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SOME COMMON FIXED POINT THEOREMS FOR FUZZY MAPPINGS

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Abstract: Zadeh [22] introduced the concept of fuzzy sets and Heilpern [11] introduced the concept of fuzzy mappings. Since then both the concepts have been generalized by various researchers. Bose and Sahani [10] and others generalized the results of Heilpern. Many authors considered class of fuzzy sets with nonempty compact (convex) $\alpha-$cut sets in a metric (linear) space, but some have given attention to class of fuzzy sets with nonempty closed and bounded $\alpha-$cut sets in a metric space. Dong Qui and Lan Shu [15] considered $\mathcal{CB}(X)$ (the class of fuzzy sets with nonempty closed and bounded $\alpha-$cut sets) equipped with the generalized Hausdorff metric. They proved some common fixed point theorems for fuzzy mappings $F_i : \mathcal{CB}(X) \to \mathcal{CB}(X)$ satisfying certain inequalities. Dong Qui et al [16] considered the class of fuzzy sets, with nonempty compact $\alpha-$cut sets ($\mathcal{C}(X)$) and fuzzy mappings $F_i : \mathcal{C}(X) \to \mathcal{C}(X)$. This paper deals with several common fixed point theorems of such fuzzy mappings satisfying various (different) inequalities and these extend the works of various researchers. Following Alber and Guerre-Delabriere [2], we extend the concept of $\phi-$weakly contractive condition to the fuzzy mappings and prove a common fixed point theorem for a pair of such fuzzy mappings. Also we prove a fixed point theorem for a fuzzy mapping which is a $(\delta, L)-$weak contraction (introduced by Berinde and Berinde [6]).

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1. Introduction

L. Zadeh [22] introduced the concept of fuzzy sets in 1965 and Heilpern [11] introduced the concept of fuzzy mappings in 1981. Since then both the concepts, that of fuzzy sets and that of fuzzy mappings have been generalized by various researchers. The concept of fuzzy sets play a very important role in many scientific and engineering applications, and the Hausdorff metric plays an important role in set-valued analysis. Bose and Sahani [10] generalized the fixed point results for fuzzy contraction mappings of Heilpern to common fixed points of generalized fuzzy contraction mappings and to fuzzy non-expansive mappings. Other workers also extended Heilpern's result in different directions. Many authors generalized or extended the work of Bose and Sahani, especially Lee and Cho [13], Lee et al [14], Vijayaraju and Marudai [24], Arora and Sharma [3], etc.

Many authors considered the class of fuzzy sets with nonempty $\alpha-$cut sets in a complete metric space (or nonempty compact convex $\alpha-$cut sets in a complete metric linear space), but some have given their attention to the class of fuzzy sets with nonempty closed bounded $\alpha-$cut sets in a complete metric space. Recently Dong Qui and Lan Shu [15] established the completeness of $\mathcal{CB}(X)$ with respect to the completeness of the metric space X, where $\mathcal{CB}(X)$ denotes the class of fuzzy sets with nonempty bounded closed $\alpha-$cut sets equipped with the generalized Hausdorff metric d_∞ which takes the supremum on the Hausdorff distances between the corresponding $\alpha-$cut sets. Also they proved some common fixed point theorems for a family of fuzzy mappings $F_i : \mathcal{CB}(X) \to \mathcal{CB}(X)$, $i \in \mathbb{Z}^+$, satisfying certain inequalities. Here we discuss some common fixed point theorems for fuzzy mappings satisfying different kind of inequalities. We consider both types of fuzzy sets with nonempty closed bounded $\alpha-$cut sets and with nonempty compact $\alpha-$cut sets. The later type of fuzzy sets was also considered by Dong Qui, Lan Shu, and Jian Guan [16] recently. In this paper we prove three common fixed point theorems concerning an infinite family of fuzzy mappings or a pair of fuzzy mappings satisfying three different type inequalities.

Alber and Guerre-Delabriere [2] introduced the concept of weakly contractive mappings as a generalization of contractions. Bae [4] proved some fixed point theorems for weakly contractive multi-valued mappings. Azam and Beg [1] proved a common fixed point theorem for a pair of fuzzy mappings $F_i : X \to W(X)$ satisfying a weakly contractive condition. $W(X)$ denotes the collection of fuzzy sets on the metric linear space which are approximate quantities and have nonempty compact and convex $\alpha-$cut sets. Zhang and Song

[23] proved a common fixed point theorem for a pair of single valued mappings of X into itself satisfying a generalized $\phi-$weak contractive condition. Beg and Abbas [5] proved some fixed point theorems for single valued mappings which are weakly contractive with respect to a mapping f. Recently Bose and Roy-chowdhury [8] generalized the results of Azam and Beg, Zhang and Song, and Beg and Abbas. Here we present a common fixed point theorem for a fuzzy mapping $F : \mathcal{C}(X) \to \mathcal{C}(X)$ satisfying a weakly contractive condition.

Berinde and Berinde [6] extended the notion of $(\delta, L)-$weak contraction from single value mappings to multi-valued mappings and Kamran [12] further extended the notion of $(\delta, L)-$weak contraction to multi-valued $f-$weak contraction and generalized multi-valued $f-$weak contraction. Bose and Roy-chowdhury [9] recently proved some generalization/extension of the works of Berinde and Berinde, and Kamran. In this paper, we present a fixed point theorem for a fuzzy mapping $F : \mathcal{CB}(X) \to \mathcal{CB}(X)$ which is a $(\delta, L)-$weak contraction.

2. Preliminaries

Let (X, d) be a metric space and let $CB(X)$ be the set of all closed bounded subsets of X. We denote the closure of a set $A \subseteq X$ as $\overline{A}$. The Hausdorff metric on $CB(X)$ is defined as

$$H(A, B) = \max\left\{\sup_{x\in B}\inf_{x\in A} d(x, y), \sup_{x\in A}\inf_{x\in B} d(x, y)\right\} = \max\left\{\rho(B, A), \rho(a, B)\right\},$$

where A, $B \in CB(X)$ and

$$\rho(A, B) = \sup_{x\in A}\inf_{x\in B} d(x, y) = \sup_{x\in A} d(x, B)$$

is the Hausdorff separation of A from B. It can be shown that $\rho(A, B) = 0$ if and only if $A \subseteq B$, and $H(A, B) = 0$ if and only if $A = B$. A fuzzy set μ on X is defined by its membership function $\mu(x)$ which is a mapping from X into $[0, 1] = I$. A $\alpha-$cut of μ is $[\mu]^\alpha = \{x \in X : \mu(x) \geq \alpha\}$, where $0 < \alpha \leq 1$, and we separately specify the support of $[\mu]^0$ of μ to be the closure of the union of $[\mu]^\alpha$ for $0 < \alpha \leq 1$. Denote by $\mathcal{CB}$(X) the totality of fuzzy sets $\mu : X \to I$, which satisfy that for each $\alpha \in I$, the $\alpha-$cut of μ is nonempty closed bounded subset of X.

Let μ_1, $\mu_2 \in \mathcal{CB}$(X). Then $\mu_1 \subseteq \mu_2$, if and only if $\mu_1(x) \leq \mu_2(x)$ for each $x \in X$. That is, $\mu_1 \subseteq \mu_2$ if and only if $[\mu_1]^\alpha \subseteq [\mu_2]^\alpha$ for all $\alpha \in I$. The $d_\infty-$metric (called supremum or generalized Hausdorff metric) is induced by the Hausdorff metric H as

$$d_\infty(\mu_1,\mu_2) = \sup_{0\le\alpha\le1} H\left([\mu_1]^\alpha,[\mu_2]^\alpha\right) = \max\left\{\rho_\infty(\mu_1,\mu_2),\rho_\infty(\mu_2,\mu_1)\right\},$$

where $\mu_1,\ \mu_2 \in \mathcal{CB}\,(\mathrm{X})$, and

$$\rho_\infty\left(\mu_1,\mu_2\right) = \sup_{0\le\alpha\le1} \rho\left([\mu_1]^\alpha,[\mu_2]^\alpha\right)$$

is the Hausdorff separation of μ_1 from μ_2.

Let $\{\mu_n\}$ be a sequence in $\mathcal{CB}\,(\mathrm{X})$. Then μ_n converges with respect to the $d_\infty-$metric if and only if $[\mu_n]^\alpha$ converges uniformly in $\alpha\in I$ with respect to the Hausdorff metric.

Definition 1. Let X,Y be any metric spaces. A mapping F is said to be a fuzzy mapping from the space $\mathcal{CB}\,(\mathrm{X})$ into $\mathcal{CB}\,(\mathrm{Y})$, that is $F(\mu)\in\mathcal{CB}\,(\mathrm{Y})$ for each $\mu\in\mathcal{CB}\,(\mathrm{X})$. The element $\mu*\in\mathcal{CB}\,(\mathrm{X})$ is said to be a fixed point of a fuzzy self-mapping F of $\mathcal{CB}\,(\mathrm{X})$ if and only if $\mu*\subseteq F(\mu*)$.

Lemma 1. *The metric space $(CB(X),H)$ is complete provided X is complete.*

Lemma 2. (see [15]) *Let $\mu_1,\ \mu_2,\ \mu_3\in\mathcal{CB}\,(\mathrm{X})$. Then the following hold:*

(i) $\rho_\infty(\mu_1,\mu_2)=0$ if and only if $\mu_1\subseteq\mu_2$,

(ii) $d_\infty(\mu_1,\mu_2)=0$ if and only if $\mu_1=\mu_2$,

(iii) if $\mu_1\subseteq\mu_2$ then $\rho_\infty(\mu_1,\mu_3)\le d_\infty(\mu_2,\mu_3)$,

(iv) $\rho_\infty(\mu_1,\mu_3)\le d_\infty(\mu_1,\mu_2)+\rho_\infty(\mu_2,\mu_3)$.

Lemma 3. (see [18]) *Let A be a set in X and let $\{A_\alpha:\alpha\in I\}$ be a family of subsets of A such that*

(i) $A_0=A$,

(ii) $\alpha\le\beta$ implies $A_\beta\subseteq A_\alpha$,

(iii) $\alpha_1\le\alpha_2\le .....,\ \lim_{n\to\infty}\alpha_n=\alpha$ implies $A_\alpha=\cap_{k=1}^{\infty}A_{\alpha_k}$.

Then the function $\phi:X\to I$ defined by $\phi(x)=\sup\{\alpha\in I:x\in A_\alpha\}$ has the property that $[\phi]^\alpha=A_\alpha$. Conversely, for any fuzzy set μ in X, the family $\{[\mu]^\alpha\}$ of $\alpha-$level sets of μ satisfies the above conditions (i)-(iii).

The function ϕ is actually defined on the set A, but we can extend it to X by defining $\phi(x)=0$ for all $x\in X-A$. Lemma 3 is known as Negoita-Ralescu Representation Theorem.

Theorem A. (see [15]) *The metric space $(\mathcal{CB}\,(\mathrm{X}),d_\infty)$ is complete when X is complete.*

Lemma 4. *Let (X,d) be a metric space and $A,\ B\in CB(X)$. Then* *(i) for any* $\varepsilon > 0$ *and any* $x \in A$*, there exists a* $y \in B$ *such that* $d(x, y) \leq H(A, B) + \varepsilon$,

(ii) for any $x \in A$ *and any* $\beta > 1$*, there exists a* $y \in B$ *such that* $d(x, y) \leq \beta H(A, B)$.

Theorem B. (see [15]) *Let* (X, d) *be a metric space and* $\mu_1, \mu_2 \in \mathcal{CB}(\mathrm{X})$*. Then*

(i) for any $\varepsilon > 0$ *and any* $\mu_3 \in \mathcal{CB}(\mathrm{X})$ *satisfying* $\mu_3 \subseteq \mu_1$*, there exists a* $\mu_4 \in \mathcal{CB}(\mathrm{X})$ *such that* $\mu_4 \subseteq \mu_2$ *and* $d_\infty(\mu_3, \mu_4) \leq d_\infty(\mu_1, \mu_2) + \varepsilon$,

(ii) for any $\beta > 1$ *and any* $\mu_3 \in \mathcal{CB}(\mathrm{X})$ *satisfying* $\mu_3 \subseteq \mu_1$*, there exists a* $\mu_4 \in \mathcal{CB}(\mathrm{X})$ *such that* $\mu_4 \subseteq \mu_2$ *and* $d_\infty(\mu_3, \mu_4) \leq \beta d_\infty(\mu_1, \mu_2)$.

Let (X, d) *be a metric space. Denote by* $\mathcal{C}(X)$ *the totality of fuzzy sets* $\mu : X \to I$ *which satisfy that for each* $\alpha \in I$*, the* $\alpha-$*cut set of* μ *is nonempty compact subset in* X*. Let* $C(X)$ *be the set of all non-empty compact subsets of* X.

Lemma 5. *The metric space* $(C(X), H)$ *is complete provided* X *is complete.*

Lemma 6. (see [16]) *Let* μ_1, μ_2, $\mu_3 \in \mathcal{C}(X)$*, then (i), (ii), (iii), and (iv) of Lemma 2 hold.*

Theorem C. (see [16]) *The metric space* $(\mathcal{C}(X), d_\infty)$ *is complete provided* X *is complete.*

Theorem D. (see [16]) *Let* (X, d) *be a compact metric space and* μ_1, $\mu_2 \in \mathcal{C}(X)$*. Then for any* $\mu_3 \in \mathcal{C}(X)$ *satisfying* $\mu_3 \subseteq \mu_1$*, there exists a* $\mu_4 \in \mathcal{C}(X)$ *such that* $\mu_4 \subseteq \mu_2$ *and* $d_\infty(\mu_3, \mu_4) \leq d_\infty(\mu_1, \mu_2)$.

Definition 2. Let (X, d) be a compact metric space. F is said to be a fuzzy mapping (also) if and only if F is a mapping from the space $\mathcal{C}(X)$ into $\mathcal{C}(Y)$, where Y is another metric space, i.e., $F(\mu) \in \mathcal{C}(X)$ for each $\mu \in \mathcal{C}(X)$.

Definition 3. (see [2]) Let (X, d) be a metric space. A mapping $F : X \to X$ is said to be weakly contractive or a $\phi-$weak contraction if $d(Fx, Fy) \leq d(x, y) - \phi(d(x, y))$ for each $x, y \in X$, where $\phi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that ϕ is positive on $(0, \infty), \phi(0) = 0$.

A mapping $F : X \to X$ is said to be a generalized $\phi-$weak contraction if $d(Fx, Fy) \leq M(x, y) - \phi(M(x, y))$, where

$$M(x, y) = \max\left\{d(x, y), d(Fx, x), d(Fy, y), \frac{1}{2}[d(y, Fx) + d(x, Fy)]\right\}.$$

A mapping $F : X \to \mathcal{P}(X)$ (where $\mathcal{P}(X)$ is the set all non-empty subsets of X) is said to be a multi-valued generalized $\phi-$weak contraction if $H\left(Fx, Fy\right) \leq M(x,y) - \phi\left(M(x,y)\right)$, for all $x,\ y \in X$, where $M(x,y) = \max\left\{d(x,y),\ d(Fx,x),\ d\left(Fy,y\right),\ \frac{1}{2}[d(y,Fx+d(x,Fy)]\right\}$.

Definition 4. (see [6]) Let (X,d) be a metric space. A mapping $(F : X \to X$ is called a weak contraction or $(\delta, L)-$weak contraction if there exists two constants, $\delta \in (0,1)$ and $L \geq 0$, such that $d\left(Fx, Fy\right) \leq \delta.d(x,y) + Ld\left(y, Fx\right)$, for all $x,\ y \in X$.

Let $\mathcal{P}(X)$ be the family of all non-empty subsets of X and let $F : X \to \mathcal{P}(X)$ be a multi-valued mapping. F is said to be a multi-valued $(\delta, L)-$weak contraction if there exist two constants $\delta \in (0,1)$ and $L \geq 0$ such that $H\left(Fx, Fy\right) \leq \delta d(x,y) + Ld\left(y, Fx\right)$, for all $x,\ y \in X$.

Both the definitions above can be extended to fuzzy mappings which are self mappings of $\mathcal{CB}(X)/\mathcal{C}(X)$ into itself and we present two fixed point theorems concerning such fuzzy mappings also.

3. Main Results

We now state and prove five theorems. While generalizing a result of Bose and Sahani [10], Arora and Sharma [3] considered a sequence of fuzzy mappings satisfying a different inequality pairwise in a linear metric space setting. Bose [7] has extended their results to fuzzy mappings for the class of fuzzy sets with nonempty closed and bounded $\alpha-$cut sets instead of nonempty compact and convex $\alpha-$cut sets. For the first theorem, we use a slight variation of the same inequality satisfied by the family of fuzzy mappings pairwise.

Theorem 1. *Let (X,d) be a complete metric space and let $\{F_i\}$ be a sequence of fuzzy self mappings of $\mathcal{CB}\left(\mathrm{X}\right)$. If there exists a constant $q, 0 < q < \frac{1}{2}$, such that for each $\mu_1, \mu_2 \in \mathcal{CB}\left(\mathrm{X}\right)$, and for arbitrary positive integers, i, j*

$$d_\infty(F_i(\mu_1), F_j(\mu_2)) \leq q \times \max\{d_\infty(\mu_1,\mu_2),$$
$$\rho_\infty(\mu_1, F_i(\mu_1), \rho_\infty(\mu_2, F_j(\mu_2), \rho_\infty(\mu_2, F_i(\mu_1), \rho_\infty(\mu_1, F_j(\mu_2)\}\,,$$

then there exists a $\mu* \in \mathcal{CB}\left(\mathrm{X}\right)$ such that $\mu* \subseteq F_i(\mu*)$ for all $i \in \mathbb{Z}^+$.

Proof. Let $\mu_0 \in \mathcal{CB}(\mathrm{X})$ and $\mu_1 \subseteq F_1(\mu_0)$. By Theorem B, there exists $\mu_2 \in \mathcal{CB}\left(\mathrm{X}\right)$ such that $\mu_2 \subseteq F_2(\mu_1)$ and $d_\infty(\mu_1, \mu_{2)} \leq d_\infty(F_1(\mu_0), F_2(\mu_1)) + q$. Again by Theorem B, we can find $\mu_3 \in \mathcal{CB}\left(\mathrm{X}\right)$ such that $\mu_3 \subseteq F_3(\mu_2)$ and $d_\infty(\mu_2, \mu_{3)} \leq d_\infty(F_2(\mu_1), F_3(\mu_2)) + \frac{q^2}{1-q}$. By induction, we produce a sequence $\{\mu_n\}$ of elements in $\mathcal{CB}\left(\mathrm{X}\right)$ such that$\mu_{n+1} \subseteq F_{n+1}(\mu_n)$, $n = 0,1,2,3...$ and $d_\infty(\mu_n, \mu_{n+1}) \leq d_\infty(F_n(\mu_{n-1}), F_{n+1}((\mu_n)) + \frac{q^n}{(1-q)^{n-1}}$. We now show that $\{\mu_n\}$

is a Cauchy sequence in $\mathcal{CB}\left(\mathrm{X}\right)$. We have

$$d_\infty(\mu_n,\mu_{n+1}) \leq q\max\{d_\infty(\mu_{n-1},\mu_n), \\ d_\infty(\mu_{n-1},\mu_n), d_\infty(\mu_n,\mu_{n+1}), 0, d_\infty(\mu_{n-1},\mu_{n+1})\} + \frac{q^n}{(1-q)^{n-1}}.$$

Thus we have $d_\infty(\mu_n,\mu_{n+1}) \leq \frac{q}{(1-q)} d_\infty(\mu_{n-1},\mu_n) + \frac{q^n}{(1-q)^n} \leq \lambda^n d_\infty(\mu_0,\mu_1) + n\lambda^n$, where $\lambda = \frac{q}{(1-q)}$.

Since $\sum_{n=1}^{\infty}\lambda^n < \infty$ and $\sum_{n=1}^{\infty} n\lambda^n < \infty$, $\{\lambda_n\}$ is a Cauchy sequence in $\mathcal{CB}\left(\mathrm{X}\right)$. By completeness of $\left(\mathcal{CB}\left(\mathrm{X}\right), d_\infty\right)$ there exists $\mu*$ such that $\mu_i \to \mu*$ in $\mathcal{CB}\left(\mathrm{X}\right)$. By Lemma 2, we have

$$\begin{aligned} \rho_\infty\left(\mu*, F_i\left(\mu*\right)\right) &\leq d_\infty\left(\mu*, \mu_j\right) + \rho_\infty\left(\mu_j, F_i(\mu*)\right) \\ &\leq d_\infty\left(\mu*, \mu_j\right) + d_\infty\left(F_j(\mu_{j-1}), F_i(\mu*)\right). \end{aligned}$$

After some simplification and taking limit as $j \to \infty$, we have

$$\begin{aligned} \rho_\infty\left(\mu*, F_i\left(\mu*\right)\right) &\leq q.\max\{0, 0, \rho_\infty(\mu*, F_i(\mu*)), 0, \rho_\infty(\mu*, F_i(\mu*))\} \\ &\leq q\rho_\infty(\mu*, F_i(\mu*)). \end{aligned}$$

Since $q < \frac{1}{2}$, this implies that $\rho_\infty\left(\mu*, F_i\left(\mu*\right)\right) = 0$. By Lemma 2, we have $\mu* \subseteq F_i(\mu*)$ for all i. This completes the proof. □

Corollary 1. *Let (X,d) be a complete metric space and let F be a fuzzy self-mappings of $\mathcal{CB}\left(\mathrm{X}\right)$. If there exists a constant q, $0 < q < \frac{1}{2}$, such that for each $\mu_1, \mu_2 \in \mathcal{CB}\left(\mathrm{X}\right)$, and for arbitrary positive integers, i, j*

$$d_\infty\left(F\left(\mu_1\right), F\left(\mu_2\right)\right) \leq \\ q\max\{d_\infty(\mu_1,\mu_2), \rho_\infty(\mu_1, F(\mu_1), \rho_\infty(\mu_2, F(\mu_2), \rho_\infty(\mu_2, F(\mu_1), \rho_\infty(\mu_1, F(\mu_2)\},$$

then there exists a $\mu* \in \mathcal{CB}\left(\mathrm{X}\right)$ such that $\mu* \subseteq F(\mu*)$.

Proof. We can define a sequence of fuzzy self-mappings of $\mathcal{CB}(X)$ as $F_i = F$, for all $i \in \mathbb{Z}^+$in Theorem 1. □

Corollary 2. *Let (X,d) be a complete metric space and let F_i be a fuzzy self-mappings of $\mathcal{CB}\left(\mathrm{X}\right)$. If for each $\mu_1, \mu_2 \in \mathcal{C}\left(\mathrm{X}\right)$, such that*

$$d_\infty(F_i(\mu_1), F_j(\mu_2)) \leq a_1 d(\mu_1,\mu_2) + a_2\rho_\infty(\mu_1, F_i(\mu_1) + a_3\rho_\infty(\mu_2, F_j(\mu_2) \\ + a_4\rho_\infty(\mu_1, F_j(\mu_2) + a_5\rho_\infty(\mu_2, F_i(\mu_1),$$

where a_1, a_2, a_3, a_4, a_5 are non-negative and $a_1 + a_2 + a_3 + a_4 + a_5 < \frac{1}{2}$. Then there exists $\mu* \in \mathcal{C}\left(\mathrm{X}\right)$ such that $\mu* \subseteq F_i(\mu*)$ for all $i \in \mathbb{Z}^+$.

Proof. Let $a_1 + a_2 + a_3 + a_4 + a_5 = q$. Then the result follows from Theorem

1. ∎

Remark 1. The above theorem holds for a pair of self-mappings F, G of $\mathcal{CB}(X)$ satisfying the same inequality. The proof is similar. We choose the sequence $\{\mu_n\}$ as follows:

$$\mu_{2n+1} \in F(\mu_{2n}) \quad \text{and} \quad \mu_{2n+2} \in G(\mu_{2n+1}).$$

Following Rashwan and Ahmed [19], we consider the set G of all continuous functions $g : [0,\infty)^5 \to [0,\infty)$ with the following properties:

(i) g is increasing in the 2-nd, 3-th, 4-th, and 5-th variable.

(ii) If $u, v \in [0,\infty)$ are such that $u \le g(v,v,u,u+v,0)$ or $u \le g(v,u,v,0,u+v)$, then $u \le hv$ where $0 < h < 1$ is a given constant.

(iii) If $u \in [0,\infty)$ is such that $u \le g(u,0,0,u,u)$, then $u = 0$.

Now we state our next result.

Theorem 2. *Let (X,d) be a compact metric space and let $\{F_i\}$ be a sequence of fuzzy self-mappings of $\mathcal{C}(\mathcal{X})$.*

If there is a $g \in G$ such that for all $\mu_1, \mu_2 \in \mathcal{C}$ (X),

$$\begin{aligned} & d_\infty(F_i(\mu_1), F_j(\mu_2)) \\ & \le g\left\{d_\infty(\mu_1,\mu_2), \rho_\infty(\mu_1, F_i(\mu_1), \rho_\infty(\mu_2, F_j(\mu_2), \rho_\infty(\mu_1, F_j(\mu_2), \rho_\infty(\mu_2, F_i(\mu_1)\right\}, \end{aligned}$$

then there exists a $\mu* \in \mathcal{C}$ (X) such that $\mu* \subseteq F_i(\mu*)$ for all $i \in \mathbb{Z}^+$.

Proof. Let $\mu_0 \in \mathcal{C}(X)$ and $\mu_1 \subseteq F_1(\mu_0)$. By Theorem D , there exists $\mu_2 \in \mathcal{C}(X)$ such that $\mu_2 \subseteq F_2(\mu_1)$ and $d_\infty(\mu_1,\mu_2) \le d_\infty(F_1(\mu_0), F_2(\mu_1))$. Again we can find $\mu_3 \in \mathcal{C}(X)$ such that $\mu_3 \subseteq F_3(\mu_2)$ and $d_\infty(\mu_2,\mu_3) \le d_\infty(F_2(\mu_1), F_3(\mu_2))$. By induction we have a sequence $\{\mu_n\}$ of elements in $\mathcal{C}(X)$ such that $\mu_{n+1} \subseteq F_{n+1}(\mu_n)$, $n = 0,1,2,3,\cdots$ and

$$\begin{aligned} d_\infty(\mu_n,\mu_{n+1}) &\le d_\infty(F_n(\mu_{n-1}), F_{n+1}(\mu_n)) \\ &\le g\,(d_\infty(\mu_{n-1},\mu_n), \rho_\infty(\mu_{n-1}, F_n(\mu_{n-1})), \rho_\infty(\mu_n, F_{n+1}(\mu_n)), \\ &\qquad \rho_\infty(\mu_{n-1}, F_{n+1}(\mu_n)), \rho_\infty(\mu_n, F_n(\mu_{n-1})))\,. \end{aligned}$$

This implies

$$\begin{aligned} & d_\infty\,(\mu_n,\mu_{n+1}) \\ & \quad \le g\left(d_\infty(\mu_{n-1},\mu_n), d_\infty(\mu_{n-1},\mu_n), d_\infty(\mu_n,\mu_{n+1})), d_\infty(\mu_{n-1},\mu_{n+1}),\right. \\ & \qquad\qquad d_\infty\,(\mu_n,\mu_n)) \\ & \quad \le g\,(d_\infty(\mu_{n-1},\mu_n), d_\infty(\mu_{n-1},\mu_n), d_\infty(\mu_n,\mu_{n+1})), d_\infty(\mu_{n-1},\mu_n) \\ & \qquad\qquad + d_\infty(\mu_n,\mu_{n+1}), 0)\,. \end{aligned}$$

This implies that $d_\infty(\mu_n,\mu_{n+1}) \le h d_\infty(\mu_{n-1},\mu_n)$ by property (ii) of the g function where $h < 1$.

We have

$$d_\infty(\mu_{k+m}, \mu_k)) \leq \sum_{i=k}^{k+m-1} d_\infty(\mu_i, \mu_{i+1}) \leq \sum h^k d_\infty(\mu_0, \mu_1)$$
$$= \frac{h^k}{1-h} d_\infty(\mu_0, \mu_1) \to 0 \quad \text{as} \quad k \to \infty.$$

That is, $\{\mu_n\}$ is Cauchy and there exists $\mu* \in \mathcal{C}(X)$ such that $\mu_n \to \mu*$.

Next we show that

$$\begin{aligned}
\rho_\infty(\mu*, F_i(\mu*)) &\leq d_\infty(\mu*, \mu_j) + \rho_\infty(\mu_j, F_i(\mu*)) \\
&\leq d_\infty(\mu*, \mu_j) + d_\infty(F_i(\mu*), F_j(\mu_{j-1})) \\
&\leq d_\infty(\mu*, \mu_j) + g\left(d_\infty(\mu^*, \mu_{j-1}), \rho_\infty(\mu^*, F_i(\mu^*)),\right. \\
&\quad \rho_\infty(\mu_{j-1}, F_j(\mu_{j-1})), \rho_\infty(\mu^*, F_j(\mu_{j-1})), \rho_\infty(\mu_{j-1,} F_i(\mu^*))) \\
&\leq d_\infty(\mu*, \mu_j) + g\left(d_\infty(\mu^*, \mu_{j-1}), \rho_\infty(\mu^*, F_i(\mu^*)), d_\infty(\mu_{j-1}, \mu_j)\right), \\
&\quad d_\infty(\mu^*, \mu_j), d_\infty(\mu*, \mu_{j-1}) + \rho_\infty(\mu *, F_i(\mu*)))
\end{aligned}$$

This result in the following:

$\rho_\infty(\mu*, F_i(\mu*)) \leq g(0, \rho_\infty(\mu*, F_i(\mu*)), 0, 0, \rho_\infty(\mu*, F_i(\mu*)))$ when $j \to \infty$ and this implies that

$\rho_\infty(\mu*, F_i(\mu*)) = 0$ by property (ii) of g function. Hence by Lemma 6, we have $\mu* \subseteq F_i(\mu*)$ for all i. □

We have several corollaries of Theorem 2.

Corollary 3. *Let X be a compact metric space and $\{F_i\}$ be a sequence of fuzzy self mappings of $C(X)$. If there exists a constant $0 \leq q < 1$, such that for all μ_1, $\mu_2 \in \mathcal{C}$ (X),*

$$\begin{aligned}
d_\infty(F_i(\mu_1), F_j(\mu_2)) &\leq q \max\{d_\infty(\mu_1, \mu_2), \rho_\infty(\mu_1, F_i(\mu_1)), \rho_\infty(\mu_2, F_j(\mu_2)), \\
&\qquad \frac{1}{2}[\rho_\infty(\mu_1, F_j(\mu_2)) + \rho_\infty(\mu_2, F_i(\mu_1)]\},
\end{aligned}$$

then there exists a $\mu* \in \mathcal{C}$ (X) such that $\mu* \subseteq F_i(\mu*)$ for all $i \in \mathbb{Z}^+$.

Proof. Take $g(x_1, x_2, x_3, x_4, x_5) = q \max\left\{x_1, x_2, x_3, \frac{1}{2}[x_4 + x_5]\right\} \in G$. By above theorem the result follows. □

Corollary 4. *Let X be a compact metric space and $\{F_i\}$ be a sequence of fuzzy self mappings of $C(X)$. If there exists a constant $0 \leq q < \frac{1}{2}$, such that for all μ_1, $\mu_2 \in \mathcal{C}$ (X),*

$$\begin{aligned}
d_\infty(F_i(\mu_1), F_j(\mu_2)) &\leq q \max[d_\infty(\mu_1, \mu_2), \rho_\infty(\mu_1, F_i(\mu_1), \rho_\infty(\mu_2, F_j(\mu_2), \\
&\qquad \rho_\infty(\mu_1, F_j(\mu_2), \rho_\infty(\mu_2, F_i(\mu_1)],
\end{aligned}$$

then there exists a $\mu* \in \mathcal{C}\,(\mathrm{X})$ *such that* $\mu* \subseteq F_i(\mu*)$ *for all* $i \in \mathbb{Z}^+$.

Proof. Take $g\,(x_1, x_2, x_3, x_4, x_5) = q \max\{x_1, x_2, x_3, x_4, x_5\} \in G$. The result follows by an application of Theorem 2. Also we note this a particular case of Theorem 1. □

Corollary 5. *Let* X *be a compact metric space and* $\{F_i\}$ *be a sequence of fuzzy self mappings of* $C(X)$. *If there exists a constant* $0 < q < 1$, *such that for all* μ_1, $\mu_2 \in \mathcal{C}\,(\mathrm{X})$,

$$d_\infty\left(F_i\left(\mu_1\right), F_j\left(\mu_2\right)\right) \quad \leq \quad q\left[\rho_\infty\left(\mu_1, F_i\left(\mu_1\right)\right)\rho_\infty\left(\mu_2, F_j\left(\mu_2\right)\right)\right]^{\frac{1}{2}},$$

then there exists a $\mu* \in \mathcal{C}\,(\mathrm{X})$ *such that* $\mu* \subseteq F_i\,(\mu*)$ *for all* $i \in \mathbb{Z}^+$.

Proof. We consider the function $g\,(x_1, x_2, x_3, x_4, x_5) = q\,[x_2x_3]^{\frac{1}{2}}$. Since $g \in G$, we can apply Theorem 2 to get this result. □

Remark 2. Corollary 5 is true when $\{F_i\}$ is a sequence of fuzzy self-mappings of $\mathcal{CB}(X)$ as proved in Theorem 5 by Qui and Shu [15].

Corollary 6. *Let* X *be a compact metric space and* $\{F_i\}$ *be a sequence of self-mappings of* $\mathcal{C}(X)$. *If for each* μ_1, $\mu_2 \in \mathcal{C}\,(\mathrm{X})$, *such that*

$$\begin{aligned} d_\infty(F_i(\mu_1), F_j(\mu_2)) \quad &\leq \quad a[\rho_\infty(\mu_1, F_i(\mu_1) + \rho_\infty(\mu_2, F_j(\mu_2)] \\ &\quad +b[\rho_\infty(\mu_1, F_j(\mu_2) + \rho_\infty(\mu_2, F_i(\mu_1)] + cd_\infty(\mu_1, \mu_2), \end{aligned}$$

where a, b, c *are non-negative and* $2a+2b+c < 1$. *Then there exists* $\mu* \in \mathcal{C}\,(\mathrm{X})$ *such that* $\mu* \subseteq F_i(\mu*)$ *for all* $i \in \mathbb{Z}^+$.

Proof. Let $g : [0, \infty)^5 \to [0, \infty)$ be defined by $g\,(x_1, x_2, x_3, x_4, x_5) = a\,[x_2 + x_3] + b\,[x_4 + x_5] + cx_1$. Since $g \in G$, the result follows from Theorem 2. □

Corollary 7. *Let* X *be a compact metric space and* $\{F_i\}$ *be a sequence of self-mappings of* $\mathcal{C}(X)$. *If there exists a constant* q, $0 \leq q < 1$, *such that for all* $\mu_1, \mu_2 \in \mathcal{C}\,(\mathrm{X})$,

$$\begin{aligned} d_\infty\left(F_i\left(\mu_1\right), F_j\left(\mu_2\right)\right) \leq q \max\Bigg\{ & d_\infty\left(\mu_1, \mu_2\right), \frac{1}{2}\left[\rho_\infty(\mu_1, F_i(\mu_1) + \rho_\infty(\mu_1, F_i(\mu_1)\right], \\ & \frac{1}{2}\left[\rho_\infty(\mu_1, F_j(\mu_2) + \rho_\infty(\mu_2, F_i(\mu_1)\right]\Bigg\}, \end{aligned}$$

then there exists $\mu* \in \mathcal{C}\,(\mathrm{X})$ *such that* $\mu* \subseteq F_i(\mu*)$ *for all* $i \in \mathbb{Z}^+$.

Proof. Let $g : [0, \infty)^5 \to [0, \infty)$ be defined by

$$g\,(x_1, x_2, x_3, x_4, x_5) \quad = \quad q \max\left\{x_1, \frac{1}{2}\left[x_2 + x_3\right], \frac{1}{2}\left[x_4 + x_5\right]\right\}.$$

Since $g \in G$, by Theorem 2, we get the desired result. □

Corollary 8. *Let X be a complete metric space and $\{F_i\}$ be a sequence of self-mappings of $\mathcal{C}(X)$. If there exists a constant q, $0 \leq q < 1$, such that for all $\mu_1,\ \mu_2 \in \mathcal{C}\,(\mathrm{X})$,*

$$d_\infty\left(F_i(\mu_1), F_j(\mu_2)\right) \leq a\frac{\rho_\infty(\mu_2, F_i(\mu_2)[1+\rho_\infty(\mu_1, F_j(\mu_1)]}{1+d_\infty(\mu_1,\mu_2)} + bd_\infty(\mu_1,\mu_2)$$

for all $\mu_1 \neq \mu_2$, $a,\ b > 0$, and $a+b<1$. Then there exists $\mu* \in \mathcal{C}\,(\mathrm{X})$ such that $\mu* \subseteq F_i(\mu*)$ for all $i \in \mathbb{Z}^+$.

Proof. Let $g : [0,\infty)^5 \to [0,\infty)$ be defined by

$$g\left(x_1, x_2, x_3, x_4, x_5\right) = a\frac{x_3\left[1+x_2\right]}{1+x_1} + bx_1.$$

Since $g \in G$, by Theorem 2, we get the desired result. □

Next theorem is an extension of the work of Sedghi et al [20] in a compact metric space setting. In the following let $K : [0,\infty) \to [0,\infty)$, $K(0) = 0$, $\sum_{n=1}^{\infty} K^n(t) < \infty$ for all $t \in (0,\infty)$ ($\Longrightarrow K(t) < t$) and K is nondecreasing and continuous and $\phi : [0,\infty)^5 \to [0,\infty)$ is a continuous and increasing in each coordinate variable and $\phi\left(t,t,t,at,bt\right) \leq t$ for every $t \in [0,\infty)$, where $a+b=2$.

Theorem 3. *Let (X,d) be a compact metric space and let $\{F_i\}$ be a sequence of fuzzy self-mappings of $\mathcal{C}\,(X)$.*

If there exist a $K : [0,\infty) \to [0,\infty)$ and a $\phi : [0,\infty)^5\phi : [0,\infty)^5 \to [0,\infty)$ as specified above such that for all $\mu_1,\ \mu_2 \in \mathcal{C}\,(\mathrm{X})$,

$$\begin{aligned} d_\infty(F_i(\mu_1), F_j(\mu_2)) \leq\ & K\left(\phi\left\{d_\infty\left(\mu_1,\mu_2\right), \rho_\infty\left(\mu_1, F_i\left(\mu_1\right)\right), \rho_\infty\left(\mu_2, F_j\left(\mu_2\right)\right),\right.\right. \\ & \left.\left.\rho_\infty\left(\mu_2, F_i\left(\mu_1\right)\right), \rho_\infty\left(\mu_1, F_j\left(\mu_2\right)\right)\right\}\right), \end{aligned}$$

then there exists a $\mu* \in \mathcal{C}\,(\mathrm{X})$ such that $\mu* \subseteq F_i(\mu*)$ for all $i \in \mathbb{Z}^+$.

Proof. Let $\mu_0 \in \mathcal{C}(X)$ and $\mu_1 \subseteq F_1(\mu_0)$. By Theorem D, there exists $\mu_2 \in \mathcal{C}(X)$ such that $\mu_2 \subseteq F_2(\mu_1)$ and $d_\infty(\mu_1,\mu_2) \leq d_\infty(F_1(\mu_0), F_2(\mu_1))$. Again we can find $\mu_3 \in \mathcal{C}(X)$ such that $\mu_3 \subseteq F_3(\mu_2)$ and

$$\begin{aligned} d_\infty(\mu_2,\mu_3) &\leq d_\infty(F_2(\mu_1), F_3(\mu_2)) \\ &\leq K\left(\phi\left\{d_\infty(\mu_1,\mu_2), \rho_\infty(\mu_1, F_2(\mu_1), \rho_\infty(\mu_2, F_3(\mu_2),\right.\right. \\ &\qquad \left.\left.\rho_\infty(\mu_2, F_2(\mu_1), \rho_\infty(\mu_1, F_3(\mu_2))\right\}\right) \\ &\leq K\left(\phi\left\{d_\infty(\mu_1,\mu_2), d_\infty(\mu_1,\mu_2), d_\infty(\mu_2,\mu_3),\right.\right. \\ &\qquad \left.\left.d_\infty(\mu_2,\mu_2), d_\infty(\mu_1,\mu_3)\right\}\right) \text{ by Lemma 2} \\ &\leq K\left(\phi\left\{d_\infty(\mu_1,\mu_2), d_\infty(\mu_1,\mu_2), d_\infty(\mu_2,\mu_3),\right.\right. \\ &\qquad \left.\left.0, d_\infty(\mu_1,\mu_2)+d_\infty(\mu_2,\mu_3)\right\}\right). \end{aligned}$$

If $d_\infty\left(\mu_1,\mu_2\right) < d\left(\mu_2,\mu_3\right)$ then by the inequality above, we have

$$d_\infty(\mu_2,\mu_3) \le K\left(\phi\left\{d_\infty(\mu_3,\mu_2), d_\infty(\mu_3,\mu_2), d_\infty(\mu_2,\mu_3),\right.\right.$$
$$\left.\left. 0, d_\infty(\mu_3,\mu_2)+d_\infty(\mu_2,\mu_3)\right\}\right) \le K(d_\infty(\mu_3,\mu_2)) < d_\infty(\mu_3,\mu_2),$$

which is a contradiction.

Hence $d_\infty(\mu_1,\mu_2) > d(\mu_2,\mu_3)$. Thus we get $d_\infty(\mu_2,\mu_3) \le K(d_\infty(\mu_1,\mu_2)) \le K(d_\infty(\mu_0,\mu_1))$.

By induction we have a sequence $\{\mu_n\}$ of elements in $\mathcal{C}(X)$ such that $\mu_{n+1} \subseteq F_{n+1}(\mu_n)$, $n = 0,\ 1,\ 2,\ 3,\ \cdots$ and

$$\begin{aligned}d_\infty(\mu_n,\mu_{n+1}) &\le d_\infty(F_n(\mu_{n-1}), F_{n+1}(\mu_n))\\ &\le K\left(\phi\left\{d_\infty(\mu_{n-1},\mu_n), \rho_\infty(\mu_{n-1}, F_n(\mu_{n-1})), \rho_\infty(\mu_n, F_{n+1}(\mu_n)),\right.\right.\\ &\qquad \left.\left.\rho_\infty(\mu_{n-1}, F_{n+1}(\mu_n)), \rho_\infty(\mu_n, F_n(\mu_{n-1}))\right\}\right).\end{aligned}$$

This implies

$$\begin{aligned}d_\infty(\mu_n,\mu_{n+1}) \quad \le \quad & K\left(\phi\left\{d_\infty(\mu_{n-1},\mu_n), d_\infty(\mu_{n-1},\mu_n), d_\infty(\mu_n,\mu_{n+1})),\right.\right.\\ & \left.\left. d_\infty(\mu_{n+1}\mu_n), d_\infty(\mu_n,\mu_n)\right\}\right).\end{aligned}$$

This implies that

$$\begin{aligned}d_\infty(\mu_n,\mu_{n+1}) \quad &\le \quad K\left(d_\infty(\mu_{n-1},\mu_n)\right)\\ &\le \quad K^2\left(d_\infty(\mu_{n-2},\mu_{n-1})\right) \le ..... \le K^n\left(d_\infty(\mu_0,\mu_1)\right).\end{aligned}$$

This implies that $\{\mu_n\}$ is Cauchy and by completeness of $\mathcal{C}(X)$, there exists $\mu* \in \mathcal{C}(X)$ such that $\mu_n \to \mu*$.

Next we show that

$$\begin{aligned}\rho_\infty(\mu*, F_i(\mu*)) \quad &\le \quad d_\infty(\mu*,\mu_j) + \rho_\infty(\mu_j, F_i(\mu*))\\ &\le \quad d_\infty(\mu*,\mu_j) + d_\infty(F_i(\mu*), F_j(\mu_{j-1})).\end{aligned}$$

This result in the following:

$$\begin{aligned}&\le d_\infty(\mu*,\mu_j)\\ &+K\left(\phi\left\{d_\infty(\mu^*,\mu_{j-1}), \rho_\infty(\mu^*, F_i(\mu^*)), \rho_\infty(\mu_{j-1}, F_j(\mu_{j-1})),\right.\right.\\ &\qquad \left.\left.\rho_\infty(\mu_{j-1}, F_i(\mu^*)), \rho_\infty(\mu^*, F_j(\mu_{j-1}))\right\}\right)\\ &\le d_\infty(\mu*,\mu_j)\\ &+K\left(\phi\left\{d_\infty(\mu^*,\mu_{j-1}), \rho_\infty(\mu^*, F_i(\mu^*)), \rho_\infty(\mu_{j-1}, \mu_j)),\right.\right.\\ &\qquad \left.\left.\rho_\infty(\mu_{j-1}, F_i(\mu^*)), \rho_\infty(\mu^*, \mu_j)\right\}\right)\end{aligned}$$

and taking limit as $j \to \infty$this implies that

$$\begin{aligned}\rho_\infty(\mu*, F_i(\mu*)) \quad &\le \quad K\left(\phi\left\{0, \rho_\infty(\mu^*, F_i(\mu^*)), 0, \rho_\infty(\mu^*, F_i(\mu^*)), 0\right\}\right)\\ &\le \quad K\left(\phi\left\{\rho_\infty(\mu^*, F_i(\mu^*)), \rho_\infty(\mu^*, F_i(\mu^*)), \rho_\infty(\mu^*, F_i(\mu^*)),\right.\right.\\ &\qquad \left.\left.\rho_\infty(\mu^*, F_i(\mu^*)), \rho_\infty(\mu^*, F_i(\mu^*))\right\}\right)\end{aligned}$$

$$\leq \quad K\left(\rho_{\infty}(\mu^{*}, F_i(\mu^{*})) < \rho_{\infty}\left(\mu^{*}, F_i\left(\mu^{*}\right)\right)\right)$$

and this implies that $\rho_{\infty}(\mu*, F_i(\mu*)) = 0$.

By Lemma 6, we have $\mu* \subseteq F_i(\mu*)$ for all i. □

Corollary 9. *Let (X, d) be a complete metric space and let $\{F_i\}$ be a sequence of fuzzy self-mappings of $\mathcal{C}\left(\mathrm{X}\right)$.*

If there exist a $K : [0, \infty) \to [0, \infty)$ and a $\phi : [0, \infty)^5 \phi : [0, \infty)^5 \to [0, \infty)$ as specified above such that for all $\mu_1, \mu_2 \in \mathcal{C}\left(\mathrm{X}\right)$,

$$d_{\infty}\left(F_i\left(\mu_1\right), F_j\left(\mu_2\right)\right)$$
$$\leq K\left(\max\left\{d_{\infty}\left(\mu_1, \mu_2\right), \rho_{\infty}\left(\mu_1, F_i\left(\mu_1\right)\right), \rho_{\infty}\left(\mu_2, F_j\left(\mu_2\right)\right), \frac{1}{2}\left[\rho_{\infty}\left(\mu_2, F_i\left(\mu_1\right)\right), +\rho_{\infty}\left(\mu_1, F_j\left(\mu_2\right)\right)\right]\right\}\right),$$

then there exists a $\mu* \in \mathcal{C}\left(\mathrm{X}\right)$ such that $\mu* \subseteq F_i(\mu*)$ for all $i \in \mathbb{Z}^+$.

Proof. $\phi\left(t_1, t_2, t_3, t_4, t_5\right) = \max\left\{t_1, t_2, t_3, \frac{t_4+t_5}{2}\right\}$ satisfies the required condition imposed on ϕ. Hence the result follows from Theorem 3. □

Remarks 3. 1) The proof of Theorem 2.7 given by Sedghi et al [20] is true when $\alpha-$cut sets are compact, not when $\alpha-$cut sets are nonempty, closed and bounded as claimed by the authors.

2) Remark 2.9 in [20] is wrong since $\phi\left(t_1, t_2, t_3, t_4, t_5\right) = \max\left\{t_1, t_2, t_3, t_4, t_5\right\}$ does not satisfy the required condition. However

$$\phi\left(t_1, t_2, t_3, t_4, t_5\right) = \max\left\{t_1, t_2, t_3, \frac{t_4 + t_5}{2}\right\}$$

satisfies the required condition imposed on ϕ.

3) Furthermore the proof of Theorem 1 of Turkoglu and Rhoades [21] is unfortunately wrong (equation (2), p. 118) and this might be the reason for Sedghi et al to replace max by ϕ function. I have discussed this case in another paper.

Let (X, d) be a complete metric space. A fuzzy mapping $F : \mathcal{C}\left(X\right) \to \mathcal{C}\left(X\right)$ is said to be weakly contractive or a $\phi-$weak contraction if $d_{\infty}(F(\mu_1), F(\mu_2)) \leq d_{\infty}(\mu_1, \mu_2) - \phi\left(d_{\infty}\left(\mu_1, \mu_2\right)\right)$ where $\phi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that $\phi(t) > 0$ for $t \in (0, \infty)$ and $\phi\left(0\right) = 0$. We now prove a common fixed point theorem concerning a pair of such fuzzy mappings.

Theorem 4. *Let (X, d) be a compact metric space and let F_1, $F_2 : \mathcal{C}(X) \to \mathcal{C}(X)$ be two fuzzy mappings satisfing the following condition:*

$d_{\infty}\left(F_1\left(\mu_1\right), F_2\left(\mu_2\right)\right) \leq d_{\infty}\left(\mu_1, \mu_2\right) - \phi\left(d_{\infty}\left(\mu_1, \mu_2\right)\right)$ *for all* μ_1, $\mu_2 \in \mathcal{C}(X)$.

Then there exists a $\mu^* \in \mathcal{C}(X)$ *such that* $\mu^* \subseteq F_1(\mu^*)$ *and* $\mu^* \subseteq F_2(\mu^*)$.

Proof. Let $\mu_0 \in \mathcal{C}(X)$ and $\mu_1 \subseteq F_1(\mu_0)$. Then by Theorem D, there exists $\mu_2 \subseteq F_2(\mu_1)$ and

$$d_\infty(\mu_1, \mu_2) \le d_\infty(F_1(\mu_0), F_2(\mu_1)) \le d_\infty(\mu_0, \mu_1) - \phi(d_\infty(\mu_0, \mu_1)).$$

Again by Theorem D, we can find $\mu_3 \in \mathcal{C}(X)$ such that $\mu_3 \subseteq F_1(\mu_2)$ and

$$d_\infty(\mu_2, \mu_3) \le d_\infty(F_1(\mu_1), F_2(\mu_2)) \le d_\infty(\mu_1, \mu_2) - \phi(d_\infty(\mu_1, \mu_2)).$$

Continuing this process, we produce a sequence $\{\mu_n\}$ of points in $\mathcal{C}(X)$ as follows:

$\mu_n \subseteq F_1(\mu_{n-1})$, n is odd , choose $\mu_{n+1} \subseteq F_2(\mu_n)$ such that

$$\begin{aligned} d_\infty(\mu_n, \mu_{n+1}) &\le d_\infty(F_1(\mu_{n-1}), F_2(\mu_n)) \\ &\le d_\infty(\mu_{n-1}, \mu_n) - \phi(d_\infty(\mu_{n-1}, \mu_n)) \le d_\infty(\mu_{n-1}, \mu_n). \end{aligned}$$

That is, $d_\infty(\mu_n, \mu_{n+1}) \le d_\infty(\mu_{n-1}, \mu_n)$

The sequence $\{d_\infty(\mu_n, \mu_{n+1})\}$ is a decreasing sequence of bounded real numbers and therefore converges to a limit l. Suppose $l > 0$. This implies that

$$d_\infty(\mu_{n+1}, \mu_{n+2}) \le d_\infty(\mu_n, \mu_{n+1}) - \phi(l)$$

and

$$d_\infty(\mu_{n+N}, \mu_{n+N+1}) \le d_\infty(\mu_n, \mu_{n+1}) - N\phi(l)$$

and this leads to a contracdition for N large enough if $\phi(l) > 0$. Hence $l = 0$. That is, $d_\infty(\mu_n, \mu_{n+1}) \to 0$ as $n \to \infty$. As in Bose and Roychowdhury [8], it can be proved that $\{d_\infty(\mu_n, \mu_{n+1})\}$ is a Cauchy sequence. By completeness of $\mathcal{C}(X)$, $\{\mu_n\}$ converges to $\mu^* \in \mathcal{C}(X)$. To prove that $\mu^* \subseteq F_1(\mu^*)$, consider the following:

$$\rho_\infty(\mu^*, F_1(\mu^*)) \le d_\infty(\mu^*, \mu_j) + \rho_\infty(\mu_j, F_1(\mu^*)),$$

where j is even, say $n = 2k$.

$$\begin{aligned} &\le d_\infty(\mu^*, \mu_j) + d_\infty(F_2(\mu_{j-1}), F_1(\mu^*)) \\ &\le d_\infty(\mu^*, \mu_j) + d_\infty(\mu^*, \mu_{j-1}) - \phi(d_\infty(\mu^*, \mu_{j-1})). \end{aligned}$$

By taking $j \to \infty$, we have $\rho_\infty(\mu^*, F_1(\mu^*)) = 0$ and this implies that $\mu^* \subseteq F_1(\mu^*)$, by Lemma 2. Similarly we can show that $\mu^* \subseteq F_2(\mu^*)$. □

Remark 4. We can extend Theorem 4 to an infinite family of such fuzzy mappings where the fuzzy mappings F_i, F_j satisfy the same weak contractive condition instead of F_1 and F_2.

Next we prove a fixed point theorem for a fuzzy mapping $T : \mathcal{CB}(X) \to \mathcal{CB}(X)$ which is a $(\delta, L)-$weak contraction.

Theorem 5. *Let* $T : \mathcal{CB}(X) \to \mathcal{CB}(X)$ *be a fuzzy mapping which satisfies* *the following condition:*

$$d_\infty(T(\mu_1),T(\mu_2)) \leq \delta d_\infty(\mu_1,\mu_2)+L\rho_\infty(\mu_2,T(\mu_1)),$$

where $0<\delta<1$, $L>0$ but arbitrary. Then there exists a $\mu^*\in\mathcal{CB}(X)$ such that $\mu^*\subseteq T(\mu^*)$.

Proof. Let $\mu_0\in\mathcal{CB}(X)$ and $\mu_1\subseteq T(\mu_0)$. By Theorem B, there exists $\mu_2\in\mathcal{CB}(X)$ such that $\mu_2\subseteq T(\mu_1)$ and

$$d_\infty(\mu_1,\mu_2) \leq \beta d_\infty(T(\mu_0),T(\mu_1)),$$

where $\beta\geq 1$ but chosen such that $\beta\delta<1$.

$$\leq \beta\delta d_\infty(\mu_0,\mu_1)+L\beta\rho_\infty(\mu_1,T\mu_0)$$

Since $\rho_\infty(\mu_1,T\mu_0)=0$ by Lemma 2, we have

$$d_\infty(\mu_1,\mu_2) \leq \beta\delta d_\infty(\mu_0,\mu_1).$$

Let $\beta\delta=h$. Again by Theorem B, there exists $\mu_3\in\mathcal{CB}(X)$ such that $\mu_3\subseteq T(\mu_2)$ and

$$\begin{aligned} d_\infty(\mu_2,\mu_3) &\leq \beta d_\infty(T(\mu_1),T(\mu_2)\leq \beta\delta d_\infty(\mu_1,\mu_2)+\beta L\rho_\infty(\mu_2,T(\mu_1)) \\ &\leq hd_\infty(\mu_1,\mu_2)\leq h^2 d_\infty(\mu_0,\mu_1). \end{aligned}$$

Similarly, we have

$$d_\infty(\mu_n,\mu_{n+1}) \leq h^n d_\infty(\mu_0,\mu_1).$$

This leads to

$$d_\infty(\mu_n,\mu_{n+m}) \leq h^n\frac{(1-h^m)}{(1-h)}d_\infty(\mu_0,\mu_1).$$

Since $h<1$, $\{\mu_n\}$ is a Cauchy sequence in $\mathcal{CB}(X)$ and by virtue of the completeness of $\mathcal{CB}(X)$, the sequence $\{\mu_n\}$ converges to μ^* in $\mathcal{CB}(X)$.

To show that $\mu^*\subseteq T(\mu^*)$, we proceed as follows:

$$\begin{aligned} \rho_\infty(\mu^*,T(\mu^*)) &\leq d_\infty(\mu^*,\mu_j)+\rho_\infty(\mu_j,T(\mu^*)) \\ &\leq d_\infty(\mu^*,\mu_j)+d_\infty(T(\mu_{j-1}),T(\mu^*)) \\ &\leq d_\infty(\mu^*,\mu_j)+\delta d_\infty(\mu_{j-1},\mu^*)+L\rho_\infty(\mu^*,T(\mu_{j-1})) \\ &\leq d_\infty(\mu^*,\mu_j)+\delta d_\infty(\mu_{j-1},\mu^*)+Ld_\infty(\mu^*,\mu_j) \\ &\qquad +L\rho_\infty(\mu_j,T(\mu_{j-1})) \text{ by Lemma 2.} \end{aligned}$$

Taking limit as $j\to\infty$, we have $\rho_\infty(\mu^*,T(\mu^*))=0$. By Lemma 2, we have $\mu^*\subseteq T(\mu^*)$. □

Remark 5. Bose and Roychowdhury [8, 9] have considered a different type of weakly contractive fuzzy mappings and (δ,L) −fuzzy contractions and their fixed points (common fixed points) and related further work concerning the type of fuzzy mappings discussed in this paper will be presented in another paper.

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