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A NEW ITERATION PROCEDURE FOR STOCHASTIC NEUTRAL PARTIAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract: In this paper, we study stochastic neutral partial functional differential equations in real separable Hilbert spaces. Our aim here is to introduce a new iteration procedure for such class of equations. Using this, we investigate the existence and uniqueness of a mild solution and also the almost sure exponential stability of its sample paths. The results obtained here are less restrictive than those of Govindan, *Stochastics*, **77** (2005), 139-154.

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1. Introduction

In this paper, we consider a stochastic neutral partial functional differential equation in a real separable Hilbert space of the form:

$$d[x(t) + f(t, \pi_t x)] = [Ax(t) + a(t, \pi_t x)]dt + b(t, \pi_t x)dw(t), \quad t > 0; \tag{1}$$

$$x(t) = \varphi(t), \quad t \in [-r, 0] \quad (0 \le r < \infty);$$

where $\pi_t x = \{x(t - r + s) : 0 \le s \le r\}$, $A : D(A) \subseteq X \to X$ is the infinitesimal generator of a strongly continuous semigroup $\{S(t), t \ge 0\}$ defined on X, $a : R^+ \times C_t \to X (R^+ = [0, \infty))$, $f : R^+ \times C_t \to D((-A)^\alpha)$ and $b : R^+ \times C_t \to L(Y, X)$ are Borel measurable: and for each (t, u) are measurable; with respect

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to the σ–algebra $B_{0,t}(dw)$. Here $w(t)$ is a Y–valued Q–Wiener process and the past stochastic process $\{\varphi(t), t \in [-r, 0]\}$ has almost sure (a.s) continuous paths with $E||\varphi(\cdot, \omega)||_0^p < \infty$, $p \geq 2$.

Equation (1) was studied recently in Govindan [2]. Using global Lipschitz conditions on the nonlinear terms, existence and uniqueness of a mild solution; and the almost sure exponential stability of mild solutions were investigated in [2] by exploiting the theory of stochastic convolution integrals, see Da Prato and Zabyczyk [1], semigroup theory, see Pazy [3] and an iteration procedure introduced therein. For a motivation of a study on equation (1), we refer to Govindan [2].

In this article, our goal is to introduce yet another iteration procedure for equation (1) which, of course, is a slight variant of the one defined in [2]; and pose exactly the same questions as in [2]. The current iteration procedure is simpler and easier to handle than the one considered before in [2] and further yields more general existence and stability results, see Remarks 3.1 and 4.1 below for details. But, in order to consider this iteration procedure and to push the details through, we need some results from the semigroup theory and stochastic convolution integrals as given in Lemmas 3.2 and 4.2. The results given in these lemmas can be considered as results of independent interest.

The format of the rest of the paper is as follows. In the second section we give the preliminaries. Since we work in the same framework of [2], we shall be quite brief here. We will state all our hypotheses and also the main results from [2]. In the third section, we consider the existence and uniqueness of a mild solution of (1). In the fourth section, we discuss the almost sure asymptotic behavior of a mild solution. Finally, in the fifth section, we give an example to illustrate the theory.

2. Preliminaries

Let X, Y be real separable Hilbert spaces and $L(Y, X)$ be the space of bounded linear operators mapping Y into X. We shall use the same notation $|.|$ to denote the norms in X, Y and $L(Y, X)$. Let $(\Omega, B, P, \{B_t\}_{t\geq 0})$ be a complete probability space with an increasing right continuous family $\{B_t\}_{t\geq 0}$ of complete sub–σ–algebras of B. Let $\beta_n(t)(n = 1, 2, 3, \ldots)$ be a sequence of real–valued standard Brownian motions mutually independent defined on this probability space. Set

$$w(t)=\sum_{n=1}^{\infty}\sqrt{\lambda_n}\beta_n(t)e_n,\quad t\geq 0,$$

where $\lambda_n \geq 0$ $(n=1,2,3,\ldots)$ are nonnegative real numbers and $\{e_n\}$ $(n=1,2,3,\ldots)$ is a complete orthonormal basis in Y. Let $Q\in L(Y,Y)$ be an operator defined by $Qe_n=\lambda_n e_n$. The above Y–valued stochastic process $w(t)$ is called a Q–Wiener process.

Definition 2.1. Let $h(t)$ be an $L(Y,X)$–valued function and let λ be a sequence $\{\sqrt{\lambda_1},\sqrt{\lambda_2},\ldots\}$. Then we define

$$|h(t)|_\lambda=\left\{\sum_{n=1}^{\infty}|\sqrt{\lambda_n}h(t)e_n|^2\right\}^{1/2}.$$

If $|h(t)|_\lambda^2<\infty$, then $h(t)$ is called λ–Hilbert–Schmidt operator and let $\sigma(\lambda)(Y,X)$ denote the space of all λ–Hilbert–Schmidt operators from Y to X.

Definition 2.2. Let $\Phi:[0,\infty)\to\sigma(\lambda)(Y,X)$ be a predictable, B_t–adapted process. Then, for any Φ satisfying $\int_0^t E|\Phi(s)|_\lambda^2 ds<\infty$ we define the X–valued stochastic integral $\int_0^t\Phi(s)dw(s)\in X$ with respect to $w(t)$ by

$$\left(\int_0^t\Phi(s)dw(s),h\right)=\int_0^t\langle\Phi^*(s)h,dw(s)\rangle,\quad h\in X,$$

where Φ^* is the adjoint operator of Φ.

Given a stochastic process $\{x(t),t\geq 0\}$, $B_{r,s}(x)$ denotes the least sub–σ–algebra of the σ–algebra B generated by $\{x(t),r\leq t\leq s\}$. $B_{r,s}(dw)$ denotes the least sub–σ–algebra of B generated by the increments $\{w(s)-w(t),r\leq t\leq s\}$ of the Y–valued Q–Wiener process $\{w(t),t\geq 0\}$ with $w(0)=0$. The least sub–σ–algebra of B corresponding to the past process is denoted by $B_{-r,0}(\varphi)$ with respect to which φ is measurable and not be dependent on $B_{0,\infty}(dw)$. The least sub–σ–algebra that contains the σ–algebras $B_1,B_2,\ldots,B_n$ is denoted by $B_1\vee B_2\vee\cdots\vee B_n$.

Let C_T be the space of continuous functions $x:[-r,T]\to X(0<T<\infty)$ with the norm $||x||_T=\sup_{-r\leq s\leq T}|x(s)|$. Let $B_T=B_T(\varphi)$ be the space of measurable random processes $\phi(t,\omega)$ with a.s. continuous paths; for each $t\in[0,T],\phi(t,\omega)$ is measurable with respect to $B_{0,t}(dw)$ and $\phi(s,\omega)=\varphi(s,\omega)$ for $-r\leq s\leq 0$ with the norm $||\phi||_{B_T}=(E||\phi(\cdot,\omega)||_T^p)^{1/p}$. B_T is a Banach space.

Definition 2.3. A semigroup is said to be exponentially stable if there exist positive constants M,a such that $||S(t)||\leq M\exp(-at)$, $t\geq 0$, where $||\cdot||$ denotes the operator norm in X. If $M=1$, then the semigroup is said to be a contraction semigroup. A semigroup $\{S(t), t \geq 0\}$ is said to be uniformly bounded if $||S(t)|| \leq M$, for all $t \geq 0$, where $M \geq 1$ is some constant.

If $\{S(t), t \geq 0\}$ is an analytic semigroup with infinitesimal generator A such that $0 \in \rho(A)$ (the resolvent of A) then it is possible to define the fractional power $(-A)^\alpha$, for $0 < \alpha \leq 1$ as a closed linear operator on its domain $D((-A)^\alpha)$. Further, the subspace $D((-A)^\alpha)$ is dense in X and the expression

$$||x||_\alpha = |(-A)^\alpha x|, \quad x \in D((-A)^\alpha),$$

defines a norm on $D((-A)^\alpha)$.

Next, we recall the mild solution of (1).

Definition 2.4. A stochastic process $\{x(t), t \in [-r, T]\}(0 < T < \infty)$ is called a mild solution of equation (1) if

i) $x(t)$ is B_t–adapted and predictable with $\int_0^T |x(t)|^2 dt < \infty$, a.s.,

ii) $x(t) = \varphi(t)$, $t \in [-r, 0]$ a.s. and $B_{-r,t}(x) \vee B_{0,t}(dw)$ does not depend on $B_{t,\infty}(dw)$ for each $t \in [0, T]$, and

iii) $x(t)$ satisfies the integral equation

$$\begin{aligned} x(t) &= S(t)[\varphi(0) + f(0, \varphi)] - f(t, \pi_t x) \\ &- \int_0^t AS(t-s) f(s, \pi_s x) ds + \int_0^t S(t-s) a(s, \pi_s x) ds \\ &+ \int_0^t S(t-s) b(s, \pi_s x) dw(s), \quad \text{a.s.}, \quad t \in [0, T]. \end{aligned}$$

Let the following assumptions hold a.s.:

(H1) $-A$ is the infinitesimal generator of an analytic semigroup of bounded linear operators $\{S(t), t \geq 0\}$ in X and that the semigroup is uniformly bounded.

(H2) For $p \geq 2$, the functions $a(t, u)$ and $b(t, u)$ satisfy the Lipschitz and linear growth conditions:

$$\begin{aligned} |a(t, \pi_t x) - a(t, \pi_t y)|^p &\leq C_1 ||x - y||_t^p, \quad C_1 > 0; \\ |b(t, \pi_t x) - b(t, \pi_t y)|_\lambda^p &\leq C_2 ||x - y||_t^p, \quad C_2 > 0; \\ |a(t, \pi_t x)|^p + |b(t, \pi_t x)|_\lambda^p &\leq C_3 (1 + ||x||_t^p), \quad C_3 > 0. \end{aligned}$$

(H3) $f(t, u)$ is a function continuous in t and satisfies:

$$\begin{aligned} |(-A)^\alpha f(t, \pi_t x) - (-A)^\alpha f(t, \pi_t y)| &\leq C_4 ||x - y||_t, \quad C_4 > 0; \\ |(-A)^\alpha f(t, \pi_t x)| &\leq C_5 (1 + ||x||_t), \quad C_5 > 0. \end{aligned}$$

Theorem 2.5. (see [2]) *Let the assumptions (H1)–(H3) hold. Suppose that for the case $p = 2$, the semigroup $\{S(t), t \geq 0\}$ is a contraction semigroup.*

Then there exists a unique mild solution $x(t)$ of equation (1) in B_T provided

$$(2+M)C||(-A)^{-\alpha}|| < 1,$$

where $C = \max\{C_4, C_5\}$, and $M = 1$ when $p = 2$.

Theorem 2.6. (see [2]) *Let the hypothesis of Theorem 2.5 hold. Suppose that the semigroup $\{S(t), t \geq 0\}$ is exponentially stable. Then the mild solution of equation (1) is exponentially stable in the quadratic mean provided*

$$a > \beta = \frac{3(C_1/a + 4C_2)}{[1 - 3C_4||(-A)^{-\alpha}||]^2}.$$

Theorem 2.7. (see [2]) *Suppose that all the conditions of Theorem 2.6 hold. Then the mild solution of (1) satisfies*

$$\limsup_{t\to\infty}(1/t)\log|x(t)| \leq (a-\beta)/4, \quad \text{a.s.}$$

3. Existence and Uniqueness of a Solution

In this section, we prove the existence and uniqueness of a mild solution of equation (1).

Let the following assumption hold:

(H4) $-A$ is the infinitesimal generator of an analytic semigroup of bounded linear operators $\{S(t), t \geq 0\}$ in X and that the semigroup is exponentially stable.

Theorem 3.1. *Let the assumptions (H2)-(H4) hold. Suppose that for $p = 2$, the semigroup $\{S(t), t \geq 0\}$ is a contraction semigroup. Then there exists a unique mild solution $x(t)$ of equation (1) provided $1/p < \alpha < 1$, and*

$$C||(-A)^{-\alpha}|| < 1,$$

where $C = \max\{C_4, C_5\}$.

To prove this theorem, let us introduce the following iteration procedure: Define for each integer $n = 1, 2, 3, \ldots$,

$$\begin{aligned} x_n(t) &= S(t)[\varphi(0) + f(0,\varphi)] - f(t, \pi_t x_n) \\ &- \int_0^t AS(t-s)f(s, \pi_s x_{n-1})ds + \int_0^t S(t-s)a(s, \pi_s x_{n-1})ds \\ &+ \int_0^t S(t-s)b(s, \pi_s x_{n-1})dw(s), \quad t \in [0,T], \end{aligned} \tag{2}$$

and for $n = 0$,

$$x_0(t) = S(t)\varphi(0), \quad t \in [0, T], \tag{3}$$

while for $n = 0, 1, 2, \ldots$

$$x_n(t) = \varphi(t), \quad t \in [-r, 0].$$

Remark 3.1. When $f \equiv 0$, the iteration scheme just introduced above reduces to the corresponding one in this special case, see Govindan [2] and the references therein. Note that the condition in Theorem 3.1 is less restrictive than the one in Theorem 2.5. In other words, the condition in Theorem 2.5 states that

$$C||(-A)^{-\alpha}|| < 1/(2 + M),$$

which is clearly more restrictive than the one in Theorem 3.1. However, in Theorem 3.1, α satisfies a resriction that $1/p < \alpha < 1$.

We will need the following lemma.

Lemma 3.2. *Let $-A$ be the infinitesimal generator of an analytic semigroup of bounded linear operators $\{S(t), t \geq 0\}$ in X. Then, for any stochastic process $F : [0, \infty) \to X$ which is strongly measurable with*

$$\int_0^T E|(-A)^\alpha F(t)|^p dt < \infty, \qquad p \geq 2 \text{ and } 0 < T \leq \infty,$$

the following inequality holds for $0 < t \leq T$:

$$E\left|\int_0^t (-A)S(t-s)F(s)ds\right|^p \leq k(p, a, \alpha) \int_0^t E|(-A)^\alpha F(s)|^p ds,$$

provided $1/p < \alpha < 1$, where

$$k(p, a, \alpha) = M_{1-\alpha}^p \frac{(p-1)^{p\alpha-1}[\Gamma((p\alpha-1)/(p-1))]^{p-1}}{(pa)^{p\alpha-1}},$$

and $\Gamma(\cdot)$ is the Gamma function.

Proof. By Pazy Theorem 6.13 on pp. 74-75 in [3] and Hölder's inequality, taking $q = p/(p-1)$ we get:

$$\begin{aligned} E\left|\int_0^t (-A)^{1-\alpha} S(t-s)(-A)^\alpha F(s)ds\right|^p \\ \leq E\left(\int_0^t M_{1-\alpha}(t-s)^{-(1-\alpha)} e^{-a(t-s)} |(-A)^\alpha F(s)| ds\right)^p \\ \leq M_{1-\alpha}^p \left(\int_0^t (t-s)^{-p(1-\alpha)/(p-1)} e^{-pa(t-s)/(p-1)} ds\right)^{p-1} \int_0^t E|(-A)^\alpha F(s)|^p ds \end{aligned}$$

$$
\begin{aligned}
&\leq M_{1-\alpha}^{p}\left(\int_0^t (t-s)^{1-q(1-\alpha)-1}e^{-qa(t-s)}ds\right)^{p-1}\int_0^t E|(-A)^{\alpha}F(s)|^p ds \\
&\quad \leq M_{1-\alpha}^{p}[\Gamma(1-q+q\alpha)/(qa)^{1-q-q\alpha}]^{p-1}\int_0^t E|(-A)^{\alpha}F(s)|^p ds \\
&\quad \leq M_{1-\alpha}^{p}\frac{(p-1)^{p\alpha-1}[\Gamma((p\alpha-1)/(p-1))]^{p-1}}{(pa)^{p\alpha-1}}\int_0^t E|(-A)^{\alpha}F(s)|^p ds. \quad \square
\end{aligned}
$$

Proof of Theorem 3.1. Let T be any fixed time with $0 < T < \infty$. It follows from (2) that

$$
\begin{aligned}
x_n(s) \;&=\; S(s)\varphi(0) + (-A)^{-\alpha}S(s)(-A)^{\alpha}f(0,\varphi) \\
&-\; (-A)^{-\alpha}(-A)^{\alpha}f(s,\pi_s x_n) \\
&+\; \int_0^s (-A)^{1-\alpha}S(s-\tau)(-A)^{\alpha}f(\tau,\pi_\tau x_{n-1})d\tau \\
&+\; \int_0^s S(s-\tau)a(\tau,\pi_\tau x_{n-1})d\tau \\
&+\; \int_0^s S(s-\tau)b(\tau,\pi_\tau x_{n-1})dw(\tau), \quad s\in[0,T].
\end{aligned}
$$

By assumptions (H4) and (H3), we obtain

$$
\begin{aligned}
|x_n(s)| \;&\leq\; M|\varphi(0)| + ||(-A)^{-\alpha}||MC_5(1+||\varphi||_0) \\
&+\; ||(-A)^{-\alpha}||C_5(1+||x_n||_t) \\
&+\; \left|\int_0^s (-A)^{1-\alpha}S(s-\tau)(-A)^{\alpha}f(\tau,\pi_\tau x_{n-1})d\tau\right| \\
&+\; \left|\int_0^s S(s-\tau)a(\tau,\pi_\tau x_{n-1})d\tau\right| \\
&+\; \left|\int_0^s S(s-\tau)b(\tau,\pi_\tau x_{n-1})dw(\tau)\right|.
\end{aligned}
$$

Note that $(-A)^{-\alpha}$, for $0 < \alpha \leq 1$ is a bounded operator, see Pazy Lemma 6.3 on p. 71 in [3]. An application of Lemma 3.2 and Lemma 2.3 in [2] (or Lemma 2.4 in [2] for the case $p = 2$) then yields

$$
\begin{aligned}
&[1 - C_5||(-A)^{-\alpha}||]^p E||x_n||_t^p \\
&\quad\leq\; 4^{p-1}\Big\{E[M|\varphi(0)| + ||(-A)^{-\alpha}||MC_5(1+||\varphi||_0) \\
&\qquad +C_5||(-A)^{-\alpha}||]^p \\
&\qquad +k(p,a,\alpha)\int_0^t E|(-A)^{\alpha}f(s,\pi_s x_{n-1})|^p ds
\end{aligned}
$$

$$
\begin{aligned}
& +M^pT^{p-1}\int_0^t E|a(s,\pi_s x_{n-1})|^p ds \\
& +M^p c(p,T)\int_0 E|b(s,\pi_s x_{n-1})|_\lambda^p ds\Big\} \\
\leq\ & \frac{4^{p-1}}{[1-C_5||(-A)^{-\alpha}||]^p}\Big\{E[M|\varphi(0)| \\
& +C_5||(-A)^{-\alpha}||(1+M(1+||\varphi||_0)]^p \\
& +k(p,a,\alpha)\int_0^t C_5^p 2^{p-1}(1+E||x_{n-1}||_s^p)ds \\
& +C_3M^p(T^{p-1}+c(p,T))\int_0^t(1+E||x_{n-1}||_s^p)ds\Big\}.
\end{aligned}
$$

Since $E||x_0||_t^p < \infty$, then so are $E||x_n||_t^p < \infty$, for all $n = 1,2,3,\ldots$, and $t \in [0,T]$. This proves the boundedness of $\{x_n\}$.

To show that $\{x_n\}$ is Cauchy in B_T, consider:

$$
\begin{aligned}
& x_1(s)-x_0(s) \\
=\ & S(s)f(0,\varphi)-f(s,\pi_s x_1)+\int_0^s(-A)S(s-\tau)f(\tau,\pi_\tau x_0)d\tau \\
& +\int_0^s S(s-\tau)a(\tau,\pi_\tau x_0)d\tau+\int_0^s S(s-\tau)b(\tau,\pi_\tau x_0)dw(\tau).
\end{aligned}
$$

By assumptions (H4) and (H3), we have

$$
\begin{aligned}
& |x_1(s)-x_0(s)| \\
\leq\ & ||(-A)^{-\alpha}||C_5(1+||\varphi||_0)+||(-A)^{-\alpha}||C_4(1+||x_1-x_0||_t) \\
& +||(-A)^{-\alpha}||C_5(1+||x_0||_t) \\
& +\left|\int_0^s(-A)^{1-\alpha}S(s-\tau)(-A)^\alpha f(\tau,\pi_\tau x_0)d\tau\right| \\
& +\left|\int_0^s S(s-\tau)a(\tau,\pi_\tau x_0)d\tau\right|+\left|\int_0^s S(s-\tau)b(\tau,\pi_\tau x_0)dw(\tau)\right|.
\end{aligned}
$$

Assumption (H2) and Lemma 2.3 in [2] (or Lemma 2.4 when $p = 2$) yield

$$
\begin{aligned}
& [1-C_4||(-A)^{-\alpha}||]^p E||x_1-x_0||_t^p \\
\leq\ & 4^{p-1}\Big\{E[||(-A)^{-\alpha}||C_5(2+||\varphi||_0+||x_0||_t)]^p \\
& +k(p,a,\alpha)\int_0^t 2^{p-1}C_5^p(1+E||x_0||_s^p)ds \\
& +M^p(T^{p-1}+C(p,T))\int_0^t C_3(1+E||x_0||_s^p)ds\Big\}.
\end{aligned}
$$

Next, consider

$$\begin{aligned} & x_n(s) - x_{n-1}(s) \\ & \quad = \quad f(s, \pi_s x_{n-1}) - f(s, \pi_s x_n) \\ & \qquad + \int_0^s (-A) S(s-\tau)[f(\tau, \pi_\tau x_{n-1}) - f(\tau, \pi_\tau x_{n-2})] d\tau \\ & \qquad + \int_0^s S(s-\tau)[a(\tau, \pi_\tau x_{n-1}) - a(\tau, \pi_\tau x_{n-2})] d\tau \\ & \qquad + \int_0^s S(s-\tau)[b(\tau, \pi_\tau x_{n-1}) - b(\tau, \pi_\tau x_{n-2})] dw(\tau). \end{aligned}$$

Estimating as before, we get

$$\begin{aligned} & [1 - C_4 ||(-A)^{-\alpha}||]^p E||x_n - x_{n-1}||_t^p \\ & \quad \leq \quad 3^{p-1} k(p, a, \alpha) \int_0^t 2^{p-1} C_4^p E||x_{n-1} - x_{n-2}||^p ds \\ & \qquad + 3^{p-1} M^p T^{p-1} C_1 \int_0^t E||x_{n-1} - x_{n-2}||^p ds \\ & \qquad + 3^{p-1} M^p c(p, T) C_2 \int_0^t E||x_{n-1} - x_{n-2}||_s^p ds. \end{aligned}$$

Hence,

$$E||x_n - x_{n-1}||_t^p \leq \frac{L}{[1 - C_4 ||(-A)^{-\alpha}||]^p} \int_0^t E||x_{n-1} - x_{n-2}||_s^p ds,$$

where

$$L = 3^{p-1} 2^{p-1} C_4^p k(p, a, \alpha) + 3^{p-1} M^p (C_1 T^{p-1} + C_2 c(p, T)).$$

Using the familiar Cauchy formula [2]:

$$\begin{aligned} & E||x_n - x_{n-1}||_t^p \\ & \quad \leq \quad \frac{L^{n-1}}{[1 - C_4 ||(-A)^{-\alpha}||]^{(n-1)p}} \int_0^t \frac{(t-s)^{n-2}}{(n-2)!} E||x_1 - x_0||_s^p ds \\ & \quad \leq \quad \frac{L^{n-1}}{[1 - C_4 ||(-A)^{-\alpha}||]^{(n-1)p}} \frac{T^{n-1}}{(n-1)!} E||x_1 - x_0||_t^p. \end{aligned}$$

This shows that $\{x_n\}$ is Cauchy in B_T. The Borel–Cantelli Lemma guarantees that $x_n(t) \to x(t)$ uniformly in t on $[0, T]$ and $x(t)$ is indeed a unique mild solution. This completes the proof. □

4. Almost Sure Exponential Stability

In this section, we consider the almost sure exponential stability of the sample paths of a trivial solution of equation (1). For this, assume from now on that $a(t,0) = b(t,0) = f(t,0) \equiv 0$ a.e.t. so that equation (1) admits a trivial solution. In the rest of the paper, we confine ourselves to the case $p = 2$.

Theorem 4.1. *Let the hypothesis of Theorem 3.1 hold. Then the mild solution of equation (1) is exponentially stable in the quadratic mean provided* $1/2 < \alpha < 1$ *and*

$$a > \beta = \frac{4[C_4^2 M_{1-\alpha}^2 \Gamma(2\alpha-1)/a^{2\alpha-1} + C_1/a + 4C_2]}{[1 - C_4||(-A)^{-\alpha}||]^2}.$$

Remark 4.1. Note that, Theorem 4.1 states that the mild solution is exponentially stable provided the condition in its statement is fulfilled together with

$$C_4||(-A)^{-\alpha}|| < 1.$$

Whereas Theorem 2.6 yields exponential stability provided the condition given therein is fulfilled together with the restriction that

$$C_4||(-A)^{-\alpha}|| < 1/3.$$

To prove this theorem, we need a lemma.

Lemma 4.2. *Let $-A$ be the infinitesimal generator of an analytic semigroup of bounded linear operators $\{S(t), t \geq 0\}$ in X. Then, for any stochastic process $F : [0,\infty) \to X$ which is strongly measurable with*

$$\int_0^T E|(-A)^\alpha F(t)|^2 dt < \infty, \qquad 0 < T \leq \infty,$$

the following inequality holds for $0 < t \leq T$:

$$E\left|\int_0^t (-A)S(t-s)F(s)ds\right|^2 \leq M_{1-\alpha}^2 \frac{\Gamma(2\alpha-1)}{a^{2\alpha-1}} \int_0^t e^{-a(t-s)} E|(-A)^\alpha F(s)|^2 ds,$$

provided $1/2 < \alpha < 1$.

Proof. From Pazy [3, Theorem 6.13], it follows that

$$E\left|\int_0^t (-A)^{1-\alpha} S(t-s)(-A)^\alpha F(s)ds\right|^2$$
$$\leq E\left(\int_0^t M_{1-\alpha} e^{-\frac{a}{2}(t-s)}(t-s)^{-(1-\alpha)} e^{-\frac{a}{2}(t-s)} |(-A)^\alpha F(s)|ds\right)^2$$

$$\leq M_{1-\alpha}^2\left(\int_0^t e^{-a(t-s)}(t-s)^{-2(1-\alpha)}ds\right)\int_0^t e^{-a(t-s)}E|(-A)^\alpha F(s)|^2ds$$
$$\leq M_{1-\alpha}^2\frac{\Gamma(2\alpha-1)}{a^{2\alpha-1}}\int_0^t e^{-a(t-s)}E|(-A)^\alpha F(s)|^2ds. \quad \square$$

Proof of Theorem 4.1. Consider

$$\begin{aligned} x(s) &= S(s)[\varphi(0)+f(0,\varphi)]-f(s,\pi_s x) \\ &+\int_0^s(-A)^{1-\alpha}S(s-\tau)(-A)^\alpha f(\tau,\pi_\tau x)d\tau \\ &+\int_0^s S(s-\tau)a(\tau,\pi_\tau x)d\tau+\int_0^s S(s-\tau)b(\tau,\pi_\tau x)dw(\tau). \end{aligned}$$

By assumptions (H4) and (H3) while observing that $f(t,0)\equiv 0$ a.e. t, we have

$$\begin{aligned} |x(s)| &\leq ||S(s)||[|\varphi(0)|+C_4||(-A)^{-\alpha}||||\varphi||_0]+C_4||(-A)^{-\alpha}||||x||_t \\ &+\left|\int_0^s(-A)^{1-\alpha}S(s-\tau)(-A)^\alpha f(\tau,\pi_\tau x)d\tau\right| \\ &+\left|\int_0^s S(s-\tau)a(\tau,\pi_\tau x)d\tau\right| \\ &+\left|\int_0^s S(s-\tau)b(\tau,\pi_\tau x)dw(\tau)\right|. \end{aligned}$$

Applying first [1, Lemma 7.2, p. 182], then Lemmas 4.1-4.2 in [2] and Lemma 4.2, we obtain

$$\begin{aligned} &[1-C_4||(-A)^{-\alpha}||]^2E||x||_t^2\leq 4\Big\{e^{-2at}E[|\varphi(0)|+C_4||(-A)^{-\alpha}||||\varphi||_0]^2 \\ &+M_{1-\alpha}^2\frac{\Gamma(2\alpha-1)}{a^{2\alpha-1}}\int_0^t e^{-a(t-s)}E|(-A)^\alpha f(s,\pi_s x)|^2ds \\ &+\frac{1}{a}\int_0^t e^{-a(t-s)}E|a(s,\pi_s x)|^2ds+4\int_0^t e^{-a(t-s)}E|b(s,\pi_s x)|_\lambda^2ds\Big\}. \end{aligned}$$

Assumption (H2) when $a(t,0)=b(t,0)\equiv 0$ a.e. t yields

$$e^{at}E||x||_t^2\leq\frac{4E[|\varphi(0)|+C_4||(-A)^{-\alpha}||||\varphi||_0]^2}{[1-C_4||(-A)^{-\alpha}||]^2}+\beta\int_0^t e^{as}E||x||_s^2ds.$$

Invoking Gronwall's Lemma, we get

$$e^{at}E||x||_t^2\leq\frac{4[1+C_4||(-A)^{-\alpha}||]^2}{[1-C_4||(-A)^{-\alpha}||]^2}E||\varphi||_0^2e^{\beta t},\qquad t\geq 0.$$

Consequently,

$$E|x(t)|^2\leq KE||\varphi||_0^2e^{-\gamma t},\quad t\geq 0,$$

where

$$\gamma = a - \beta,$$

and

$$K = \frac{4[1 + C_4||(-A)^{-\alpha}||]^2}{[1 - C_4||(-A)^{-\alpha}||]^2}. \qquad \square$$

Theorem 4.3. *Suppose that all the conditions of Theorem 4.1 hold. Then the mild solution of equation (1) satisfies*

$$\limsup_{t\to\infty}(1/t)\log|x(t)| \le -(a-\beta)/4, \quad \text{a.s.}$$

Proof. It can be proved mimicking arguments from Govindan [2]. $\square$

5. An Example

Consider the stochastic partial neutral functional differential equation with finite delays r_1, r_2 and r_3 ($r > r_i \ge 0, i = 1,2,3$):

$$\begin{aligned} d[z(t,x) + \ell_3 \int_{-r_3}^{0} z(t+u,x)du] \\ = \left[\frac{\partial^2}{\partial x^2}z(t,x) + \ell_1\int_{-r_1}^{0} z(t+u,x)du\right]dt \\ +\ell_2 z(t-r_2,x)d\beta(t), \quad t>0, \qquad (4) \\ \ell_i \ge 0, \quad i=1,2,3; \quad z(t,0) = z(t,\pi) = 0, \quad t > 0, \\ z(s,x) = \varphi(s,x), \quad \varphi(\cdot,x) \in C_t, \\ \varphi(s,\cdot) \in L^2[0,\pi], \quad -r \le s \le 0, \quad 0 \le x \le \pi; \end{aligned}$$

where $\beta(t)$ is a standard one–dimensional Wiener process and $E||\varphi||_0^2 < \infty$.

Take $X = L^2[0,\pi]$, $Y = R^1$. Define $-A : X \to X$ by $-A = \partial^2/\partial x^2$ with domain $D(-A) = \{w \in X : w, \partial w/\partial x$ are absolutely continuous, $\partial^2 w/\partial x^2 \in X$, $w(0) = w(\pi) = 0\}$. Then

$$-Aw = \sum_{n=1}^{\infty} n^2(w,w_n)w_n, \quad w \in D(-A),$$

where $w_n(x) = \sqrt{2/\pi}\sin nx$, $n = 1,2,3,\ldots$, is the orthonormal set of eigenvectors of $-A$. The operator $(-A)^{1-\alpha}$ is given by [4]:

$$|-A|^{1-\alpha}e^{-At}w = \sum_{n=1}^{\infty}(n^2)^{1-\alpha}e^{-n^2t}(w,w_n)w_n,$$

on the space $D((-A)^{1-\alpha}) = \{w \in X : \sum_{n=1}^{\infty} n(w, w_n)w_n \in X\}$, which implies

$$||(-A)^{1-\alpha}S(t)w|| \leq M_{1-\alpha}t^{-(1-\alpha)}e^{-at}||w||, \quad t > 0;$$

where $M_{1-\alpha} = [(1-\alpha)e^{-(1-a)}]^{1-\alpha}$. For details, see Taniguchi, et al [4].

It is well–known that $-A$ is the infinitesimal generator of an analytic semigroup $\{S(t), t \geq 0\}$ in X and is given by

$$S(t)w = \sum_{n=1}^{\infty} e^{-n^2 t}(w, w_n)w_n, \quad w \in X,$$

that satisfies $||S(t)|| \leq \exp(-\pi^2 t), t \geq 0$, and hence is a contraction semigroup.

Define now

$$\begin{aligned} f(t, \pi_t z) &= \ell_3 \int_{-r_3}^{0} z(t+u, x)du, \\ a(t, \pi_t z) &= \ell_1 \int_{-r_1}^{0} z(t+u, x)du, \end{aligned}$$

and

$$b(t, \pi_t z) = \ell_2 z(t - r_2, x).$$

Next,

$$\begin{aligned} ||f(t, \pi_t x)||_\alpha &= \ell_3 \left| (-A)^\alpha \int_{-r_3}^{0} z(t+u, x)du \right| \\ &\leq \ell_3 r_3 ||(-A)^{-\alpha}|| ||z||_t, \quad \text{a.s.} \end{aligned}$$

This shows that $f : R^+ \times C_t \to D((-A)^\alpha)$ with $C = \ell_3 r_3 ||(-A)^{-\alpha}||$. Similarly, $a : R^+ \times C_t \to X$ and $b : R^+ \times C_t \to L(R, X)$. Hence, equation (4) can be expressed as equation (1) with $-A, f, a$ and b as defined above.

From Pazy [3], p. 70

$$||(-A)^{-\alpha}|| \leq \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1}||S(t)||dt \leq \frac{1}{\pi^{2\alpha}}.$$

Let ℓ_3 and r_3 satisfy the relation $\ell_3 r_3/\pi^{4\alpha} < 1$, which implies that $\ell_3 r_3||(-A)^{-\alpha}||^2 < 1$. Therefore, by Theorem 3.1, equation (4) has a unique mild solution. By Theorem 4.1, the solution is also exponentially stable in the quadratic mean provided $1/2 < \alpha < 1$ and $\pi^2 > \beta$, where

$$\beta = \frac{4[\ell_3^2 r_3^2((1-\alpha)e^{\pi^2-1})^{2(1-\alpha)}\Gamma(2\alpha-1)/\pi^{2(2\alpha-1)} + \ell_1^2 r_1^2/\pi^2 + 4\ell_2^2]}{[1 - \ell_3 r_3/\pi^{4\alpha}]^2}.$$

Finally, by Theorem 4.3,

$$\limsup_{t\to\infty}(1/t)\log|x(t)| \leq -(\pi^2 - \beta)/4, \quad \text{a.s.}$$

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