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ON THE FINITE CM-GROUPS

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Abstract: There are a number of analogies between results of character degrees and conjugacy class sizes in finite groups. By analysis of the definition of monomial group (for short, M-group), we give several definitions of conjugacy class monomial groups (for short, CM-groups) and strong conjugacy class monomial groups (for short, SCM-groups). Moreover, we investigate the solvability of CM-groups and SCM-groups.

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1. Introduction

If an irreducible complex character χ of G is induced by a linear character of some subgroup, then χ is said to be monomial; furthermore, if every irreducible complex character of G is monomial, then G is said to be an M-group. A famous result [4, Theorem 5.13] due to Taketa shows that all M-groups are solvable. Berkovich further proved (see [1]) that all M$_i$-groups are solvable for $i \leq 3$. Here M_i-group G means that every irreducible character of G is induced from a character of degree at most i. König also generalized (see [5]) M-groups as follows: A finite group G is said to be a generalized M-group if every irreducible irreducible character of G has a multiple which is induced from a character ϕ of some subgroup U such that the factor $U/ker\phi$ is solvable. And it is also proved that the generalized M-groups are solvable.

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It is an interesting phenomena [2] that there exist a number of analogies between results on character degrees and conjugacy class sizes in finite groups. Roughly speaking, assume that an assertion which is expressible in terms of irreducible character degrees is true, if irreducible character degrees are replaced with conjugacy class sizes, then the resulting assertion is usually also true, and vice versa.

It is known that if the irreducible character χ is induced from a linear character ϕ of some subgroup U of G, then $\chi(1) = |G : U|$ and $U/ker\phi$ is cyclic. For any $x \in G$, the conjugacy class size $|x^G| = |G : C_G(x)|$. These basic facts inspire us to give conjugacy class analogs of the definition of M-group as follows.

Definition 1.1. Let G be a finite group. If one of the following holds, then G is called a conjugacy class monomial group (abbreviated CM-group).

1. $C_G(x)$ is solvable for every noncentral element $x \in G$.
2. Some subgroup $M \geq C_G(x)$ such that some quotient of M is a non-trivial cyclic group for every noncentral element $x \in G$.
3. $C_G(x)$ is subnormal for every noncentral element $x \in G$.

Dropping "noncentral" in the above definition, we further present the following definition. Note that the "dropping" is trivial on Definition 1.1(3).

Definition 1.2. Let G be a finite group. If one of the following holds, then G is called a strong conjugacy class monomial group (abbreviated SCM-group).

1. $C_G(x)$ is solvable for every element $x \in G$.
2. Some subgroup $M \geq C_G(x)$ such that some quotient of M is a non-trivial cyclic group for every element $x \in G$.

In this note, we discuss the solvability of CM-groups and SCM-groups under the above definitions. Observe that all of solvable groups are CM-groups and SCM-groups. All groups considered here are finite.

2. Solvability of CM-Groups

Theorem 2.1. *Let G be a CM-group respecting to Definition 1.1(3) above. Then G is a nilpotent group.*

Proof. We argue by induction on the order of G. For any noncentral element x, we get a normal subgroup M such that $C_G(x) \leq M \trianglelefteq G$. Thus

G can be expressible as a product of its normal subgroups. It is no loss to assume that M is a proper normal subgroup. We have that $C_G(y) \leq N \trianglelefteq G$ for any $y \in M - Z(M)$, thus $C_M(y) < M \cap N \trianglelefteq M$, the application of inductive argument to M yields that M is nilpotent. Therefore we conclude that G may be written as a product of its nilpotent normal subgroups. Since also each Sylow subgroup of these nilpotent normal subgroups is a characteristic subgroup of G, it follows that G can be expressed as the direct product of its Sylow subgroups, thus G is nilpotent. $\square$

Remark 2.2. Under Definition 1.1 (1) and (2) above, CM-groups need not be solvable.

A_5 is a counterexample. By [3], A_5 has 5 conjugacy classes, the orders of non-trivial centralizers are $4, 3, 5, 5$, respectively. Thus these centralizers are all Abelian. We conclude that A_5 is a counterexample to Definition 1.1(1). Furthermore, A_5 implies that even if every non-trivial centralizer is Abelian in Definition 1.1(1) above, G need not be solvable. Note that A_5 is still a minimal simple group (see [7]). Hence every proper subgroup H of A_5 is solvable and so some quotient of H is a non-trivial cyclic group.

3. Solvability of SCM-Groups

Theorem 3.1. *Assume that G is a SCM-group admitting Definition 2.1(2) above. Then G is a solvable group.*

Proof. Pick $x \in Z(G)$, then $G = C_G(x)$, thus there exists normal subgroup M such that G/M is cyclic. Take $y \in Z(M)$, then $C_G(y)/N$ is cyclic for some normal subgroup N, and so is $M/M \cap N \cong MN/N$. Repeating this process, it follows that G has a composite series whose factors are all cyclic, thus G is solvable. $\square$

Remark 3.2. If the "cyclic" condition is weakened to "solvable" in Definition 1.2(2), Theorem 3.1 is still true.

Remark 3.3. The group which satisfies Definition 1.2(1) is obvious solvable.

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