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SIMULTANEOUS INTEGRAL REPRESENTABILITY BY INFINITELY SMOOTH KERNELS WITH APPLICATION TO INTEGRAL EQUATIONS

Igor M. Novitskii
Khabarovsk Division
Institute for Applied Mathematics
Far-Eastern Branch of the Russian Academy of Sciences
54, Dzerzhinskiy Street, Khabarovsk, 680 000, RUSSIA
e-mail: novim@iam.khv.ru

Abstract: In this note, we characterize families incorporating those bounded linear operators on a separable Hilbert space $\mathcal{H}$ that can be simultaneously transformed by the same unitary equivalence transformation into Carleman integral operators on $L^2(\mathbb{R})$, whose kernels $\boldsymbol{T}\colon \mathbb{R}^2 \to \mathbb{C}$ and Carleman functions $\overline{\boldsymbol{T}(s,\cdot)}\colon \mathbb{R} \to L^2(\mathbb{R})$ are infinitely smooth and vanish at infinity together with all partial and all strong derivatives, respectively. An explicit procedure for constructing the unitary operators, from $\mathcal{H}$ onto $L^2(\mathbb{R})$, effecting such transformations is also presented. As an application, we present a smooth version of Korotkov's reduction method for general third-kind integral equations in $L^2(Y,\mu)$, whose aim is to obtain an equivalent integral equation of either the first or the second kind in $L^2(\mathbb{R})$, with an infinitely smooth Hilbert-Schmidt or Carleman kernel, respectively.

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1. Introduction and the Main Result

Throughout the note, $\mathcal{H}$ is a complex, separable, infinite-dimensional Hilbert space with norm $\|\cdot\|_{\mathcal{H}}$ and inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$, and the symbols $\mathbb{C}$, $\mathbb{N}$, and $\mathbb{Z}$, refer to the complex plane, the set of all positive integers, and the set of all integers, respectively. $\mathfrak{R}(\mathcal{H})$ denotes the Banach algebra of all bounded linear operators on $\mathcal{H}$. For an operator A in $\mathfrak{R}(\mathcal{H})$, A^* denotes the Hilbert space adjoint of A.

Throughout let (Y,μ) denote a measure space Y equipped with a positive, σ-finite, complete, separable, and not purely atomic, measure μ, and then let $L^2(Y,\mu)$ denote the Hilbert space of (equivalence classes of) μ-measurable complex-valued functions on Y equipped with the inner product $\langle f,g\rangle_{L^2(Y,\mu)}=\int_Y f(y)\overline{g(y)}\,d\mu(y)$ and the norm $\|f\|_{L^2(Y,\mu)}=\langle f,f\rangle^{1/2}_{L^2(Y,\mu)}$. When μ is the Lebesgue measure on the real line $\mathbb{R}=(-\infty,+\infty)$, we abbreviate $L^2(\mathbb{R},\mu)$ to L^2, and $d\mu(y)$ to dy.

An operator $T\in\mathfrak{R}\left(L^2(Y,\mu)\right)$ is said to be *integral* if there is a $\mu\times\mu$-measurable function $\boldsymbol{T}\colon Y\times Y\to\mathbb{C}$, a *kernel*, such that, for every $f\in L^2(Y,\mu)$,

$$(Tf)(x)=\int_Y \boldsymbol{T}(x,y)f(y)\,d\mu(y)\quad\text{for }\mu\text{-almost every }x\in Y.$$

A kernel $\boldsymbol{T}$ on $Y\times Y$ is said to be *Carleman* if $\boldsymbol{T}(x,\cdot)\in L^2(Y,\mu)$ for μ-almost every fixed x in Y. Every Carleman kernel, $\boldsymbol{T}$, induces a *Carleman function* $\boldsymbol{t}$ from Y to $L^2(Y,\mu)$ by $\boldsymbol{t}(x)=\overline{\boldsymbol{T}(x,\cdot)}$ for all x in Y for which $\boldsymbol{T}(x,\cdot)\in L^2(Y,\mu)$. An integral operator with a kernel $\boldsymbol{T}$ is called *Carleman* if $\boldsymbol{T}$ is a Carleman kernel. Recall that a bounded linear operator $U\colon\mathcal{H}\to L^2(Y,\mu)$ is *unitary* if U has range $L^2(Y,\mu)$ and $\langle Uf,Ug\rangle_{L^2(Y,\mu)}=\langle f,g\rangle_{\mathcal{H}}$ for all f, $g\in\mathcal{H}$.

This note is written in the spirit of the pioneering work by Korotkov [5] on the simultaneous reducibility of families of linear operators to integral form by means of unitary equivalence transformations and its applications to integral equations. In that paper, Korotkov showed that if operators $B_\gamma\in\mathfrak{R}(\mathcal{H})$ $(\gamma\in\mathcal{G})$ with an index set of arbitrary cardinality satisfy the condition

$$\lim_{n\to\infty}\sup_{\gamma\in\mathcal{G}}\left\|B_\gamma^*v_n\right\|_{\mathcal{H}}=0,\tag{1}$$

where $\{v_n\}$ is an orthonormal sequence in $\mathcal{H}$, then there exists a unitary operator $U\colon\mathcal{H}\to L^2(Y,\mu)$ such that, for each $\gamma\in\mathcal{G}$, the operator $T_\gamma=UB_\gamma U^{-1}$ is an integral operator on $L^2(Y,\mu)$, $(T_\gamma f)(x)=\int_Y \boldsymbol{K}_\gamma(x,y)f(y)\,d\mu(y)$ where the kernel $\boldsymbol{K}_\gamma$ is Carleman. Motivated by this result, we consider in this note the existence of a similar result when we specialize $L^2(Y,\mu)$ to be L^2 and require each Carleman kernel $\boldsymbol{K}_\gamma$ to fulfill some additional analytic conditions that are to be technically useful in many applications, including the solving of integral equations. More specifically, we prove that when $\mathcal{G}$ in (1) is at most countable, the operators B_γ can simultaneously be made to have Carleman kernels that are SK^∞ kernels in the sense of the following definition.

Definition 1. A Carleman kernel $\boldsymbol{T}\colon \mathbb{R}^2 \to \mathbb{C}$ is called an SK^∞ *kernel* [9] if it satisfies the two generally independent conditions:

(i) the function $\boldsymbol{T}$ and all its partial derivatives on $\mathbb{R}^2$ of all orders are in $C\left(\mathbb{R}^2, \mathbb{C}\right)$,

(ii) the Carleman function $\boldsymbol{t}$, $\boldsymbol{t}(s) = \overline{\boldsymbol{T}(s,\cdot)}$, and its (strong) derivatives, $\frac{d^i \boldsymbol{t}}{ds^i}$, on $\mathbb{R}$ of all orders are in $C\left(\mathbb{R}, L^2\right)$.

An SK^∞ kernel $\boldsymbol{T}$ is called a K^∞ *kernel* [9] if the conjugate transpose function $\boldsymbol{T'}$ ($\boldsymbol{T'}(s,t) = \overline{\boldsymbol{T}(t,s)}$) is also an SK^∞ kernel, that is, if additionally

(iii) the Carleman function $\boldsymbol{t'}$, $\boldsymbol{t'}(s) = \overline{\boldsymbol{T'}(s,\cdot)} = \boldsymbol{T}(\cdot, s)$, and its (strong) derivatives, $\frac{d^i \boldsymbol{t'}}{ds^i}$, on $\mathbb{R}$ of all orders are in $C\left(\mathbb{R}, L^2\right)$.

Throughout this note, $C(X,B)$, where B is a Banach space (with norm $\|\cdot\|_B$), denotes the Banach space (with the norm $\|f\|_{C(X,B)} = \sup_{x\in X} \|f(x)\|_B$) of continuous B-valued functions defined on a locally compact space X and *vanishing at infinity* (that is, given any $f \in C(X,B)$ and $\varepsilon > 0$, there exists a compact subset $X(\varepsilon, f) \subset X$ such that $\|f(x)\|_B < \varepsilon$ whenever $x \notin X(\varepsilon, f)$).

We are now ready to state our main result.

Theorem 2. *Suppose that for a family $\mathcal{B} = \{B_r \colon r \in \mathbb{N}\} \subset \mathfrak{R}(\mathcal{H})$ there exists an orthonormal sequence $\{v_n\}_{n=1}^\infty$ in $\mathcal{H}$ such that*

$$\sup_{r\in\mathbb{N}} \|B_r^* v_n\|_{\mathcal{H}} \to 0 \quad \text{as } n \to \infty. \tag{2}$$

Then there exists a unitary operator $U\colon \mathcal{H} \to L^2$ such that, for each $r \in \mathbb{N}$, the operator UB_rU^{-1} is a Carleman operator with an SK^∞ kernel.

This result has recently been published without proof in [10, Theorem 2]. For the case in which $\mathcal{B}$ is a singleton, its proof can be found in [9, Theorem 1.4]. In [6, Section 3], there is a counter-example to show that the conclusion of Theorem 2 cannot in general be extended to operator families of more than countable cardinality.

Section 2 of the present note is entirely devoted to proving Theorem 2. There we provide a direct method for constructing that unitary operator U : $\mathcal{H} \to L^2$ whose existence the theorem asserts. The method uses no spectral properties of the operators B_r, other than their joint property imposed in (2), to determine the action of U by specifying two orthonormal bases, of $\mathcal{H}$ and of L^2, one of which is meant to be the image by U of the other, the basis for L^2 may be chosen to be an infinitely smooth wavelet basis.

Section 3 of the present note deals with an application of Theorem 2 that is aimed at implanting the SK^∞ kernels into integral equations. There we follow Korotkov's method [5], applied to take into account Theorem 2, and reduce the general linear integral equation of the third kind in $L^2(Y,\mu)$ to an equivalent integral equation either of the second kind (Theorem 3) or of the first kind (Theorem 4) in L^2, with the kernel being the linear pencil of the SK^∞ kernels or of the Hilbert-Schmidt K^∞ kernels, respectively.

2. Proof of Theorem 2

The proof has two steps. The first step is to pick suitable orthonormal bases, $\{f_1, f_2, f_3, \dots\}$ for $\mathcal{H}$ and $\{u_1, u_2, u_3, \dots\}$ for L^2, and then to define a certain unitary operator from $\mathcal{H}$ onto L^2 that carries f_n into u_n for each n, and is suggested as U in the theorem. The second step is a straightforward verification that the constructed unitary operator is indeed as desired.

Step 1. Assume that $\sup_{r\in\mathbb{N}} \|B_r\| \leq 1$. This is a harmless assumption, involving no loss of generality; just replace B_r with $\|B_r\| > 1$ by $\frac{1}{\|B_r\|}\cdot B_r$. Pick a subsequence, $\{e_k\}_{k=1}^\infty$, of the sequence $\{v_n\}_{n=1}^\infty$ of (2) such that

$$M = \sum_k \sup_{r\in\mathbb{N}} \|B_r^* e_k\|_{\mathcal{H}}^{\frac{1}{4}} < \infty \tag{3}$$

(the sum notation $\sum_k$ will always be used instead of the more detailed symbol $\sum_{k=1}^\infty$). For each $r \in \mathbb{N}$, let

$$S_r = \frac{1}{r}\cdot B_r, \quad Q_r = (1-E)S_r, \quad J_r = S_r^* E, \tag{4}$$

where E is the orthogonal projection of $\mathcal{H}$ onto the closed linear span of the e_k's, and note that

$$S_r = Q_r + J_r^*. \tag{5}$$

Assume, with no loss of generality, that $\dim(1-E)\mathcal{H} = \infty$. Let $\left\{e_1^\perp, e_2^\perp, e_3^\perp, \dots\right\}$ be any orthonormal basis for the subspace $(1-E)\mathcal{H}$, and let $\{f_1, f_2, f_3, \dots\}$ be any basis for $\mathcal{H}$ including all terms of the sequences $\{e_k\}_{k=1}^\infty$ and $\left\{e_k^\perp\right\}_{k=1}^\infty$:

$$\{f_1, f_2, f_3, \dots\} = \{e_1, e_2, e_3, \dots\} \cup \left\{e_1^\perp, e_2^\perp, e_3^\perp, \dots\right\}. \tag{6}$$

It then follows from (4) and (3) that

$$\sum_n \|J_r f_n\|^2_{\mathcal{H}} \le \sum_n \|J_r f_n\|^{\frac{1}{4}}_{\mathcal{H}} = \sum_k \|S_r^* e_k\|^{\frac{1}{4}}_{\mathcal{H}} \le \sum_k \sup_{r\in\mathbb{N}} \|S_r^* e_k\|^{\frac{1}{4}}_{\mathcal{H}} \le M, \tag{7}$$

implying in particular that the operators J_r, J_r^* $(r \in \mathbb{N})$ are all Hilbert-Schmidt. For each $f \in \mathcal{H}$, let

$$d(f) = \sup_{r\in\mathbb{N}} \|J_r f\|^{\frac{1}{4}}_{\mathcal{H}} + \sup_{r\in\mathbb{N}} \|J_r^* f\|^{\frac{1}{4}}_{\mathcal{H}} + \sup_{r\in\mathbb{N}} \|\mathit{\Gamma}_r f\|_{\mathcal{H}}, \tag{8}$$

where, for each $r \in \mathbb{N}$,

$$\mathit{\Gamma}_r = \Lambda S_r, \text{ with } \Lambda = \sum_k \frac{1}{k} \left\langle \cdot, e_k^{\perp} \right\rangle_{\mathcal{H}} e_k^{\perp}. \tag{9}$$

Clearly Λ, and hence each $\mathit{\Gamma}_r$, is a Hilbert-Schmidt operator on $\mathcal{H}$. Prove that

$$d(e_k) \to 0 \quad \text{as } k \to \infty. \tag{10}$$

For this, use again the standard facts on Hilbert-Schmidt operators (see, e.g., [2, Chapter III]) to write the two following chains of relations

$$\begin{aligned} \sum_k \sup_{r\in\mathbb{N}} \|J_r^* e_k\|^2_{\mathcal{H}} &\le \sum_r \sum_k \|J_r^* e_k\|^2_{\mathcal{H}} \le \sum_r |J_r^*|_2^2 = \sum_r |J_r|_2^2 \\ &= \sum_r \sum_n \|J_r f_n\|^2_{\mathcal{H}} = \sum_r \sum_k \|S_r^* e_k\|^2_{\mathcal{H}} \\ &\le \sum_r \frac{1}{r^2} \sum_k \sup_{r\in\mathbb{N}} \|B_r^* e_k\|^2_{\mathcal{H}} \le \frac{M^8 \pi^2}{6}, \end{aligned} \tag{11}$$

$$\begin{aligned} &\sum_k \sup_{r\in\mathbb{N}} \|\mathit{\Gamma}_r e_k\|^2_{\mathcal{H}} \le \sum_r \sum_k \|\mathit{\Gamma}_r e_k\|^2_{\mathcal{H}} \le \sum_r |\mathit{\Gamma}_r|_2^2 = \sum_r |\mathit{\Gamma}_r^*|_2^2 \\ &= \sum_r \sum_n \|S_r^* \Lambda f_n\|^2_{\mathcal{H}} \le \sum_r \frac{1}{r^2} \sum_k \left\|\Lambda e_k^{\perp}\right\|^2_{\mathcal{H}} = \sum_r \frac{1}{r^2} \sum_k \frac{1}{k^2} = \frac{\pi^4}{36}, \end{aligned} \tag{12}$$

where $|\cdot|_2$ is the Hilbert-Schmidt norm. The truth of (10) is now immediate from (7), (11), and (12).

Notation. If an equivalence class $f \in L^2$ contains a function belonging to $C(\mathbb{R}, \mathbb{C})$, then we shall henceforth use $[f]$ to denote that function.

Take any orthonormal basis $\{u_1, u_2, u_3, \dots\}$ for L^2 that satisfies conditions:

(a) for each i and for each $n \in \mathbb{N}$,

$$[u_n]^{(i)} \in C(\mathbb{R}, \mathbb{C}), \tag{13}$$

where $[u_n]^{(i)}$ denotes the i-th derivative of $[u_n]$ (here and throughout, the letter i is reserved for all non-negative integers),

(b) the sequence $\{u_n\}_{n=1}^{\infty}$ decomposes into two infinite subsequences $\{g_k\}_{k=1}^{\infty}$ and $\{h_k\}_{k=1}^{\infty}$ satisfying

$$\{h_1, h_2, h_3, \dots\} = \{u_1, u_2, u_3, \dots\} \setminus \{g_1, g_2, g_3, \dots\}, \tag{14}$$

and such that if $H_{k,i} = \left\| [h_k]^{(i)} \right\|_{C(\mathbb{R},\mathbb{C})}$, $G_{k,i} = \left\| [g_k]^{(i)} \right\|_{C(\mathbb{R},\mathbb{C})}$ then, for each i,

$$\sum_k H_{k,i} < \infty, \tag{15}$$

$$\sum_k k H_{n(k),i} < \infty, \tag{16}$$

$$\sum_k d(x_k) G_{k,i} < \infty, \tag{17}$$

where $\{n(k)\}_{k=1}^{\infty}$ is a subsequence of the sequence $\{n\}_{n=1}^{\infty}$, and $\{x_k\}_{k=1}^{\infty}$ is a subsequence of the sequence $\{e_k\}_{k=1}^{\infty}$.

What follows is an example showing the explicit existence of such a basis.

Example. Let $\mathbb{N}$ be decomposed into two infinite subsequences $\{l(k)\}_{k=1}^{\infty}$ and $\{m(k)\}_{k=1}^{\infty}$, and let $\{u_1, u_2, u_3, \dots\}$ be an orthonormal basis for L^2 that has property (13), and such that, for each i,

$$\left\| [u_n]^{(i)} \right\|_{C(\mathbb{R},\mathbb{C})} \le N_n D_i \quad (n \in \mathbb{N}), \tag{18}$$

where $\{N_n\}_{n=1}^{\infty}$, $\{D_i\}_{i=0}^{\infty}$ are two sequences of positive reals, the first of which is subject to the restriction that

$$\sum_k N_{l(k)} < \infty. \tag{19}$$

Then $\{u_1, u_2, u_3, \dots\}$ is a basis of the type described in **(b)** above. Indeed, from conditions (18), (19) it follows that the requirement (15) may be satisfied for all i by taking $h_k = u_{l(k)}$ $(k \in \mathbb{N})$, and hence that a subsequence $\{n(k)\}_{k=1}^{\infty}$ of $\{n\}_{n=1}^{\infty}$ may be chosen so as to fulfill (16) for all i, with $h_{n(k)} = u_{l(n(k))}$ $(k \in \mathbb{N})$. At the same time, on account of (10) and of (18), the sequence $\{e_k\}_{k=1}^{\infty}$ does always have an infinite subsequence $\{x_k\}_{k=1}^{\infty}$ that satisfies, for each i, the requirement (17) for $g_k = u_{m(k)}$ $(k \in \mathbb{N})$.

In turn, an explicit example of a basis $\{u_1, u_2, u_3, \dots\}$ that enjoys the above three properties (13), (18), and (19), can be adopted from the wavelet theory, as follows. Let ψ be the Lemarié-Meyer wavelet,

$$[\psi](s) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{\imath\xi(\frac{1}{2}+s)} \operatorname{sgn} \xi\, b(|\xi|)\, d\xi \quad (s \in \mathbb{R})$$

with the bell function b being infinitely smooth and compactly supported on $[0, +\infty)$ (see [3, Example D, p. 62] for details). Then $[\psi]$ is of the Schwartz class $\mathcal{S}(\mathbb{R})$, so its every derivative $[\psi]^{(i)}$ is in $C(\mathbb{R}, \mathbb{C})$. In addition, the "mother wavelet" ψ generates an orthonormal basis $\{\psi_{\alpha\beta}\}_{\alpha,\,\beta\in\mathbb{Z}}$ for L^2 by

$$\psi_{\alpha\beta} = 2^{\frac{\alpha}{2}}\psi(2^{\alpha}\cdot -\beta) \quad (\alpha,\, \beta \in \mathbb{Z}).$$

In any manner, rearrange the two-indexed set $\{\psi_{\alpha\beta}\}_{\alpha,\,\beta\in\mathbb{Z}}$ into a simple sequence, so that it looks like $\{u_n\}_{n=1}^{\infty}$, clearly each term here, u_n, has property (13) for each i. Besides, it is easily verified that a norm estimate like (18) holds for each $[u_n]^{(i)}$, and the factors in the right-hand side of (18) can be taken as

$$N_n = \begin{cases} 2^{\alpha_n^2} & \text{if } \alpha_n > 0, \\ 2^{\alpha_n/2} & \text{if } \alpha_n \le 0, \end{cases} \qquad D_i = 2^{(i+1/2)^2}\left\|[\psi]^{(i)}\right\|_{C(\mathbb{R},\mathbb{C})},$$

with the convention that $u_n = \psi_{\alpha_n\beta_n}$ $(n \in \mathbb{N})$, in conformity with that rearrangement. This choice of N_n also gives $\sum_k N_{l(k)} < \infty$ (cf. (19)) whenever $\{l(k)\}_{k=1}^{\infty}$ is a subsequence of $\{l\}_{l=1}^{\infty}$ satisfying $\alpha_{l(k)} \to -\infty$ as $k \to \infty$.

Returning to the proof of the theorem, let

$$\left\{x_1^{\perp}, x_2^{\perp}, x_3^{\perp}, \dots\right\} = \left\{e_1^{\perp}, e_2^{\perp}, e_3^{\perp}, \dots\right\} \cup \left(\{e_1, e_2, e_3, \dots\} \setminus \{x_1, x_2, x_3, \dots\}\right),$$

where the subsequence $\{x_k\}_{k=1}^{\infty}$ of $\{e_k\}_{k=1}^{\infty}$ is exactly that appeared in condition **(b)** above, and recall that by (6) and (14)

$$\begin{gathered} \{f_1, f_2, f_3, \dots\} = \{x_1, x_2, x_3, \dots\} \cup \left\{x_1^{\perp}, x_2^{\perp}, x_3^{\perp}, \dots\right\}, \\ \{u_1, u_2, u_3, \dots\} = \{g_1, g_2, g_3, \dots\} \cup \{h_1, h_2, h_3, \dots\}. \end{gathered} \tag{20}$$

The time has come to construct a candidate for the desired unitary operator in the theorem. Define a unitary operator $U\colon \mathcal{H} \to L^2$ on the basis vectors by setting

$$Ux_k^{\perp} = h_k, \quad Ux_k = g_k \quad \text{for all } k \in \mathbb{N}, \tag{21}$$

in the harmless assumption that, for each $k \in \mathbb{N}$,

$$Uf_k = u_k, \quad Ue_k^{\perp} = h_{n(k)}, \tag{22}$$

where $\{n(k)\}_{k=1}^{\infty}$ is just that subsequence of $\{n\}_{n=1}^{\infty}$ which occurs in condition **(b)** above.

Step 2. This step of the proof is to show that, in fact, if the unitary operator $U\colon \mathcal{H} \to L^2$ is defined as in (21), then the operators $UB_rU^{-1}\colon L^2 \to L^2$ $(r \in \mathbb{N})$ are all simultaneously Carleman operators with SK^{∞} kernels. Since scalar factors do not alter properties (i), (ii) of Definition 1, it is sufficient, in virtue of (4), to prove instead that each of the operators $T_r = US_rU^{-1}$ $(r \in \mathbb{N})$ is Carleman, and has an SK^{∞} kernel. For this in turn it suffices, by (5), to check that the same properties are possessed by the operators $P_r = UQ_rU^{-1}$, $F_r =$

$UJ_r^*U^{-1}$ $(r \in \mathbb{N})$. The checking is straightforward, and goes by representing all pertinent kernels and Carleman functions as infinitely smooth sums of termwise differentiable series of infinitely smooth functions as follows.

Fix, to begin with, an arbitrary index $r \in \mathbb{N}$, and adopt the convention that operator notations will, from now on, be abbreviated by omitting the subscript r whenever it occurs. Note that the first summand Q in (5) admits the following representation

$$Q = \sum_k \left\langle \cdot, S^* e_k^{\perp} \right\rangle_{\mathcal{H}} e_k^{\perp},$$

with respect the orthonormal basis $\left\{e_1^{\perp}, e_2^{\perp}, e_3^{\perp}, \dots\right\}$ for $(1-E)\mathcal{H}$ (see also (4)). Accordingly, the operator P, which is the transform by U of Q, may be written as (cf. (22))

$$Pf = \sum_k \left\langle f, T^* h_{n(k)} \right\rangle_{L^2} h_{n(k)} \quad (f \in L^2), \tag{23}$$

where, by (9),

$$T^* h_{n(k)} = \sum_n \left\langle S^* e_k^{\perp}, f_n \right\rangle_{\mathcal{H}} u_n = k \sum_n \left\langle e_k^{\perp}, \mathit{\Gamma} f_n \right\rangle_{\mathcal{H}} u_n \quad (k \in \mathbb{N}), \tag{24}$$

with the series convergent in L^2. Prove that, for any fixed i, the series

$$\sum_n \left\langle e_k^{\perp}, \mathit{\Gamma} f_n \right\rangle_{\mathcal{H}} [u_n]^{(i)}(s) \quad (k \in \mathbb{N})$$

converge in the space $C(\mathbb{R}, \mathbb{C})$. Indeed, all these series are pointwise dominated on $\mathbb{R}$ by one series

$$\sum_n \|\mathit{\Gamma} f_n\|_{\mathcal{H}} \left| [u_n]^{(i)}(s) \right|,$$

which converges uniformly in $\mathbb{R}$ because its component subseries (see (20), (21))

$$\sum_k \|\mathit{\Gamma} x_k\|_{\mathcal{H}} \left| [g_k]^{(i)}(s) \right|, \quad \sum_k \left\| \mathit{\Gamma} x_k^{\perp} \right\|_{\mathcal{H}} \left| [h_k]^{(i)}(s) \right|$$

are respectively dominated by the series of (17) (see also (8)), and by the series of (15) multiplied by $\|\mathit{\Gamma}\|$. With respect to (24), the dominance argument just given yields in turn that, for each $k \in \mathbb{N}$,

$$\left\| \left[T^* h_{n(k)} \right]^{(i)} \right\|_{C(\mathbb{R},\mathbb{C})} \leq C_i k, \tag{25}$$

with a constant C_i that is independent of k.

Introduce a function $\boldsymbol{P} \colon \mathbb{R}^2 \to \mathbb{C}$ and a Carleman function $\boldsymbol{p} \colon \mathbb{R} \to L^2$, defined, for all s, $t \in \mathbb{R}$, by

$$\begin{aligned} \boldsymbol{P}(s,t) &= \sum_k \left[h_{n(k)}\right](s)\overline{\left[T^* h_{n(k)}\right](t)}, \\ \boldsymbol{p}(s) = \overline{\boldsymbol{P}(s,\cdot)} &= \sum_k \overline{\left[h_{n(k)}\right](s)} T^* h_{n(k)}. \end{aligned} \tag{26}$$

Then again, the corresponding theorems on termwise differentiation of series imply that, for all non-negative integers i and j,

$$\frac{\partial^{i+j}\boldsymbol{P}}{\partial s^i \partial t^j}(s,t) = \sum_k \left[h_{n(k)}\right]^{(i)}(s)\overline{\left[T^* h_{n(k)}\right]^{(j)}(t)},$$

$$\frac{d^i\boldsymbol{p}}{ds^i}(s) = \sum_k \overline{\left[h_{n(k)}\right]^{(i)}(s)} T^* h_{n(k)},$$

in as much as the series just displayed converge (and even absolutely) in $C\left(\mathbb{R}^2, \mathbb{C}\right)$ and $C\left(\mathbb{R}, L^2\right)$, respectively, due to (25) and (16). Thus,

$$\frac{\partial^{i+j}\boldsymbol{P}}{\partial s^i \partial t^j} \in C\left(\mathbb{R}^2, \mathbb{C}\right), \quad \frac{d^i\boldsymbol{p}}{ds^i} \in C\left(\mathbb{R}, L^2\right) \tag{27}$$

whenever i and j are non-negative integers. From (25) and (16), it also follows that the series of (23), viewed as a series with terms belonging to $C(\mathbb{R}, \mathbb{C})$, converges and even absolutely in the $C(\mathbb{R}, \mathbb{C})$ norm, and therefore that its pointwise sum is nothing else than $[Pf]$. On the other hand, the established properties of the series of (26) make it possible to write, for each temporarily fixed $s \in \mathbb{R}$, the following chain of relations

$$\begin{aligned} \sum_k \left\langle f, T^* h_{n(k)}\right\rangle_{L^2} \left[h_{n(k)}\right](s) &= \left\langle f, \sum_k \overline{\left[h_{n(k)}\right](s)} T^* h_{n(k)} \right\rangle_{L^2} \\ &= \int_{\mathbb{R}} \left(\sum_k \left[h_{n(k)}\right](s) \overline{\left[T^* h_{n(k)}\right](t)} \right) f(t)\, dt = \int_{\mathbb{R}} \boldsymbol{P}(s,t) f(t)\, dt \end{aligned}$$

whenever f is in L^2. This in conjunction with (27) implies that the operator $P = UQU^{-1}\colon L^2 \to L^2$ is a Carleman operator with the SK^∞ kernel $\boldsymbol{P}$.

Now, for the second summand in (5), write the Schmidt representation,

$$J^* = \sum_n s_n \left\langle \cdot, q_n \right\rangle_{\mathcal{H}} p_n, \tag{28}$$

where the s_n are the singular values (the eigenvalues of $(JJ^*)^{\frac{1}{2}}$), $\{p_1, p_2, p_3, \dots\}$ and $\{q_1, q_2, q_3, \dots\}$ are orthonormal sets of the singular elements (the p_n are eigenvectors for J^*J and the q_n for JJ^*). By (7) and Lemma XI.9.32 of [1]

$$\sum_n s_n^{\frac{1}{2}} < \infty. \tag{29}$$

Introduce an auxiliary operator $A \in \mathfrak{R}(\mathcal{H})$ by

$$A = \sum_n s_n^{\frac{1}{4}} \langle \cdot, p_n \rangle_{\mathcal{H}} q_n, \tag{30}$$

and note that, by the Schwarz inequality,

$$\begin{aligned} \|Af\|_{\mathcal{H}} + \|A^*f\|_{\mathcal{H}} &= \sqrt{\sum_n s_n^{\frac{1}{2}} \left|\langle f, p_n \rangle_{\mathcal{H}}\right|^2} + \sqrt{\sum_n s_n^{\frac{1}{2}} \left|\langle f, q_n \rangle_{\mathcal{H}}\right|^2} \\ &= \left\|(J^*J)^{\frac{1}{8}} f\right\|_{\mathcal{H}} + \left\|(JJ^*)^{\frac{1}{8}} f\right\|_{\mathcal{H}} \le \|Jf\|_{\mathcal{H}}^{\frac{1}{4}} + \|J^*f\|_{\mathcal{H}}^{\frac{1}{4}} \le d(f) \end{aligned} \tag{31}$$

if $\|f\|_{\mathcal{H}} = 1$. According to (28) and (30), the kernel that induces the integral Hilbert-Schmidt operator $F = UJ^*U^{-1}$ on L^2 is the sum of the bilinear series

$$\sum_n s_n^{\frac{1}{2}} UA^*q_n(s)\overline{UAp_n(t)} \left(= \sum_n s_n Up_n(s)\overline{Uq_n(t)} \right), \tag{32}$$

in the sense of almost everywhere convergence on $\mathbb{R}^2$. The functions used in this bilinear expansion may be represented by the series

$$UAp_k = \sum_n \langle p_k, A^*f_n \rangle_{\mathcal{H}} u_n, \quad UA^*q_k = \sum_n \langle q_k, Af_n \rangle_{\mathcal{H}} u_n \quad (k \in \mathbb{N})$$

convergent in L^2. Show that, for any fixed i, the functions $[UAp_k]^{(i)}$, $[UA^*q_k]^{(i)}$ ($k \in \mathbb{N}$) make sense, are all in $C(\mathbb{R}, \mathbb{C})$, and their $C(\mathbb{R}, \mathbb{C})$ norms are bounded independent of k. Indeed, all the series

$$\sum_n \langle p_k, A^*f_n \rangle_{\mathcal{H}} [u_n]^{(i)}(s), \quad \sum_n \langle q_k, Af_n \rangle_{\mathcal{H}} [u_n]^{(i)}(s) \quad (k \in \mathbb{N})$$

are dominated by one series

$$\sum_n (\|A^*f_n\|_{\mathcal{H}} + \|Af_n\|_{\mathcal{H}}) \left|[u_n]^{(i)}(s)\right|.$$

This series converges uniformly in $\mathbb{R}$, because it is composed of two uniformly convergent subseries on $\mathbb{R}$ (see (21), (22))

$$\begin{gathered} \sum_k \left(\left\|A^*x_k^{\perp}\right\|_{\mathcal{H}} + \left\|Ax_k^{\perp}\right\|_{\mathcal{H}} \right) \left|[h_k]^{(i)}(s)\right|, \\ \sum_k (\|A^*x_k\|_{\mathcal{H}} + \|Ax_k\|_{\mathcal{H}}) \left|[g_k]^{(i)}(s)\right|, \end{gathered}$$

where the first series is dominated by the series of (15) multiplied by $2\|A\|$, and the second by the series of (17), on account of (31).

Now introduce a function $\boldsymbol{F}\colon \mathbb{R}^2 \to \mathbb{C}$ and two Carleman functions $\boldsymbol{f}$, $\boldsymbol{f'}\colon \mathbb{R} \to L^2$, defined by

$$\begin{aligned} \boldsymbol{F}(s,t) &= \sum_n s_n^{\frac{1}{2}} \left[UA^*q_n\right](s)\overline{\left[UAp_n\right](t)}, \\ \boldsymbol{f}(s) = \overline{\boldsymbol{F}(s,\cdot)} &= \sum_n s_n^{\frac{1}{2}} \overline{\left[UA^*q_n\right](s)} UAp_n, \\ \boldsymbol{f'}(t) = \boldsymbol{F}(\cdot,t) &= \sum_n s_n^{\frac{1}{2}} UA^*q_n \overline{\left[UAp_n\right](t)}, \end{aligned} \tag{33}$$

whenever s, $t \in \mathbb{R}$ (cf. (32)). Then, for all non-negative integers i, j and all s, $t \in \mathbb{R}$,

$$\frac{\partial^{i+j}\boldsymbol{F}}{\partial s^i \partial t^j}(s,t) = \sum_n s_n^{\frac{1}{2}} \left[UA^*q_n\right]^{(i)}(s)\overline{\left[UAp_n\right]^{(j)}(t)},$$

$$\frac{d^i\boldsymbol{f}}{ds^i}(s) = \sum_n s_n^{\frac{1}{2}} \overline{\left[UA^*q_n\right]^{(i)}(s)} UAp_n, \quad \frac{d^j\boldsymbol{f'}}{dt^j}(t) = \sum_n s_n^{\frac{1}{2}} UA^*q_n \overline{\left[UAp_n\right]^{(j)}(t)},$$

as the series just written converge (and even absolutely) in $C\left(\mathbb{R}^2, \mathbb{C}\right)$ and in $C\left(\mathbb{R}, L^2\right)$, respectively, due to (29). It follows that $\boldsymbol{F}$ satisfies parts (i), (ii), and (iii), of Definition 1, and therefore is a Hilbert-Schmidt K^∞ kernel of F.

In accordance with (5), $T = P + F$, and hence, for each $f \in L^2$,

$$(Tf)(s) = \int_{\mathbb{R}} (\boldsymbol{P}(s,t) + \boldsymbol{F}(s,t)) f(t)\, dt \tag{34}$$

for almost every s in $\mathbb{R}$. Therefore T is a Carleman operator, and that kernel $\boldsymbol{T}$ of T, which is defined by $\boldsymbol{T}(s,t) = \boldsymbol{P}(s,t) + \boldsymbol{F}(s,t)$ whenever s and t are in $\mathbb{R}$, inherits the SK^∞ kernel properties from its terms. Consequently, the Carleman operator $T (= US_rU^{-1})$ has the SK^∞ kernel $\boldsymbol{T}$. Upon recalling that r was arbitrary, this finally implies that all the operators $T_r = US_rU^{-1}$ ($r \in \mathbb{N}$) are Carleman operators having SK^∞ kernels. That completes the proof of the theorem.

3. Application to Third-Kind Integral Equations

Throughout this section, the measure space (Y, μ) is nonatomic (that is, every μ-measurable set $E \subset Y$ can be divided into two disjoint subsets of equal measure). The *general integral equation of the third kind* in $L^2(Y, \mu)$ is an equation of the form

$$\boldsymbol{H}(x)\phi(x) - \lambda \int_Y \boldsymbol{K}(x,y)\phi(y)\,d\mu(y) = \psi(x) \quad \text{for } \mu\text{-almost all } x \in Y, \tag{35}$$

where $\boldsymbol{H} : Y \to \mathbb{C}$ (the coefficient of the equation) is a given bounded μ-measurable function, $\boldsymbol{K} : Y \times Y \to \mathbb{C}$ (the kernel of the equation) is a given $\mu \times \mu$-measurable function inducing an integral operator $K \in \mathfrak{R}\left(L^2(Y,\mu)\right)$ by $(K\phi)(x) = \int_Y \boldsymbol{K}(x,y)\phi(y)\,d\mu(y)$, the scalar λ of $\mathbb{C}$ is given, the function ψ of $L^2(Y,\mu)$ is given, and the function ϕ of $L^2(Y,\mu)$ is to be determined.

Theorem 3. *Suppose that the essential range of the coefficient $\boldsymbol{H}$ in* (35) *contains the point $\alpha \in \mathbb{C}$, that is,*

$$\mu\{y \in Y : |\boldsymbol{H}(y) - \alpha| < \varepsilon\} > 0 \quad \textit{for all } \varepsilon > 0. \tag{36}$$

Then equation (35) *is equivalent (via a unitary transformation) to a second-kind integral equation in L^2, of the form*

$$\alpha f(s) + \int_{\mathbb{R}} \left(\boldsymbol{T}_0(s,t) - \lambda \boldsymbol{T}(s,t)\right) f(t)\,dt = g(s) \quad \textit{for almost all } s \in \mathbb{R}, \tag{37}$$

where the function f of L^2 is to be determined, and both the functions $\boldsymbol{T}_0$ and $\boldsymbol{T}$ are SK^∞ kernels not depending on λ.

Proof. The proof relies primarily on the following observation by Korotkov [5, Corollary 1]: If H is the multiplication operator induced on $L^2(Y,\mu)$ by the coefficient $\boldsymbol{H}$, and I is identity operator on $L^2(Y,\mu)$, then the two-element family $\mathcal{B} = \{B_1 = H - \alpha I, B_2 = K\}$ of bounded operators on $\mathcal{H} = L^2(Y,\mu)$ satisfies the assumptions of Theorem 2. The construction of Korotkov's sequence $\{v_n\}$ fulfilling (2) for this $\mathcal{B}$ is likely to be of practical use and deserves to be expounded in some detail.

If $E \subset Y$ is a μ-measurable set of positive finite measure, the orthonormal sequence of *generalized* Rademacher functions with supports in E will be denoted by $\{R_{n,E}\}_{n=1}^{\infty}$ and is constructed iteratively through successive bisections of E as follows:

$$R_{1,E} = \frac{\chi_{E_1} - \chi_{E_2}}{\sqrt{\mu E}} \quad \text{provided } E_1 \sqcup E_2 = E \text{ with } \mu E_1 = \mu E_2 = \frac{1}{2}\mu E,$$

$$R_{2,E} = \frac{\chi_{E_{1,1}} - \chi_{E_{1,2}} + \chi_{E_{2,1}} - \chi_{E_{2,2}}}{\sqrt{\mu E}}$$

$$\text{provided } E_{i,1} \sqcup E_{i,2} = E_i \text{ with } \mu E_{i,k} = \frac{1}{4}\mu E \text{ for } i,k = 1,2,$$

$$R_{3,E} = \frac{\chi_{E_{1,1,1}} - \chi_{E_{1,1,2}} + \chi_{E_{1,2,1}} - \chi_{E_{1,2,2}} + \chi_{E_{2,1,1}} - \chi_{E_{2,1,2}} + \chi_{E_{2,2,1}} - \chi_{E_{2,2,2}}}{\sqrt{\mu E}}$$

$$\text{provided } E_{i,k,1} \sqcup E_{i,k,2} = E_{i,k} \text{ with } \mu E_{i,k,j} = \frac{1}{8}\mu E \text{ for } i,k,j = 1,2,$$

and so on indefinitely; here χ_Z denotes the characteristic function of a set Z and the unions are disjoint. A relevant result due to Korotkov states that

$$\lim_{n\to\infty} \|K^* R_{n,E}\|_{L^2(Y,\mu)} = 0 \tag{38}$$

for any integral operator $K \in \mathfrak{R}\left(L^2(Y,\mu)\right)$ and any μ-measurable $E \subset Y$ with $0 < \mu E < \infty$ (see, e.g., the proof of Theorem 3 in [4]).

Let $\{Y_n\}_{n=1}^{\infty}$ be an ascending sequence of sets of positive finite measure, such that $Y_n \uparrow Y$, let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a sequence of positive reals strictly decreasing to zero, and define $E_n = Y_n \cap \{y \in Y : \varepsilon_{n+1} < |\boldsymbol{H}(y) - \alpha| \le \varepsilon_n\}$ whenever $n \in \mathbb{N}$. Due to the assumption (36), one can always make the sets E_n to have finite nonzero measures by an appropriate choice of Y_n and ε_n ($n \in \mathbb{N}$). Having done so, let $v_n = R_{k_n,E_n}$, where, for each $n \in \mathbb{N}$, k_n is an index satisfying

$$\|B_2^* v_n\|_{L^2(Y,\mu)} = \|K^* R_{k_n,E_n}\|_{L^2(Y,\mu)} \le \frac{1}{n} \tag{39}$$

(cf. (38)). Since the E_n's are pairwise disjoint, the v_n's form an orthonormal sequence in $L^2(Y,\mu)$. Moreover, by construction of sets E_n,

$$\|B_1^* v_n\|^2_{L^2(Y,\mu)} = \frac{1}{\mu E_n} \int_{E_n} |\boldsymbol{H}(y) - \alpha|^2 \, d\mu(y) \le \varepsilon_n^2. \tag{40}$$

One can now assert from (39), (40) that $\|B_r^* v_n\|_{L^2(Y,\mu)} \to 0$ as $n \to \infty$ for $r = 1,2$. By Theorem 2, there is then a unitary operator $U : L^2(Y,\mu) \to L^2$ such that the operators $T_0 = UB_1U^{-1}$, $T = UB_2U^{-1}$ are both integral operators with SK^∞ kernels. This unitary operator can also be used to transform the integral equation (35) into an equivalent integral equation of the form (37) in such a way that $\boldsymbol{T}_0$ and $\boldsymbol{T}$ are just those SK^∞ kernels that induce T_0 and T, respectively. In operator notation, such a passage from (35) to (37) looks as follows: $U\psi = U(H - \lambda K)U^{-1}U\phi = U(\alpha I + B_1 - \lambda B_2)U^{-1}U\phi = \alpha f + (T_0 - \lambda T)f = g$ where $f = U\phi$, $g = U\psi$. The theorem is proved. □

Theorem 4. *If, with the notation and hypotheses of Theorem 3, $\alpha = 0$, then equation* (35) *is equivalent to a first-kind integral equation in L^2, of the form*

$$\int_{\mathbb{R}} \left(\boldsymbol{\Gamma}_0(s,t) - \lambda \boldsymbol{\Gamma}(s,t)\right) f(t)\, dt = w(s) \quad \textit{for almost all } s \in \mathbb{R}, \tag{41}$$

where the function f of L^2 is to be determined, and both the functions $\boldsymbol{\Gamma}_0$ and $\boldsymbol{\Gamma}$ are Hilbert-Schmidt K^∞ kernels not depending on λ.

Proof. In this case the equation (37), equivalent to (35), becomes

$$\int_{\mathbb{R}} \left(\boldsymbol{T}_0(s,t) - \lambda \boldsymbol{T}(s,t)\right) f(t)\, dt = g(s) \quad \text{for almost all } s \in \mathbb{R}. \tag{42}$$

Let $m \in L^2$ be such that $[m]$ is an infinitely differentiable, positive function all whose derivatives $[m]^{(i)}$ belong to $C(\mathbb{R},\mathbb{R})$, and let M be the multiplication operator induced on L^2 by m. Multiply both sides of equation (42) by m, to recast it into an equivalent equation of the form (41), with the same sought-for function $f \in L^2$, the new right side $w = Mg \in L^2$, and the new kernel $\boldsymbol{\Gamma}_0 - \lambda \boldsymbol{\Gamma}$, where $\boldsymbol{\Gamma}_0(s,t) = [m](s)\boldsymbol{T}_0(s,t)$, $\boldsymbol{\Gamma}(s,t) = [m](s)\boldsymbol{T}(s,t)$. It is to be proved that $\boldsymbol{\Gamma}_0$, $\boldsymbol{\Gamma}$ are Hilbert-Schmidt K^∞ kernels. The proof is further given only for $\boldsymbol{\Gamma}$, as the proof for the other kernel $\boldsymbol{\Gamma}_0$ is entirely similar. If $\boldsymbol{t}$ is the associated Carleman function of the SK^∞ kernel $\boldsymbol{T}$, then

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |\boldsymbol{\Gamma}(s,t)|^2\, dt\, ds = \int_{\mathbb{R}} m^2(s) \|\boldsymbol{t}(s)\|^2_{L^2}\, ds \leq \|\boldsymbol{t}\|^2_{C(\mathbb{R},L^2)} \|m\|^2_{L^2} < \infty,$$

implying that $\boldsymbol{\Gamma}$ is a Hilbert-Schmidt kernel and hence does induce two Carleman functions $\boldsymbol{\gamma}$, $\boldsymbol{\gamma'}\colon \mathbb{R} \to L^2$ by $\boldsymbol{\gamma}(s) = \overline{\boldsymbol{\Gamma}(s,\cdot)}$, $\boldsymbol{\gamma'}(t) = \boldsymbol{\Gamma}(\cdot,t)$. The series representation of $\boldsymbol{T}$ (see (34), (26), and (33)) gives rise to a series representation of $\boldsymbol{\Gamma}$, namely, with the notation of the proof of Theorem 2,

$$\boldsymbol{\Gamma}(s,t) = \sum_k \left[Mh_{n(k)}\right](s)\overline{\left[T^* h_{n(k)}\right](t)} + \sum_n s_n^{\frac{1}{2}} \left[MUA^* q_n\right](s)\overline{\left[UAp_n\right](t)}$$

for all $(s,t) \in \mathbb{R}^2$. Moreover, for all non-negative integers i, j and all s, $t \in \mathbb{R}$,

$$\begin{aligned} \frac{\partial^{i+j}\boldsymbol{\Gamma}}{\partial s^i \partial t^j}(s,t) = &\sum_k \left[Mh_{n(k)}\right]^{(i)}(s)\overline{\left[T^* h_{n(k)}\right]^{(j)}(t)} \\ &+ \sum_n s_n^{\frac{1}{2}} \left[MUA^* q_n\right]^{(i)}(s)\overline{\left[UAp_n\right]^{(j)}(t)}, \end{aligned} \tag{43}$$

$$\frac{d^i\boldsymbol{\gamma}}{ds^i}(s) = \sum_k \overline{\left[Mh_{n(k)}\right]^{(i)}(s)}\, T^* h_{n(k)} + \sum_n s_n^{\frac{1}{2}} \overline{\left[MUA^* q_n\right]^{(i)}(s)}\, UAp_n, \tag{44}$$

$$\frac{d^j\boldsymbol{\gamma'}}{dt^j}(t) = \sum_k Mh_{n(k)} \overline{\left[T^* h_{n(k)}\right]^{(j)}(t)} + \sum_n s_n^{\frac{1}{2}} MUA^* q_n \overline{\left[UAp_n\right]^{(j)}(t)}, \tag{45}$$

in as much as the series in the right sides of these equalities converge absolutely in $C(\mathbb{R}^2,\mathbb{C})$, as regards (43), and in $C(\mathbb{R},L^2)$, as regards (44) and (45); their dominant series are respectively:

$$\sum_k C_j \sum_{r=0}^{i} kH_{n(k),r} \cdot C_r^i, \quad \sum_n s_n^{\frac{1}{2}} \cdot \left\| [UAp_n]^{(j)} \right\|_{C(\mathbb{R},\mathbb{C})} \sum_{r=0}^{i} C_r^i \left\| [UA^* q_n]^{(r)} \right\|_{C(\mathbb{R},\mathbb{C})},$$

$$\sum_k \|T\| \sum_{r=0}^{i} H_{n(k),r} \cdot C_r^i, \quad \sum_n s_n^{\frac{1}{2}} \cdot \|A\| \sum_{r=0}^{i} C_r^i \left\| [UA^* q_n]^{(r)} \right\|_{C(\mathbb{R},\mathbb{C})},$$

$$\sum_k C_j \|m\|_{L^2} \cdot kH_{n(k),0}, \quad \sum_n s_n^{\frac{1}{2}} \cdot \|MUA^* U^{-1}\| \left\| [UAp_n]^{(j)} \right\|_{C(\mathbb{R},\mathbb{C})},$$

where $C_r^i = \binom{i}{r} \left\| [m]^{(i-r)} \right\|_{C(\mathbb{R},\mathbb{R})}$ (see also (25), (16), and (29)). Therefore $\frac{\partial^{i+j}\boldsymbol{\Gamma}}{\partial s^i \partial t^j} \in C\left(\mathbb{R}^2, \mathbb{C}\right)$, $\frac{d^i \boldsymbol{\gamma}}{ds^i}$, $\frac{d^j \boldsymbol{\gamma}'}{dt^j} \in C\left(\mathbb{R}, L^2\right)$ for all non-negative integers i, j, implying that $\boldsymbol{\Gamma}$ is a K^∞ kernel. The theorem is proved. □

Remark. To solve equation (41), Picard's theorem on the solvability of first-kind integral equations can be applied; in addition, Schmidt's singular values and elements, needed by the theorem to write down explicitly the solution to that equation, can be determined in terms of Fredholm formulae (see [7], [8], and [10], for details).

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