.2) and (2.3), we get

$$(\phi(x^2) - (\phi(x))^2)\phi(k) = \phi(k)(\phi(x^{*2}) - (\phi(x^*))^2), \text{ for all } k \in K, x \in R. \quad \square$$

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