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PRACTICAL STABILITY AND CONE VALUED LYAPUNOV FUNCTIONS FOR DIFFERENTIAL EQUATIONS WITH "MAXIMA"

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Abstract: A special type of practical stability for differential equations with "maxima" is introduced. The definitions are based on the application of two different measures as well as on the application of a scalar product on a cone. This allows us to use cone valued Lyapunov functions for investigation of stability properties of the solutions. Some sufficient conditions for d-practical stability in terms of two measures of nonlinear differential equations with "maxima" are obtained. An example illustrates the practical application of the proved results.

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1. Introduction

It is well-known (see [3]) that stability and even asymptotic stability themselves are neither necessary nor sufficient to ensure practical stability. The desired state of a system may be mathematically unstable; however, the system may oscillate sufficiently close to the desired state, and its performance is deemed acceptable. The practical stability is neither weaker nor stronger than the usual stability; an equilibrium can be stable in the usual sense, but not practically stable, and vice versa. For example an aircraft may oscillate around a mathematically unstable path, yet its performance may be acceptable. Prac-

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tical stability is, in a sense, a uniform boundedness of the solution relative to the initial conditions, but the bound must be sufficiently small.

We will study practical stability of differential equations with “maxima”. Note that stability for differential equations with “maxima” is studied by D.D. Bainov et al [4]. In this paper practical stability will be investigated by the help with cone valued multi-dimensional Lypunov functions, Note that in the practical applications such kind of functions are comparatively easier for construction. To avoid applications of comparison systems of differential equations and to apply scalar differential equations, we introduce scalar product on a cone and appropriate modifications of stability definitions.

Let $r > 0$ be a given number, $t_0 \in \mathbb{R}_+$, and $\phi \in C([-r, 0], \mathbb{R}^n)$. Consider the nonlinear differential equations with “maxima”

$$x' = F(t, x(t), \max_{s\in[t-r,t]} x(s)) \quad \text{for} \quad t \geq t_0 \tag{1}$$

with initial condition

$$x(t) = \phi(t - t_0) \quad \text{for} \quad t \in [t_0 - r, t_0], \tag{2}$$

where $x \in \mathbb{R}^n$, $F : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $F = (F_1, F_2, \dots, F_n)$, $r > 0$ is a constant, $\phi \in C([-r, 0], \mathbb{R}^n)$ and $t_0 \in \mathbb{R}_+$ is a fixed point.

We denote by $x(t; t_0, \phi)$ the solution of the initial value problem (1), (2). In our further investigations we will assume that solution $x(t; t_0, \phi)$ is defined on $[t_0 - r, \infty)$ for any initial function $\phi \in C([-r, 0], \mathbb{R}^n)$.

Let $x, y \in \mathbb{R}^n$. Denote by $(x \bullet y)$ the dot product of both vectors x and y.

Let $\mathcal{K} \subset \mathbb{R}^n$ be a cone. Consider the set

$$\mathcal{K}^* = \{\varphi \in \mathbb{R}^n : \ (\varphi \bullet x) \geq 0 \text{ for any } \ x \in \mathcal{K}\}.$$

We assume that $\mathcal{K}^*$ is a cone.

Introduce the following sets

$$\begin{aligned} K &= \{a \in C(\mathbb{R}_+, \mathbb{R}_+) \ : \ a(s) \text{ is strictly increasing and } a(0) = 0\}, \\ CK &= \{b \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+) \ : \ b(t) \in K \text{ for any fixed } t \in [0, \infty)\}, \\ \mathcal{G} &= \{h \in C([-r, \infty) \times \mathbb{R}^n, \mathcal{K}) \ : \ \inf_{x\in\mathbb{R}^n} h(t, x) = 0 \text{ for each } t \geq -r\}. \end{aligned} \tag{3}$$

Let $h_0, h \in \mathcal{G}$, $\varphi_0 \in \mathcal{K}^*$, $\phi \in C([-r, 0], \mathbb{R}^n)$, and $t \in [0, \infty)$. Define

$$H(t, \phi, \varphi_0) = \sup\{(\varphi_0 \bullet h(t + s, \phi(s))) : \quad s \in [-r, 0]\} \tag{4}$$

and

$$H_0(t, \phi, \varphi_0) = \sup\{(\varphi_0 \bullet h_0(t + s, \phi(s))) : \quad s \in [-r, 0]\}. \tag{5}$$

Let ρ be positive constant, $\varphi_0 \in \mathcal{K}^*$, $h \in \mathcal{G}$. Define set:

$$\tilde{\mathcal{S}}(h, \rho, \varphi_0) = \{(t, x) \in [0, \infty) \times \mathbb{R}^n : \quad (\varphi_0 \bullet h(t, x)) < \rho\}.$$

In our further investigations we will use following comparison scalar ordinary differential equations

$$u' = g(t, u), \tag{6}$$

where $u \in \mathbb{R}$, $g(t, 0) \equiv 0$.

We will use some properties of the functions from class $\mathcal{G}$.

Definition 1. Let $h, h_0 \in \mathcal{G}$. Function h_0 is *uniformly φ_0-finer* than h with a constant $\delta > 0$ and a function $p \in K$ if for any point $(t, x) \in [-r, \infty) \times \mathbb{R}^n$ such that $(\varphi_0 \bullet h_0(t, x)) < \delta$ the inequality $(\varphi_0 \bullet h(t, x)) \le p((\varphi_0 \bullet h_0(t, x))$ holds.

Lemma 1. *Let $\varphi_0 \in \mathcal{K}^*$, $h, h_0 \in \mathcal{G}$, and h_0 is uniformly φ_0-finer than h with a constant δ and a function $p \in K$. Then for any $\phi \in C([-r, 0], \mathbb{R}^n)$ and $t \in \mathbb{R}_+$ the inequality $H_0(t, \phi, \varphi_0) < \delta$ implies $H(t, \phi, \varphi_0) \le p(H_0(t, \phi, \varphi_0))$, where functions H_0, H are defined by (4), (5).*

Proof. Let the vector $\phi \in C([-r, 0], \mathbb{R}^n)$ and the point $t \in \mathbb{R}_+$ be such that $H_0(t, \phi, \varphi_0) < \delta$, i.e. $\sup_{s \in [-r,0]}(\varphi_0 \bullet h_0(t + s, \phi(s))) < \delta$. Since h_0 is uniformly finer than h then for every $s \in [-r, 0]$ the inequality $(\varphi_0 \bullet h(t + s, \phi(s))) \le p((\varphi_0 \bullet h_0(t + s, \phi(s)))) \le p(H_0(t, \phi, \varphi_0))$ holds, i.e.

$$H(t, \phi, \varphi_0) \le p(H_0(t, \phi, \varphi_0)). \qquad \square$$

We will introduce following class of functions:

Definition 2. We will say that function $V(t, x) : \Omega \times \mathbb{R}^n \to \mathcal{K}$, $\Omega \subset \mathbb{R}_+$, $V = (V_1, V_2, \ldots, V_n)$, belongs to the class $\mathcal{L}$ if:

1. $V(t, x) \in C^1(\Omega \times \mathbb{R}^n, \mathcal{K})$;
2. There exist constants $M_i > 0$, $i = 1, 2, \ldots, n$, such that $|V_i(t, x) - V_i(t, y)| \le M_i ||x - y||$ for any $t \in \Omega$, $x, y \in \mathbb{R}^n$.

Let function $V \in \mathcal{L}$, $V = (V_1, V_2, \ldots, V_n)$ and $\phi \in C([-r, 0], \mathbb{R}^n)$. We define a derivative $\mathcal{D}_{(1)}V(t, x)$ of the function V along the system (1) by the equalities

$$\mathcal{D}_{(1)}V_i(t, \phi(0)) = \frac{\partial V_i(t, \phi(0))}{\partial t} + \sum_{j=1}^{n} \frac{\partial V_i(t, \phi(0))}{\partial x_j} F_j(t, \phi(0), \sup_{s \in [-r,0]} \phi(s)),$$
$$i = 1, 2, \ldots, n, \tag{7}$$

where $\mathcal{D}_{(1)}V(t, x) = (\mathcal{D}_{(1)}V_1(t, x), \mathcal{D}_{(1)}V_2(t, x), \ldots, \mathcal{D}_{(1)}V_n(t, x))$.

We will introduce the definition of a new type of stability for differential equations with "maxima", based on the ideas of stability in terms of two measures (see [2]) and a dot product.

Definition 3. Let the vector $\varphi_0 \in \mathcal{K}^*$, the functions $h, h_0 \in \mathcal{G}$, and the constants $\lambda, A: \ 0 < \lambda < A$ be given. The system of differential equations with "maxima" (1) is said to be

(S1) *d-practically stable with respect to* (λ, A) in terms of measures h_0 and h with a vector φ_0 if there exists $t_0 \geq 0$ such that for any $\phi \in C([-r, 0], \mathbb{R}^n)$ inequality $H_0(t_0, \phi, \varphi_0) < \lambda$ implies $(\varphi_0 \bullet h(t, x(t; t_0, \phi))) < A$ for $t \geq t_0$, where the function H_0 is defined by (5), $x(t; t_0, \phi)$ is a solution of differential equations with "maxima"(1) with initial condition (2);

(S2) *d-uniformly practically stable with respect to* (λ, A) in terms of measures h_0 and h with a vector φ_0 if for any $t_0 \geq 0$ and $\phi \in C([-r, 0], \mathbb{R}^n)$ inequality $H_0(t_0, \phi, \varphi_0) < \lambda$ implies $(\varphi_0 \bullet h(t, x(t; t_0, \phi))) < A$ for $t \geq t_0$.

Note in the case $\varphi_0 = (1, 1, \ldots, 1)$, $h(t, x) = h_0(t, x) \equiv (|x_1|, |x_2|, \ldots, |x_n|)$, where $x = (x_1, x_2, \ldots, x_n)$, d-practical stability, given in Definition 3, reduces to practical stability of differential equations with "maxima":

Definition 4. The system of differential equations with "maxima" (1) is said to be

(S3) *practically stable with respect to* (λ, A) there exists $t_0 \geq 0$ such that for any $\phi \in C([-r, 0], \mathbb{R}^n)$ inequality $\sup_{s \in [t_0 - r, t_0]} ||\phi(s)|| < \lambda$ implies $||x(t; t_0, \phi)|| < A$ for $t \geq t_0$,where $x(t; t_0, \phi)$ is a solution of differential equations with "maxima"(1) with initial condition (2);

(S4) *uniformly practically stable with respect to* (λ, A) if for any $t_0 \geq 0$ and $\phi \in C([-r, 0], \mathbb{R}^n)$ inequality $\sup_{s \in [t_0 - r, t_0]} ||\phi(s)|| < \lambda$ implies $||x(t; t_0, \phi))|| < A$ for $t \geq t_0$.

Note that in the case $r = 0$, the above given definitions reduce to definitions for practical stability of ordinary differential equations, given in the book [1].

In the further investigations we will use following comparison result:

Lemma 2. *Let the following conditions be fulfilled:*

1. The vector $\varphi_0 \in \mathcal{K}^*$ *and the function* $V(t, x) : [t_0, T] \times \mathbb{R}^n \to \mathcal{K}$, $V \in \mathcal{L}$ *are such that for any function* $\psi \in C([-r, 0], \mathbb{R}^n)$ *and any number* $t \in [t_0, T]$ *such that* $(\varphi_0 \bullet V(t, \psi(0))) \geq (\varphi_0 \bullet V(t + s, \psi(s)))$ *for* $s \in [-r, 0)$ *the inequality*

$$\Big(\varphi_0 \bullet \mathcal{D}_{(1)} V(t, \psi(0))\Big) \leq g(t, (\varphi_0 \bullet V(t, \psi(0))))$$

holds, where $g \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+), g(t, 0) \equiv 0$.

2. The function $x(t) = x(t; t_0, \varphi)$ is a solution of (1) with initial condition $x(t_0 + s) = \varphi(s)$, $s \in [-r, 0]$, that is defined for $t \in [t_0 - r, T]$ where $\varphi \in C([-r, 0], \mathbb{R}^n)$.

3. The function $u^*(t) = u^*(t; t_0, u_0)$ is the maximal solution of (6) with initial condition $u^*(t_0) = u_0$, that is defined for $t \in [t_0, T]$.

Then the inequality $\max_{s\in[-r,0]}(\varphi_0 \bullet V(t_0+s, \varphi(s))) \leq u_0$ implies the validity of the inequality $(\varphi_0 \bullet V(t, x(t))) \leq u^*(t)$ for $t \in [t_0, T]$.

Proof. Let $u_n(t)$ be the maximal solution of the initial value problem

$$u' = g(t, u) + \frac{1}{n},$$
$$u(t_0) = u_0 + \frac{1}{n}, \tag{8}$$

where $\max_{s\in[-r,0]}(\varphi_0 \bullet V(t_0+s, \varphi(s))) \leq u_0$ and n is a natural number. Assume that $u_n(t)$ is defined for $t \in [t_0, T]$.

Define a function $m(t) \in C([t_0, T], \mathbb{R}_+)$ by the equality

$$m(t) = (\varphi_0 \bullet V(t, x(t))).$$

Because of the fact that $u^*(t; t_0, u_0) = \lim_{n\to\infty} u_n(t)$ it is enough to prove that for any natural number n the inequality

$$m(t) \leq u_n(t) \quad \text{for} \quad t \in [t_0, T] \tag{9}$$

holds.

Note that for any natural number n inequality $m(t_0) < u_n(t_0)$ holds.

Assume inequality (9) is not true. Let n be a natural number such that there exists a point $\eta \in (t_0, T) : \ m(\eta) > u_n(\eta)$. Let $t_n^* = \max\{t \in [t_0, T] : m(s) < u_n(s) \text{ for } s \in [t_0, t)\}$, $t_n^* < T$.

Therefore

$$m(t_n^*) = u_n(t_n^*), \quad m(t) < u_n(t) \quad \text{for} \quad t \in [t_0, t_n^*), \quad m(t) \geq u_n(t) \quad \text{for } t \in (t_n^*, t_n^* + \delta), \tag{10}$$

where $\delta > 0$ is enough small number.

From inequalities (10) it follows that

$$m'(t_n^*) \geq u_n'(t_n^*) = g(t, u_n(t_n^*)) + \frac{1}{n} = g(t, m(t_n^*)) + \frac{1}{n}. \tag{11}$$

From $g(t, u) + \frac{1}{n} > 0$ on $[t_n^* - r, t_n^*] \cap [t_0, T]$ it follows the function $u_n(t)$ is nondecreasing on $[t_n^* - r, t_n^*] \cap [t_0, T]$.

If $t_n^* - r \geq t_0$ then $m(t_n^*) = v_n(t_n^*) \geq v_n(s) > m(s)$ for $s \in [t_n^* - r, t_n^*)$.

If $t_n^* - r < t_0$, then as above $m(t_n^*) > m(s)$ for $s \in [t_0, t_n^*)$ and $m(t_n^*) = v_n(t_n^*) \geq v_n(t_0) = u_0 + \frac{1}{n} > u_0 \geq \sup_{s\in[-r,0]} V(t_0 + s, \phi(s)) \geq m(s)$ for $s \in [t^* - r, t_0)$.

Therefore, $m(t_n^*) > m(s)$ for $s \in [t_n^* - r, t_n^*)$.

According to condition 1 of Lemma 2 using standard argument we get $m'(t_n^*) \leq g(t, m(t_n^*)) < g(t, m(t_n^*)) + \frac{1}{n}$ that contradicts (11). Therefore the inequality (9) holds and hence the conclusion of Lemma 2 follows. □

We will obtain sufficient conditions for d-practical stability in terms of two measures of systems of differential equations with “maxima”. We will employ Lyapunov functions from class $\mathcal{L}$. The proof is based on Razumikhin method combined by comparison method, employed scalar ordinary differential equations.

Theorem 1. *Let the following conditions be fulfilled:*

1. The function $F \in C(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, $F(t, 0, 0) \equiv 0$.

2. The functions $h_0, h \in \mathcal{G}$, *the vector* $\varphi_0 \in \mathcal{K}^*$.

3. There exists a function $V(t, x) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathcal{K}$, $V(t, x) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathcal{K}$, $V \in \mathcal{L}$ *such that:*

(i) $b((\varphi_0 \bullet h(t, x))) \leq (\varphi_0 \bullet V(t, x)) \leq a((\varphi_0 \bullet h_0(t, x)))$, $(t, x) \in \tilde{\mathcal{S}}(h, A, \varphi_0)$, *where* $a, b \in K$ *and* $a(\lambda) < b(A)$;

(ii) for any function $\psi \in C([-r, 0], \mathbb{R}^n)$ *and any number* $t \geq 0$ *such that* $(\varphi_0 \bullet V(t, \psi(0))) \geq (\varphi_0 \bullet V(t+s, \psi(s)))$ *for* $s \in [-r, 0)$ *and* $(t, \psi(0))) \in \tilde{\mathcal{S}}(h, A, \varphi_0)$ *the inequality*

$$\Big(\varphi_0 \bullet \mathcal{D}_{(1)} V(t, \psi(0))\Big) \leq g(t, (\varphi_0 \bullet V(t, \psi(0))))$$

holds, where $g \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$, $g(t, 0) \equiv 0$, $\rho > 0$ *is a constant.*

4. For any initial function $\phi \in C([-r, 0], \mathbb{R}^n)$ *the solution of the initial value problem for systems of differential equations with “maxima” (1),(2) exists on* $[t_0 - r, \infty)$, $t_0 \geq 0$.

5. For any initial point $(t_0, u_0) \in \mathbb{R}_+ \times \mathbb{R}$ *the solution of scalar equation (6) exists on* $[t_0, \infty)$, $t_0 \geq 0$.

6. The scalar differential equation (6) is practically stable with respect to $(a(\lambda), b(A))$.

Then the system of differential equations with “maxima” (1) is d-practically stable with respect to (λ, A) *in terms of measures* h_0 *and* h *with a vector* φ_0.

Proof. From condition 5 it follows that there exists a point $t_0 \geq 0$ such that

$|u_0| < a(\lambda)$ implies

$$|u(t;t_0,u_0)| < b(A) \quad \text{for} \quad t \geq t_0, \tag{12}$$

where $u(t;t_0,u_0)$ is a solution of scalar differential equation (6) with initial condition $u(t_0) = u_0$.

Choose a function $\phi \in C([-r,0],\mathbb{R}^n)$ such that $H_0(t_0,\phi,\varphi_0) < \lambda$ and let $x(t;t_0,\phi)$ be a solution of (1) with initial condition (2).

Let $u_0 = \max_{s\in[-r,0]}(\varphi_0 \bullet V(t_0+s,\phi(s)))$. From Lemma 2 follows the validity of the inequality

$$(\varphi_0 \bullet V(t,x(t;t_0,\phi))) \leq u^*(t;t_0,u_0) \quad \text{for} \quad t \geq t_0. \tag{13}$$

From condition (i) we obtain

$$(\varphi_0 \bullet V(t_0+s,\phi(s))) \leq a((\varphi_0 \bullet h_0(t_0+s,\phi(s)))) \leq a(H_0(t_0,\phi,\varphi_0)) < a(\lambda). \tag{14}$$

From inequalities (12) and (14) follows that

$$(\varphi_0 \bullet V(t,x(t;t_0,\phi))) \leq u^*(t;t_0,u_0) < b(A) \quad \text{for} \quad t \geq t_0. \tag{15}$$

From inequality (15) and condition (i) we get for $t \geq t_0$

$$b((\varphi_0 \bullet h(t,x(t;t_0,\phi)))) \leq (\varphi_0 \bullet V(t,x(t;t_0,\phi))) \leq u^*(t;t_0,u_0) < b(A), \tag{16}$$

or

$$(\varphi_0 \bullet h(t,x(t;t_0,\phi))) < A. \tag{17}$$

□

Theorem 2. *Let the following conditions be fulfilled:*

1. The conditions 1, 2, 3, 4, and 5 of Theorem 1 are satisfied.

2. The scalar differential equation (6) is uniformly practically stable with respect to $(a(\lambda),b(A))$.

Then the system of differential equations with "maxima" (1) is uniformly d-practically stable with respect to (λ,A) in terms of measures h_0 and h with the vector φ_0.

Proof. From condition 2 it follows that for every point $t_0 \geq 0$ and $|u_0| < a(\lambda)$ the inequality

$$|u(t;t_0,u_0)| < b(A) \quad \text{for} \quad t \geq t_0 \tag{18}$$

holds, where $u(t;t_0,u_0)$ is a solution of scalar differential equation (6) with initial condition $u(t_0) = u_0$.

We will prove that for every point $t_0 \geq 0$ and every function $\phi \in C[[-r,0],\mathbb{R}^n]$

such that $H_0(t_0, \phi, \varphi_0) < \lambda$ and the inequality

$$(\varphi_0 \bullet h(t, x(t; t_0, \phi))) < A \quad \text{for} \quad t \geq t_0 \tag{19}$$

holds, where $x(t; t_0, \phi)$ is a solution of (1) with initial condition (2).

Assume the claim is not true. From condition (i) follows that for every function $\phi \in C([-r, 0], \mathbb{R}^n)$ such that $H_0(t_0, \phi, \varphi_0) < \lambda$ the inequalities

$$b((\varphi_0 \bullet h(t_0 + s, \phi(t_0 + s))) \leq a((\varphi_0 \bullet h_0(t_0 + s, \phi(t_0 + s))) < a(\lambda) < b(A),$$
$$s \in [-r, 0]$$

hold, or $(\varphi_0 \bullet h(t, \phi(t))) < A$ on the initial interval $[t_0 - r, t_0]$. According to the assumption there exists a point $t_0 \geq 0$ and a corresponding solution $x(t; t_0, \phi)$ of (1), (2) with $H_0(t_0, \phi, \varphi_0) < \lambda$ and $t^* > t_0$ such that

$$\begin{aligned} &(\varphi_0 \bullet h(t, x(t; t_0, \phi))) < A \quad \text{for} \quad t \in [t_0 - r, t^*), \\ &(\varphi_0 \bullet h(t^*, x(t^*; t_0, \phi))) = A, \\ &(\varphi_0 \bullet h(t, x(t; t_0, \phi))) \geq A \quad \text{for} \quad t \in (t^*, t^* + \Delta], \end{aligned} \tag{20}$$

where $\Delta > 0$ is a small enough number.

Let $u_0^* = \max_{s \in [-r,0]} (\varphi_0 \bullet V(t_0 + s, \phi(s)))$. From Lemma 2 and condition (ii) it follows the validity of the inequality

$$(\varphi_0 \bullet V(t, x(t; t_0, \phi))) \leq u^*(t; t_0, u_0^*) \quad \text{for} \quad t \in [t_0, t^*], \tag{21}$$

where $u^*(t; t_0, u_0^*)$ is a solution of scalar differential equation (6) with initial condition $u(t_0) = u_0^*$.

From condition (i) we obtain for $s \in [-r, 0]$

$$(\varphi_0 \bullet V(t_0 + s, \phi(s))) \leq a((\varphi_0 \bullet h_0(t_0 + s, \phi(s)))) \leq a(H_0(t_0, \phi, \varphi_0)) < a(\lambda). \tag{22}$$

Inequality (22) proves that $|u_0^*| < a(\lambda)$ and therefore, according to inequality (18) we get

$$u^*(t; t_0, u_0) < b(A) \quad \text{for} \quad t \in [t_0, t^*]. \tag{23}$$

From inequality (23), the choice of the point t^*, and condition (i) we get

$$\begin{aligned} b(A) &= b((\varphi_0 \bullet h(t^*, x(t^*; t_0, \phi)))) \leq (\varphi_0 \bullet V(t^*, x(t^*; t_0, \phi))) \\ &\leq u^*(t^*; t_0, u_0) < b(A). \end{aligned} \tag{24}$$

The obtained contradiction proves the validity of inequality (19). □

As partial cases of the above results we obtain sufficient conditions for practical stability and for uniform practical stability of differential equations with “maxima”.

Theorem 3. *Let the following conditions be fulfilled:*

1. The function $F \in C(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, $F(t, 0, 0) \equiv 0$.

3. There exists a function $V(t,x) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathcal{K}$, $V(t,x) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathcal{K}$, $V \in \mathcal{L}$ such that

(i) $b(||x||) \leq \sum_{i=1}^n V_i(t,x) \leq a(||x||)$ for $t \in \mathbb{R}_+$, $||x|| < A$, where $a, b \in K$ and $a(\lambda) < b(A)$;

(ii) for any function $\psi \in C([-r,0],\mathbb{R}^n)$ and any number $t \geq 0$ such that $\sum_{i=1}^n V_i(t,\psi(0)) \geq \sum_{i=1}^n V_i(t+s,\psi(s))$ for $s \in [-r,0)$ and $||\psi(0))|| < A$ the inequality

$$\sum_{i=1}^n \mathcal{D}_{(1)} V(_i t, \psi(0)) \leq g(t, \sum_{i=1}^n V_i(t,\psi(0)))$$

holds, where $g \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$, $g(t,0) \equiv 0$, $\rho > 0$ is a constant.

4. For any initial function $\phi \in C([-r,0],\mathbb{R}^n)$ the solution of the initial value problem for systems of differential equations with "maxima" (1),(2) exists on $[t_0 - r, \infty)$, $t_0 \geq 0$.

5. For any initial point $(t_0, u_0) \in \mathbb{R}_+ \times \mathbb{R}$ the solution of scalar equation (6) exists on $[t_0, \infty)$, $t_0 \geq 0$.

Then if scalar differential equation (6) is practically stable with respect to $(a(\lambda), b(A))$ then the system of differential equations with "maxima" (1) is practically stable with respect to (λ, A);

If scalar differential equation (6) is uniformly practically stable with respect to $(a(\lambda), b(A))$ then the system of differential equations with "maxima" (1) is uniformly practically stable with respect to (λ, A).

Now we will illustrate the application of the obtained above sufficient conditions on an example.

Example 1. Consider the following system of differential equations with "maxima"

$$\begin{aligned} x'(t) &= -x(t)\Big(x^2(t) + y^2\Big)\sin^2 t + e^{-t} \max_{s\in[t-r,t]} x(s) \\ y'(t) &= -y(t)\Big(x^2(t) + y^2\Big)\sin^2 t + e^{-t} \max_{s\in[t-r,t]} y(s), \quad t \geq t_0, \end{aligned} \tag{25}$$

with initial conditions

$$x(t) = \phi_1(t - t_0), \quad y(t) = \phi_2(t - t_0) \quad \text{for } t \in [t_0 - r, t_0], \tag{26}$$

where $x, y \in \mathbb{R}$, $r > 0$ is enough small constant, $t_0 \geq 0$.

Let $h_0(t,x,y) = (|x|, \frac{|y|}{2})$, $h(t,x,y) = (x^2, \frac{y^2}{2})$.

Consider $V : \mathbb{R}^2 \to \mathcal{K}$, $V = (V_1, V_2)$, $V_1(x,y) = \frac{1}{2}(x+y)^2$, $V_2(x,y) = \frac{1}{4}(x-y)^2$, where $\mathcal{K} = \{(x,y) : x \geq 0, y \geq 0\} \subset \mathbb{R}^2$ is a cone.

Now, let us consider the vector $\varphi_0 = (1, 2)$. Then $(\varphi_0 \bullet h(t, x, y)) = x^2 + y^2$, $(\varphi_0 \bullet V(x, y)) = \frac{1}{2}(x+y)^2 + \frac{1}{2}(x-y)^2 = x^2 + y^2$ and $(\varphi_0 \bullet h_0(t, x, y)) = |x| + |y|$.

It is easy to check the validity of condition (i) of Theorem 1 for functions $b(s) = s \in K$ and $a(s) = s^2 \in K$.

Let function $\psi \in C([-r, 0], \mathbb{R}^2)$, $\psi = (\psi_1, \psi_2)$ be such that the inequality

$$\begin{aligned}(\varphi_0 \bullet V(\psi_1(0), \psi_2(0))) &= \psi_1^2(0) + \psi_2^2(0) \\ &\geq \psi_1^2(s) + \psi_2^2(s) = (\varphi_0 \bullet V(\psi_1(s), \psi_2(s)) \quad \text{for } s \in [-r, 0) \quad (27)\end{aligned}$$

holds.

Then

$$\begin{aligned}\psi_1(0) \max_{s\in[t-r,t]} \psi_1(s) &\leq |\psi_1(0)| \, |\max_{s\in[t-r,t]} \psi_1(s)| = \sqrt{(\psi_1(0))^2} \sqrt{(\max_{s\in[t-r,t]} \psi_1(s))^2} \\ &\leq \sqrt{(\varphi_0 \bullet V(\psi_1(0), \psi_2(0)))} \sqrt{(\varphi_0 \bullet V(\psi_1(s), \psi_2(s))))} \\ &\leq (\varphi_0 \bullet V(\psi_1(0), \psi_2(0)))\end{aligned}$$

and

$$\begin{aligned}\psi_2(0) \max_{s\in[t-r,t]} \psi_2(s) &\leq |\psi_2(0)| \, |\max_{s\in[t-r,t]} \psi_2(s)| = \sqrt{(\psi_2(0))^2} \sqrt{(\max_{s\in[t-r,t]} \psi_2(s))^2} \\ &\leq \sqrt{(\varphi_0 \bullet V(\psi_1(0), \psi_2(0))} \sqrt{(\varphi_0 \bullet V(\psi_1(s), \psi_2(s)))} \\ &\leq (\varphi_0 \bullet V(\psi_1(0), \psi_2(0))).\end{aligned}$$

Therefore if inequality (27) is fulfilled, then

$$\begin{aligned}\Big(\varphi_0 \bullet \mathcal{D}_{(25)} V(\psi_1(0), \psi_2(0))\Big) &= -\Big((\psi_1(0))^2 + (\psi_2(0))^2\Big)^2 \sin^2 t \\ &\quad + e^{-t}\Big(\psi_1(0) \max_{s\in[t-r,t]} \psi_1(s) + \psi_2(0) \max_{s\in[t-r,t]} \psi_2(s)\Big) \\ &\leq 2e^{-t}(\varphi_0 \bullet V(\psi_1(0), \psi_2(0))). \quad (28)\end{aligned}$$

Consider the scalar comparison equation $u' = 2e^{-t}u$ with initial condition $u(t_0) = u_0$, which solution is $u(t) = u_0 \; e^{e^{-t_0} - e^{-t}} \leq u_0 e^{e^{-t_0}}$ for $t \geq t_0$. Let $\lambda = 1, A = e$. Then $a(\lambda) = 1, b(A) = e$ and for $t_0 > \ln(\ln(\frac{A}{\lambda})) = 0$ if $|u_0| < 1$ then $|u(t)| \leq |u_0| e^{e^{-t_0}} < e = A$, i.e. the solution is uniformly practically stable, and therefore, according to Theorem 1 the system of differential equations with "maxima" (25) is uniformly d-practically stable in terms of both measures h_0, h with the vector $\varphi_0 = (1, 2)$ i.e. inequality $\max_{t\in[-r,0]} \Big(|\phi_1(t)| + |\phi_2(t)|\Big) < 1$ implies $x^2(t) + y^2(y) < e$ for $t \geq 0$.

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