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# DECOMPOSITIONS OF MIXED GRAPHS USING PARTIAL ORIENTATIONS OF $P_4$ AND $S_3$

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**Abstract:** In this paper, we give necessary and sufficient conditions for the existence of a decomposition of the $\lambda$-fold mixed complete graph into partial orientations of $P_4$ and $S_3$. Simple direct constructions are given in each case.

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**Key Words:** graph theory, graph decompositions

## 1. Introduction

Let $K_v$ ($D_v$) denote the complete graph (digraph) on $v$ vertices. A *decomposition* $\mathscr{D}$ of a graph (digraph) $H = (V, E)$ is a partition of the edge (arc) set $E$ ($A$) of $H$. For each part $B$ of the partition, the subgraph of $H$ induced by $B$ is called a *block* of the partition. This paper will be concerned with the case that all blocks are isomorphic to a single *block prototype* $G$. In this case, we say that $\mathscr{D}$ is a $G$-decomposition of $H$. This situation is denoted $G|H$.

A *mixed graph* on $v$ vertices is an ordered pair $(V, C)$ where $V$ is a set of vertices, $|V| = v$, and $C$ is a set of unordered and ordered pairs, denoted $[x, y]$ and $(x, y)$ respectively, of distinct elements of $V$. An ordered pair, $(x, y) \in C$, is called an *arc* of $(V, C)$. An unordered pair, $[x, y] \in C$, is called an *edge* of graph $(V, C)$. The *complete mixed graph* on $v$ vertices, denoted $M_v$, is the mixed graph $(V, C)$ where, for every pair of distinct vertices $v_1, v_2 \in V$, we have

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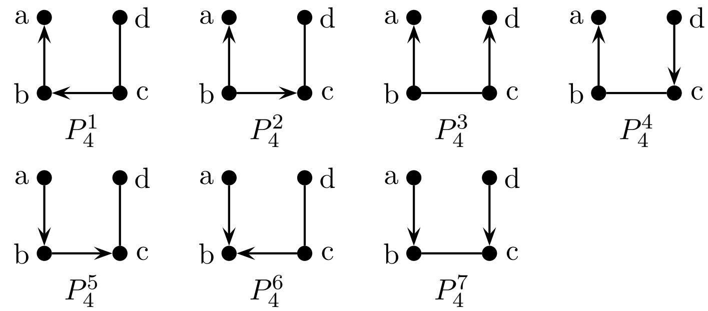


Figure 1: Partial orientations of $P_4$

$\{(v_1, v_2), (v_2, v_1), [v_1, v_2]\} \subset C$. The $\lambda$-*fold complete mixed graph* on $v$ vertices, denoted $\lambda M_v$, is the mixed multigraph $(V, C)$ where, for every pair of distinct vertices $v_1, v_2 \in V$, we have $\lambda$ copies of $\{(v_1, v_2), (v_2, v_1), [v_1, v_2]\} \subset C$. A *partial orientation of* $G$ is obtained from $G$ by replacing each edge $v_1v_2 \in E(G)$ with either $(v_1, v_2)$, $(v_2, v_1)$, or $[v_1, v_2]$. We restrict our attention to partial orientations with twice as many arcs as edges as this is the case with $M_v$. For all undefined graph theory terminology and notation, refer to [4].

We are inspired by the above comments to study decompositions of $\lambda M_v$ into partial orientations of $P_4$ and $S_3$, i.e., the path on four vertices and the star on four vertices, respectively. Combined with [2], this will give necessary and sufficient conditions for all connected graphs with three edges. We note that $v \geq 4$ is necessary in all cases [1]. Unless otherwise noted, if $v$ is odd, we let $V(M_v) = \mathbb{Z}_v$, i.e., the integers modulo $v$. Similarly, if $v$ is even, let $V(M_v) = \mathbb{Z}_{v-1} \cup \{\infty\}$. In either case, we will assume that all computations on the vertices are done modulo $v$ (when $v$ is odd) or modulo $v - 1$ (when $v$ is even).

The *converse* of a directed (mixed) graph $G$, denoted $G^c$, is obtained from $G$ by reversing the orientation on all arcs, i.e., $(v_1, v_2) \in G^c$ iff $(v_2, v_1) \in G$. We note that $G|H$ iff $G^c|H^c$ [1].

## 2. Decompositions Involving Partial Orientations of $P_4$

There are seven partial orientations of $P_4$ as shown in Figure 1. We let $[a, b, c, d]_i$ denote the $P_4^i$-block with vertex set $\{a, b, c, d\}$, as illustrated in Figure 1.

**Theorem 1.** *There exists a* $P_4^1$ *or* $P_4^5$*-decomposition of* $\lambda M_v$ *iff* $v \geq 4$.

*Proof.* We proceed by direct construction for $\lambda = 1$.

Let $v \equiv 0 \pmod 2$, say $v = 2t+2$ for $t \geq 1$. The required decomposition is given by the set of blocks $[i+j+3, i+2j+3, i+j+1, i]_1$, $[i+3, i+1, i, \infty]_1$, and $[i+2, \infty, i+1, i]_1$ for $i = 0, ..., 2t$ and $j = 1, ..., t-1$.

Let $v \equiv 1 \pmod 2$, say $v = 2t+1$ for $t \geq 2$. The required decomposition is given by the set of blocks $[i+j+3, i+2j+3, i+j+1, i]_1$ and $[i+4, i+2, i+1, i]_1$ for $i = 0, ..., 2t$ and $j = 1, ..., t-1$.

As $P_4^{5c} \cong P_4^1$, the same necessary and sufficient conditions apply. □

**Theorem 2.** *There exists a $P_4^2$ or $P_4^6$-decomposition of $\lambda M_v$ iff $\lambda(v-1) \equiv 0 \pmod 2$ and $v \geq 4$.*

*Proof.* Note that the directed part of $P_4^2$ will not decompose $D_v$ when $v \equiv 0 \pmod 2$ and $\lambda \equiv 1 \pmod 2$, see [3]. We proceed by direct construction on the remaining cases.

Let $v \equiv 1 \pmod 2$, say $v = 2t+1$, where $t \geq 2$. The required decomposition is given by the set of blocks $[i-j, i, i+j+2, i+1]_2$ and $[i+1, i, i+2, i+3]_2$ for $i = 0, ..., 2t$, and $j = 1, ..., t-1$.

Let $v \equiv 0 \pmod 2$, say $v = 2t+2$, where $t \geq 1$. It suffices to give the required construction when $\lambda = 2$. This decomposition is given by taking the blocks $[i-j, i, i+j+2, i+1]_2$ for $i = 0, ..., t-1$ and $j = 1, ..., 2t$ each twice along with the blocks $[i+2, \infty, i, i+1]_2$, $[i+2, i, \infty, i+1]_2$, $[\infty, i, i+1, i+2]_2$, and $[i+2, i, i+1, \infty]_2$ for $i = 0, ..., 2t$.

As $P_4^{6c} \cong P_4^2$, the same necessary and sufficient conditions apply. □

**Theorem 3.** *There exists a $P_4^3$ or $P_4^7$-decomposition of $\lambda M_v$ iff $v \geq 4$.*

*Proof.* We proceed by direct construction for $\lambda = 1$.

Let $v \equiv 0 \pmod 2$, say $v = 2t+2$ for $t \geq 2$. The required decomposition is given by the set of blocks $[i+j+2, i, i+j+1, i+1]_3$, $[i+1, \infty, i, i+2]_3$, and $[\infty, i, i+1, i+2]_3$ for $i = 0, ..., 2t$ and $j = 1, ..., t-1$.

If $v = 5$, the decomposition is given by the blocks $[i+1, i, i+3, i+2]_3$ and $[i+2, i, i+1, i+4]_3$ for $i = 0, ..., 4$.

Let $v \equiv 1 \pmod 2$, say $v = 2t+1$ for $t \geq 3$. The required decomposition is given by the set of blocks $[i+j+2, i, i+j+1, i+1]_3$, $[i+1, i, i+2t, i+1]_3$, and $[i+3, i, i+2, i+4]$ for $i = 0, ..., 2t$ and $j = 2, ..., t-1$.

Since $P_4^{7c} \cong P_4^3$, the same necessary and sufficient conditions apply. □

**Theorem 4.** *There exists a $P_4^4$-decomposition of $\lambda M_v$ iff $v \geq 4$.*

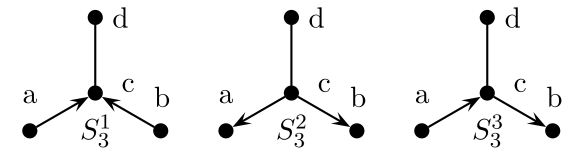


Figure 2: Partial orientations of $S_3$

*Proof.* We proceed by direct construction for $\lambda = 1$.

Let $v \equiv 0 \pmod 2$, say $v = 2t + 2$ for $t \geq 1$. The required decomposition is given by the set of blocks $[i+j+3, i+2j+3, i+j+2, i]_3$, $[i+2, \infty, i+1, i]_3$, and $[\infty, i+1, i+2, i]_3$ for $i = 0, ..., 2t$ and $j = 1, ..., t-1$.

Let $v \equiv 1 \pmod 2$, say $v = 2t + 1$ for $t \geq 2$. The required decomposition is given by the set of blocks $[i+j+3, i+2j+3, i+j+2, i]_3$, and $[i+4, i+2, i+1, i]_3$ for $i = 0, ..., 2t$ and $j = 1, ..., t-1$. □

## 3. Decompositions Involving Partial Orientations of $S_3$

There are three partial orientations of $S_3$ as shown in Figure 2. The $S_3^i$-block with vertex set $\{a, b, c, d\}$ will be denoted $[c; a, b, d]_i$ as illustrated in Figure 2.

**Theorem 5.** *There exists a $S_3^1$ or $S_3^2$-decomposition of $\lambda M_v$ iff $\lambda(v-1) \equiv 0 \pmod 2$ and $v \geq 4$.*

*Proof.* Since the directed part of $S_3^1$ will not decompose $D_v$ for $v \equiv 0 \pmod 2$, see [3], it follows that $v \equiv 1 \pmod 2$ or $\lambda \equiv 0 \pmod 2$ is necessary.

Suppose that $v = 2t+1$, $\lambda = 1$, and $t \geq 2$. The blocks $[i; i+t, i+t+1, i+1]_1$ and $[i; i+j, i-j, i+j+1]_1$ for $i = 0, ..., 2t$ and $j = 1, ..., t-1$ form the required decomposition.

Suppose that $v = 2t + 2$, $\lambda = 2$, and $t \geq 1$. Take one copy of each of the blocks $[\infty; i+1, i+2, i]_1$, $[i+2; i+1, i, \infty]_1$, $[i+1, \infty, i, i+2]_1$, and $[i+2; \infty, i, i+1]_1$ for $i = 0, ..., 2t$. We also take two copies of each of the blocks $[i+j+2; i, i+2, i+2j+3]_1$ for $i = 0, ..., 2t$ and $j = 1, ..., t-1$.

Since $S_3^{2c} = S_3^1$, the same necessary and sufficient conditions apply. □

**Lemma 6.** *There is no $S_3^3$-decomposition of $\lambda M_4$ when $\lambda$ is odd.*

*Proof.* Suppose that such a decomposition exists. Let $k_i$ denote the number of times that vertex $i$ in $\lambda M_4$ is the center of one of the $S_3^3$-blocks. Note that the degree, in-degree, and out-degree of each vertex in $\lambda M_4$ is $3\lambda$. Hence for

$j \neq i$, there are $3\lambda - k_i$ incidences between vertex $i$ and vertex $j$ that are not accounted for by the blocks centered at vertex $i$. From this it follows that $k_j = 3\lambda - k_i$ for all $i$ and $j$. In particular, $k_\ell = 3\lambda - k_i = k_j$. This implies that $k_i = k_j$ for all $i$ and $j$. Thus $k_i = 3\lambda - k_i$ or equivalently $2k_i = 3\lambda$. Ergo, $\lambda$ must be even. □

**Theorem 7.** *There exists a $S_3^3$-decomposition of $\lambda M_v$ iff $v \geq 4$ except when $(\lambda, v) = (2k+1, 4)$.*

*Proof.* There is no $S_3^3$-decomposition of $\lambda M_4$ when $\lambda$ is odd by Lemma 6. Hence we proceed with constructing the remaining cases.

Let $v \equiv 0 \pmod 2$, say $v = 2t+2$, where $t \geq 2$. The required decomposition is given by the set of blocks $[i+j+2; i, i+2, i+1]_3$, $[\infty; i+1, i+2, i]_3$, and $[i+1; i, i+3, i+2]_3$ for $i = 0, ..., 2t$ and $j = 1, ..., t-1$.

Let $v \equiv 1 \pmod 2$, say $v = 2t+1$, where $t \geq 2$. The required decomposition is given by the set of blocks $[i+j+2; i, i+2, i+1]_3$ and $[i+1; i, i+3, i+2]_3$ for $i = 0, ..., 2t$ and $j = 1, ..., t-1$.

For $v = 4$, it suffices to give constructions for $\lambda = 2$. The required construction is given by the set of blocks $[i; i+2, i+1, \infty]_3$, $[\infty; i+1, i+2, i]_3$, $[i; \infty, i+2, i+1]_3$, and $[i; i+1, \infty, i+2]_3$ for $i = 0, 1, 2$ and all computations are done modulo 3. □

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