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CRITERIA FOR OSCILLATION OF A CLASS OF SECOND ORDER NON-LINEAR FORCED DIFFERENCE EQUATIONS

S. Lourdu Marian[1 §], A. George Maria Selvam[2]
[1]Department of Master of Computer Applications
Saveetha Engineering College
Thandalam, Chennai, 602 105, INDIA
e-mail: lm_jerrome@rediffmail.com
[2]Department of Mathematics
Sacred Heart College
Tirupattur, Vellore Dist., Tamil Nadu, 635 601, INDIA
e-mail: agm_shc@rediffmail.com

Abstract: In this paper, we investigate the oscillatory behavior of solutions of the second order non-linear forced difference equation

$$\Delta[a(n)\phi(\Delta x(n))] + q(n+1)f(x(n+1)) = r(n).$$

We also obtain sufficient conditions for any non-oscillatory solution $x(n)$ of the above equation satisfying $\lim_{n\to\infty} \inf |x(n)| = 0$.

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1. Introduction

In this paper, we consider the oscillation of the solution of the following non-linear forced difference equations of the form

$$\Delta[a(n)\phi(\Delta x(n))] + q(n+1)f(x(n+1)) = r(n). \tag{1}$$

We assume that:

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§Correspondence author

(H_1) $a(n)$ is a sequence of positive real numbers and $q(n), r(n)$ are sequences of real numbers where $q(n)$ is not identically zero for $n \geq n_0$;

(H_2) $\phi \in C'(\mathbb{R}, \mathbb{R})$ is an odd and strictly increasing function and ϕ is sub multiplicative ie: $\phi(x, y) \leq \phi(x)\phi(y)$ for $x, y > 0$;

(H_3) $f : \mathbb{R} \to \mathbb{R}, xf(x) > 0, f'(x) \geq 0$ for $x \neq 0$ and $f(u) - f(v) = g(u, v)(u - v)$ for all $u, v \neq 0$ where g is a non-negative function.

By a solution of (1), we mean a sequence $\{x(n)\}$ which is defined for $n \geq n_0 \geq 0$ and satisfies (1) for $n \geq n_0 \geq 0$. A solution $\{x(n)\}$ of equation (1) is said to be oscillatory if the terms $\{x(n)\}$ of the solution are neither eventually positive nor eventually negative. Otherwise, the solution is said to be non-oscillatory. Equation (1) is said to be oscillatory if all its solutions are oscillatory.

The theory of difference equations and their applications have been of great interest to researchers around the world, see for example the monographs [1] and [4]. The oscillatory behavior of some difference equations have been investigated in [8]-[11]. It is interesting to study second order nonlinear difference equations because they have physical applications.

In [5], Saker studied the oscillatory behavior of forced Emden-Fowler super linear difference equations of the form

$$\Delta^2 x(n-1) + q(n)x^{\gamma}(n) = g(n).$$

In [6], Saker obtained sufficient conditions for the oscillations of second order forced nonlinear dynamic equations

$$(a(t)x^{\Delta}(t))^{\Delta} + p(t)f(x^{\sigma}) = r(t).$$

In [8], E. Thandapani discussed oscillation and non-oscillation of quasilinear difference equations of the form

$$\Delta\left[a(n-1)\left|\Delta y(n-1)\right|^{\alpha-1}\Delta y(n-1)\right] + q(n)f(y(n)) = 0, n = 1, 2, \ldots.$$

In [7] E. Thandapani et al obtained conditions under which the following second order difference equation without the forcing term

$$\Delta^2 y(n) + a(n+1)f(y(n+1)) = 0$$

is oscillatory.

Throughout, we shall use some of the following notations; $\mathbb{N} = \{0, 1, 2, \ldots\}$, the set of all natural numbers including zero, $\mathbb{N}_{n_0} = \{n_0, n_0 + 1 \ldots\}$ where $n_0 \in \mathbb{N}$ and $\mathbb{N}_{n_0}^{\alpha} = \{n_0, n_0 + 1, \ldots, \alpha\}$.

2. Basic Lemmas

In this section, we provide some lemmas which play crucial role in proving the main results in Section 3.

Lemma 1. (Theorem 2.2.6, [3]) *Let the function* $K(n,s,y):\mathbb{N}_{n_0}\times\mathbb{N}_{n_0}\times\mathbb{R}\to\mathbb{R}$ *be such that for each fixed* n,s *the function* $K(n,s,y)$ *is non-decreasing. Furthermore, let* $\{p(n)\}$ *be a given sequence and* $\{u(n)\}$, $\{v(n)\}$ *be sequences satisfying, for* $n\in\mathbb{N}_{n_0}$

$$u(n)\geq(\leq)p(n)+\sum_{s=n_0}^{n-1}K(n,s,u(s))$$

and

$$v(n)\geq(\leq)p(n)+\sum_{s=n_0}^{n-1}K(n,s,v(s)).$$

Then $u(n)\geq(\leq)v(n)$ *for all* $n\in\mathbb{N}_{n_0}$, *where* $\mathbb{N}_{n_0}=\{n_0,n_0+1,\ldots\}$.

The following result is extracted from [2].

Lemma 2. (Lemma 2.1, [2]) *If* ϕ *is sub multiplicative on* $[0,\infty)$, *then* ϕ *satisfies* $\phi\left(\frac{y}{x}\right)\geq\frac{\phi(y)}{\phi(x)}$ *for all* $x,y>0$ *and the inverse function* ψ *of* ϕ *is super multiplicative on* $[0,\infty)$, *that is,* $\psi(x,y)\geq\psi(x)\psi(y)$ *for all* $x,y\geq 0$. *Moreover,* ψ *satisfies* $\psi\left(\frac{y}{x}\right)\leq\frac{\psi(y)}{\psi(x)}$ *for all* $x,y>0$.

3. Main Results

In this section we establish conditions under which the equation (1) oscillates.

Lemma 3. *Suppose that* $x(n)$ *is a positive (negative) solution of (1) for* $n\in\mathbb{N}_{n_0}^{\alpha}$ *and there exists* $n_1\in\mathbb{N}_{n_0}^{\alpha}$ *and* $m>0$ *such that*

$$\begin{aligned}-\frac{a(n_0)\phi(\Delta x(n_0))}{f(x(n_0))}&+\sum_{i=n_0}^{n-1}\left(q(i+1)-\frac{r(i)}{f(x(i+1))}\right)\\&+\sum_{i=n_0}^{n_1-1}\frac{a(i)\phi(\Delta x(i))g(x(i+1),x(i))\Delta x(i)}{f(x(i))f(x(i+1))}\geq m\end{aligned}\tag{2}$$

for all $n \in \mathbb{N}^{\alpha}_{n_1}$*. Then*

$$a(n)\phi(\Delta x(n)) \leq (\geq) - mf(x(n_1)), \quad n \in \mathbb{N}^{\alpha}_{n_1}. \tag{3}$$

Proof. Suppose the contrary that (1) has a non-oscillatory solution $x(n)$. Without loss of generality we assume that $x(n)$ is a positive solution of (1) for $n \in \mathbb{N}(n_0)$.

Now from equation (1), we obtain

$$\Delta\left[\frac{a(n)\phi(\Delta x(n))}{f(x(n))}\right] = \frac{\Delta[a(n)\phi(\Delta x(n))]}{f(x(n+1))} - \frac{[a(n)\phi(\Delta x(n))]g(x(n+1),x(n))\Delta x(n)}{f(x(n))f(x(n+1))}$$
$$= \frac{r(n) - q(n+1)f(x(n+1))}{f(x(n+1))} - \frac{[a(n)\phi(\Delta x(n))]g(x(n+1),x(n))\Delta x(n)}{f(x(n))f(x(n+1))},$$

$$\Delta\left[\frac{a(n)\phi(\Delta x(n))}{f(x(n))}\right] = \frac{r(n)}{f(x(n+1))} - q(n+1) - \frac{[a(n)\phi(\Delta x(n))]g(x(n+1),x(n))\Delta x(n)}{f(x(n))f(x(n+1))}. \tag{4}$$

Summing (4) from n_0 to $n-1$, we obtain,

$$\frac{a(n)\phi(\Delta x(n))}{f(x(n))} = \frac{a(n_0)\phi(\Delta x(n_0))}{f(x(n_0))} + \sum_{i=n_0}^{n-1}\left[\frac{r(i)}{f(x(i+1))} - q(i+1)\right] - \sum_{i=n_0}^{n-1}\frac{[a(i)\phi(\Delta x(i))]\Delta x(i)g(x(i+1),x(i))}{f(x(i))f(x(i+1))}$$

$$-\frac{a(n)\phi(\Delta x(n))}{f(x(n))} = -\frac{a(n_0)\phi(\Delta x(n_0))}{f(x(n_0))} + \sum_{i=n_0}^{n-1}\left[q(i+1) - \frac{r(i)}{f(x(i+1))}\right] + \sum_{i=n_0}^{n-1}\frac{[a(i)\phi(\Delta x(i))]\Delta x(i)g(x(i+1),x(i))}{f(x(i))f(x(i+1))}.$$

By using (2), we obtain,

$$-\frac{a(n)\phi(\Delta x(n))}{f(x(n))} \geq m + \sum_{i=n_1}^{n-1}\frac{[a(i)\phi(\Delta x(i))]\Delta x(i)g(x(i+1),x(i))}{f(x(i))f(x(i+1))} \quad \text{for } n \in \mathbb{N}^{\alpha}_{n_1}. \tag{5}$$

We shall now consider the following two cases.

Case (i). Suppose that $x(n)$ is positive for $n \in \mathbb{N}^{\alpha}_{n_0}$. Then it follows from

(5) that

$$-a(n)\phi(\Delta x(n)) > 0, \quad n \in \mathbb{N}_{n_1}^{\alpha},$$

or equivalently $\Delta x(n) < 0, n \in \mathbb{N}_{n_1}^{\alpha}$. Let $u(n) = -a(n)\phi(\Delta x(n))$. Then (5) becomes

$$u(n) \geq mf(x(n)) + \sum_{i=n_1}^{n-1} \frac{-u(i)\Delta x(i)g(x(i+1), x(i))}{f(x(i))f(x(i+1))} f(x(n)) \text{ for } n \in \mathbb{N}_{n_1}^{\alpha}. \quad (6)$$

Define $K(n,s,y) = \frac{f(x(n))g(x(s+1),x(s))[-\Delta x(s)]}{f(x(s))f(x(s+1))} y,\ n, s \in \mathbb{N}_{n_1}^{\alpha}, y \in \mathbb{R}_0$. Notice that, for each fixed n and s, the function $K(n,s,y)$ is nondecreasing in y. Letting $p(n) = mf(x(n))$ and applying Lemma 1 to inequality (6), we obtain,

$$u(n) \geq v(n),\ n \in \mathbb{N}_{n_1}^{\alpha}, \quad (7)$$

where $v(n)$ is a solution of the equation

$$v(n) = p(n) + \sum_{s=n_1}^{n-1} K(n,s,v(s)),\ n \in \mathbb{N}_{n_1}^{\alpha}. \quad (8)$$

Multiplying the equality (8) by $\frac{1}{f(x(n))}$ and then applying the operator Δ, we obtain

$$\Delta\left[\frac{v(n)}{f(x(n))}\right] = \Delta\left[m + \sum_{i=n_1}^{n-1} \frac{[-\Delta x(i)]g(x(i+1), x(i))}{f(x(i))f(x(i+1))} v(i)\right],$$

$$\Delta\left[\frac{v(n)}{f(x(n))}\right] = \frac{[-\Delta x(n)]g(x(n+1), x(n))}{f(x(n))f(x(n+1))} v(n). \quad (9)$$

On the other hand

$$\Delta\left[\frac{v(n)}{f(x(n))}\right] = \frac{\Delta v(n)}{f(x(n+1))} - \frac{[-\Delta x(n)]g(x(n+1), x(n))}{f(x(n))f(x(n+1))} v(n). \quad (10)$$

Then (9) and (10) imply that $\frac{\Delta v(n)}{f(x(n))} = 0$ so that $v(n) = v(n_1) = mf(x(n_1))$ for all $n \in \mathbb{N}_{n_1}^{\alpha}$. This and (7) imply (3) holds.

Case (ii). Suppose that $x(n)$ is negative for $n \in \mathbb{N}_{n_1}^{\alpha}$. Then (5) gives $a(n)\phi(\Delta x(n)) > 0$, $n \in \mathbb{N}_{n_1}^{\alpha}$, or equivalently $\Delta x(n) > 0, n \in \mathbb{N}_{n_1}^{\alpha}$. Let $u(n) = a(n)\phi(\Delta x(n))$. It follows from (5) that

$$u(n) \geq -mf(x(n)) + \sum_{s=n_1}^{n-1} \frac{(-f(x(n))\Delta x(s)g(x(s+1), x(s))u(s)}{f(x(s))f(x(s+1))}, \quad n \in \mathbb{N}_{n_1}^{\alpha}.$$

Defining $K(n,s,y)$ as in Case (i), we note that for fixed $n, s \in \mathbb{N}_{n_1}^{\alpha}$, $K(n,s,y)$ is non-decreasing with respect to y. Applying Lemma 1 with $p(n) = -mf(x(n))$, we get (7), where $v(n)$ is a solution of

$$v(n) = -mf(x(n)) + \sum_{s=n_1}^{n-1} \frac{[-f(x(n))]\Delta x(s) g(x(s+1), x(s))}{f(x(s)) f(x(s+1))} v(s), \quad n \in \mathbb{N}_{n_1}^{\alpha}.$$

As in the proof of Case (i), we see that $\Delta v(n) = 0$ and hence $v(n) = v(n_1) = -mf(x(n_1)), n \in \mathbb{N}_{n_1}^{\alpha}$. Then (7) immediately reduces to (3).

This completes the proof. □

We assume that the following conditions hold in the following theorems:

(C_1) $\lim\limits_{|x|\to\infty} |f(x)| = \infty$.

(C_2) $\sum\limits_{i=n_0}^{\infty} |r(i)| < \infty$.

(C_3) $-\infty < \sum\limits_{i=n_0}^{\infty} q(i+1) < \infty$.

(C_4) $\sum\limits_{i=n_0}^{\infty} \psi\left(\frac{1}{a(i)}\right) = \infty$, where ψ is the inverse function of ϕ.

Theorem 4. *Let conditions (C_1)-(C_4) hold and let $x(n)$ be a non-oscillatory solution of (1) such that $\lim\limits_{n\to\infty} \inf |x(n)| > 0$. Then:*

$$\sum_{i=n_0}^{\infty} \frac{a(i)\phi(\Delta x(i)) g(x(i+1), x(i)) \Delta x(i)}{f(x(i)) f(x(i+1))} < \infty \tag{11}$$

and

$$\begin{aligned} \frac{a(n)\phi(\Delta x(n))}{f(x(n))} &= \sum_{i=n}^{\infty} \left(q(i+1) - \frac{r(i)}{f(x(i+1))} \right) \\ &\quad + \sum_{i=n}^{\infty} \frac{a(i)\phi(\Delta x(i)) g(x(i+1), x(i)) \Delta x(i)}{f(x(i)) f(x(i+1))} \end{aligned} \tag{12}$$

for sufficiently large n.

Proof. Since $\lim\limits_{n\to\infty} \inf |x(n)| > 0$, there exists an integer $n_1 \in \mathbb{N}_{n_0}$ and two constants m_1 and m_2 such that $|x(n)| \geq m_1$ and $|f(x(n))| \geq m_2$ for $n \in \mathbb{N}_{n_1}$. Hence from (C_2), it follows that

$$\left| \sum_{s=n_1}^{n-1} \frac{r(s)}{f(x(s))} \right| \leq \sum_{s=n_1}^{n-1} \left| \frac{r(s)}{f(x(s))} \right| \leq \frac{1}{m_2} \sum_{s=n_1}^{n-1} |r(s)| \leq m_3 \text{ for } n \in \mathbb{N}_{n_1}, \tag{13}$$

where m_3 is a positive constant.

Suppose that (11) does not hold. Then (2) holds for $n \in \mathbb{N}_{n_1}$, if n_1 is sufficiently large. Suppose $x(n)$ is positive for all $n \in \mathbb{N}_{n_1}$. Applying Lemma 2, $a(n)\phi(\Delta x(n)) \le -mf(x(n_1))$ and $\Delta x(n) < 0, n \in \mathbb{N}_{n_1}$ for some $m > 0$. Then it follows from Lemma 2 that

$$\Delta x(n) \le \psi\left(\frac{-mf(x(n_1))}{a(n)}\right) \le -\psi[mf(x(n_1))]\psi\left(\frac{1}{a(n)}\right). \tag{14}$$

Summing the inequality (14) from n_1 to $n-1$, we obtain

$$x(n) \le x(n_1) - \psi[mf(x(n_1))]\sum_{i=n_1}^{n-1}\psi\left(\frac{1}{a(i)}\right).$$

But by condition (C_4), we obtain a contradiction to the fact that $x(n) > 0$ for all $n \in \mathbb{N}_{n_1}$.

The case when $x(n)$ is negative for $n \ge n_1$ follows by a similar argument.

This completes the proof of inequality (11).

Summing (4) from n_0 to $n-1$, we obtain

$$\begin{aligned}\frac{a(n)\phi(\Delta x(n))}{f(x(n))} &= \frac{a(n_0)\phi(\Delta x(n_0))}{f(x(n_0))} - \sum_{i=n_0}^{\infty}\left(q(i+1) - \frac{r(i)}{f(x(i+1))}\right)\\ &- \sum_{i=n_0}^{\infty}\frac{a(i)\phi(\Delta x(i))\Delta x(i)g(x(i+1),x(i))}{f(x(i))f(x(i+1))} + \sum_{i=n}^{\infty}\left(q(i+1) - \frac{r(i)}{f(x(i+1))}\right)\\ &+ \sum_{i=n}^{\infty}\frac{a(i)\phi(\Delta x(i))\Delta x(i)g(x(i+1),x(i))}{f(x(i))f(x(i+1))},\end{aligned}$$

$$\begin{aligned}\frac{a(n)\phi(\Delta x(n))}{f(x(n))} =& \beta + \sum_{i=n}^{\infty}\left(q(i+1) - \frac{r(i)}{f(x(i+1))}\right)\\ &+ \sum_{i=n}^{\infty}\frac{a(i)\phi(\Delta x(i))\Delta x(i)g(x(i+1),x(i))}{f(x(i))f(x(i+1))},\end{aligned} \tag{15}$$

where

$$\begin{aligned}\beta = &\ \frac{a(n_0)\phi(\Delta x(n_0))}{f(x(n_0)} - \sum_{i=n_0}^{\infty}\left(q(i+1) - \frac{r(i)}{f(x(i+1))}\right)\\ &- \sum_{i=n_0}^{\infty}\frac{a(i)\phi(\Delta x(i))\Delta x(i)g(x(i+1),x(i))}{f(x(i))f(x(i+1))}.\end{aligned}$$

In order to prove (12), we need to show that $\beta = 0$. If $\beta < 0$, in view of (C_3), (11) and (14), we can choose n_1 so large that

$$\left|\sum_{i=n}^{\infty} q(i+1)\right| \le -\frac{\beta}{6}, \quad n \in \mathbb{N}_{n_1}, \tag{16}$$

$$\left|\sum_{i=n}^{\infty}\frac{r(i)}{f(x(i+1))}\right| \leq -\frac{\beta}{6}, \quad n \in \mathbb{N}_{n_1}, \tag{17}$$

and

$$\left|\sum_{i=n}^{\infty}\frac{a(i)\phi(\Delta x(i))\Delta x(i)g(x(i+1),x(i))}{f(x(i))f(x(i+1))}\right| \leq -\frac{\beta}{6}. \tag{18}$$

If we take $n = n_0$ in (15), then we obtain

$$\begin{aligned}\frac{a(n_0)\phi(\Delta x(n_0))}{f(x(n_0))} =& \beta + \sum_{i=n_0}^{\infty}\left(q(i+1) - \frac{r(i)}{f(x(i+1))}\right)\\ &+ \sum_{i=n_0}^{\infty}\frac{a(i)\phi(\Delta x(i))\Delta x(i)g(x(i+1),x(i))}{f(x(i))f(x(i+1))}.\end{aligned} \tag{19}$$

Using (16)-(19) and the fact that $g(u,v) \geq 0$ for $u, v > 0$, we obtain

$$\begin{aligned}&-\frac{a(n_0)\phi(\Delta x(n_0))}{f(x(n_0))}\\ &+\sum_{i=n_0}^{n-1}\left(q(i+1) - \frac{r(i)}{f(x(i+1))}\right) + \sum_{i=n_0}^{n-1}\frac{a(i)\phi(\Delta x(i))g(x(i+1),x(i))}{f(x(i))f(x(i+1))}\\ &= -\beta - \sum_{i=n_0}^{\infty}\left(q(i+1) - \frac{r(i)}{f(x(i+1))}\right) - \sum_{i=n_0}^{\infty}\frac{a(i)\phi(\Delta x(i))g(x(i+1),x(i))}{f(x(i))f(x(i+1))}\\ &> -\beta + \frac{\beta}{6} + \frac{\beta}{6} + \frac{\beta}{6} = -\frac{\beta}{2} = m_0 > 0 \text{ for all } n \in \mathbb{N}_1.\end{aligned}$$

Thus condition (2) of Lemma 3 is satisfied. Then we can apply Lemma 2 to obtain a contradiction as in the proof of the inequality (11). If $\beta > 0$, then it follows from (15) that

$$\lim_{n\to\infty}\frac{a(n)\phi(\Delta x(n))}{f(x(n))} = \beta > 0$$

which implies that $x(n)\Delta x(n) > 0$ for sufficiently large n since ϕ is an odd function.

Hence for sufficiently large n_1, we have

$$\frac{a(n)\phi(\Delta x(n))}{f(x(n))} \geq \frac{\beta}{2}, \text{ for all } n \in \mathbb{N}_{n_1} \tag{20}$$

which implies that $\Delta x(n) > 0$, eventually.

Since $\lim_{n\to\infty}\frac{f(x(n))}{f(x(n+1))} = 1$, then for $0 < \varepsilon < 1$, there exists $n_1 \geq n_0$ such that $\frac{f(x(n))}{f(x(n+1))} \geq \varepsilon$ for $n \in \mathbb{N}_{n_1}$.

Hence

$$\frac{a(n)\phi(\Delta x(n))}{f(x(n+1))} = \frac{a(n)\phi(\Delta x(n))}{f(x(n))}\frac{f(x(n))}{f(x(n+1))} \geq \frac{\beta}{2}\varepsilon, \quad n \in \mathbb{N}_{n_1} \tag{21}$$

Define $h(t) = f(x(n)) + (t-n)\Delta f(x(n)), n \leq t \leq n+1$. Then $h'(t) = \Delta f(x(n)) > 0$ and $h(n) = f(x(n)), n \in \mathbb{N}_{n_1}$.

Hence from (11) and (21), we obtain

$$\infty > \sum_{i=n_1}^{\infty} \frac{a(i)\phi(\Delta x(i))g(x(i+1), x(i))\Delta x(i)}{f(x(i))f(s(i+1))}$$

$$\geq \frac{\beta}{2}\varepsilon \sum_{i=n_1}^{\infty} \frac{\Delta f(x(i))}{f(x(i))} = \frac{\beta}{2}\varepsilon \sum_{i=n_1}^{\infty} \frac{h'(t)}{f(x(i))} \geq \frac{\beta}{2}\varepsilon \sum_{i=n_1}^{\infty} \int_i^{i+1} \frac{h'(t)}{h(t)}$$

$$= \frac{\beta}{2}\varepsilon \lim_{n\to\infty} \ln\left(\frac{h(n)}{h(n_1)}\right).$$

Hence $\ln h(t) < \infty$ which implies that $f(x(n)) < \infty$ as $n \to \infty$. From (C_1), $\{x(n)\}$ is bounded.

On the other hand, it follows from (20), the monotonicity of f and $x(n)\Delta x(n) > 0$ for $n \in \mathbb{N}_{n_1}$, that

$$\Delta x(n) \geq \psi\left(\frac{\beta}{2}\frac{f(x(n_1))}{a(n)}\right) \geq \psi\left(\frac{\beta}{2}f(x(n_1))\right)\psi\left(\frac{1}{a(n)}\right), \quad n \in \mathbb{N}_{n_1},$$

if $x(n) > 0$ and

$$\Delta x(n) \leq \psi\left(\frac{\beta}{2}\frac{f(x(n_1))}{a(n)}\right) \leq \psi\left(\frac{\beta}{2}f(x(n_1))\right)\psi\left(\frac{1}{a(n)}\right), \quad n \in \mathbb{N}_{n_1},$$

if $x(n) < 0$.

Hence, condition (C_4) implies that $\lim_{n\to\infty} |x(n)| = \infty$ which contradicts the boundedness of $x(n)$.

Thus the proof is complete. □

We note that if (C_2) and (C_3) hold, then

$$h_0(n) = \sum_{j=n}^{\infty} (q(j+1) - \ell\,|r(j)|), \quad n \in \mathbb{N}_{n_0},$$

is finite for any positive constant ℓ. Assume that $h_0(n) \geq 0$ for sufficiently large

n. Define for $m \in \mathbb{N}$, the series

$$h_1(n) = \sum_{j=n}^{\infty} h_0(j)\psi\left(\frac{h_0(s)}{a(s)}\right),$$

$$h_{m+1}(n) = \sum_{j=n}^{\infty} [h_0(j) + \lambda h_m(j)]\,\psi\left(\frac{h_0(j) + \lambda h_m(j)}{a(j)}\right).$$

Theorem 5. *Let conditions (C_1), (C_2), (C_3), (C_4) hold and $h_0(n) \geq 0$ for $\ell \in [0,1]$ and n large enough. Suppose that for any eventually positive (or negative) solution $x(n)$, there exists a constant $\lambda > 0$ such that*

$$\frac{g(u,v)\psi(u)}{f(u)} \geq \lambda > 0, \quad u,v \neq 0. \tag{22}$$

If there exists a positive integer N such that $h_m(n)$ is finite for $m = 1, 2, \ldots N-1$ and $h_N(n)$ is infinite, then $x(n)$ is oscillatory or $\lim_{n\to\infty} \inf |x(n)| = 0$.

Proof. Suppose, to the contrary, that $x(n)$ is a non-oscillatory solution of (1) such that $\lim_{n\to\infty} \inf |x(n)| > 0$. Hence by Theorem 3, $x(n)$ satisfies conditions (11) and (12). Furthermore there exist $n_1 \in \mathbb{N}_{n_0}$ and $m_1 > 0$, $m_2 > 0$ such that $|x(n)| \geq m_1$ and $|f(x(n))| \geq m_2$ for $n \in \mathbb{N}_{n_1}$. Summing (4) from n_0 to $n-1$ and taking limit as $n \to \infty$, we obtain

$$\begin{aligned}\frac{a(n)\phi(\Delta x(n))}{f(x(n))} \geq& h_0(n) + \sum_{j=n}^{\infty} \frac{a(j)\phi(\Delta x(j))g(x(j+1), x(j))\Delta x(j)}{f(x(j))f(x(j+1))} \\ \geq& h_0(n) \geq 0, \quad n \in \mathbb{N}_{n_1}.\end{aligned} \tag{23}$$

Hence $x(n)\Delta x(n) > 0$ for $n \in \mathbb{N}_{n_1}$.

Thus

$$\Delta x(n) \geq \psi\left(\frac{h_0(n)f(x(n))}{a(n)}\right), \quad n \in \mathbb{N}_{n_1}, \tag{24}$$

if $x(n) > 0$ and

$$\Delta x(n) \leq \psi\left(\frac{h_0(n)f(x(n))}{a(n)}\right), \quad n \in \mathbb{N}_{n_1}, \tag{25}$$

if $x(n) < 0$.

It follows from Lemma 2, (22), (23) and either (24) or (25) that

$$\sum_{j=n}^{\infty} \frac{a(j)\phi(\Delta x(j))g(x(j+1), x(j))\Delta x(j)}{f(x(j))f(x(j+1))} \geq \sum_{j=n}^{\infty} \frac{h_0(j)g(x(j+1), x(j))\Delta x(j)}{f(x(j+1))}$$

$$\geq \sum_{j=n}^{\infty} \frac{h_0(j)g(x(j+1),x(j))}{f(x(j+1))}\psi\left(\frac{h_0(j)f(x(j))}{a(j)}\right)$$

$$\geq \sum_{j=n}^{\infty} \frac{h_0(j)g(x(j+1),x(j))}{f(x(j+1))}\psi\left(f(x(j))\right)\psi\left(\frac{h_0(j)}{a(j)}\right)$$

$$\geq \lambda \sum_{j=n}^{\infty} h_0(j)\psi\left(\frac{h_0(j)}{a(j)}\right) = \lambda h_1(n), \quad n \in \mathbb{N}_{n_1},$$

$$\sum_{j=n}^{\infty} \frac{a(j)\phi(\Delta x(j))g(x(j+1),x(j))\Delta x(j)}{f(x(j))f(x(j+1))} \geq \lambda h_1(n), \quad n \in \mathbb{N}_{n_1}. \tag{26}$$

If $N = 1$, then the right hand side of (26) is infinite which contradicts (11). Thus we obtain our conclusion. If $N > 1$, it follows from (23) and (26) that

$$\frac{a(n)\phi(\Delta x(n))}{f(x(n))} \geq h_0(n) + \lambda h_1(n) > 0, \quad n \in \mathbb{N}_{n_1}.$$

As before, we obtain

$$\Delta x(n) \geq \psi\left(\frac{f(x(n))}{a(n)}\left[h_0(n) + \lambda h_1(n)\right]\right) \geq \psi\left(f(x(n)\right)\psi\left(\frac{h_0(n) + \lambda h_1(n)}{a(n)}\right), \quad n \in \mathbb{N}_{n_1}$$

if $x(n) > 0$ and

$$\Delta x(n) \leq \psi\left(\frac{f(x(n))}{a(n)}\left[h_0(n) + \lambda h_1(n)\right]\right) \geq \psi\left(f(x(n)\right)\psi\left(\frac{h_0(n) + \lambda h_1(n)}{a(n)}\right), \quad n \in \mathbb{N}_{n_1}$$

if $x(n) < 0$.

Hence

$$\sum_{j=n}^{\infty} \frac{a(j)\phi(\Delta x(j))g(x(j+1),x(j))\Delta x(j)}{f(x(j))f(x(j+1))} \geq \lambda \sum_{j=n}^{\infty} \left[h_0(j) + \lambda h_1(j)\right]\psi\left(\frac{h_0(j) + \lambda h_1(j)}{a(j)}\right) = \lambda h_2(n).$$

If $N = 2$, then once again we get a contradiction. A similar argument yields a contradiction for any integer $N > 2$. This completes the proof of the theorem. □

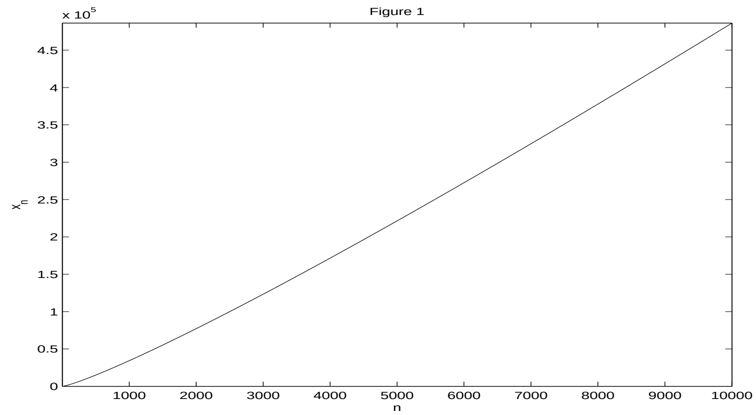

Figure 1: Graph of the solution of equation (27)

4. Examples

In this section, we provide examples to illustrate the results established in the above section. The examples are accompanied by graphs representing the solutions of the respective difference equations. Graphs are generated by *MATLAB*.

Example 1. We consider the following difference equation

$$\Delta\left[\frac{1}{n}\Delta x(n)\right] + \frac{1}{(n+1)^3}x(n+1) = -\frac{1}{n(n+1)^2}, \quad n \geq 1, \tag{27}$$

in which $a(n) = \frac{1}{n}$, $q(n+1) = \frac{1}{(n+1)^3}$, $r(n) = -\frac{1}{n(n+1)^2}$, $\phi(u) = u$ and $f(u) = u$. We find that

$$\sum_{j=1}^{\infty} |r(j)| = \sum_{j=1}^{\infty} \frac{1}{j(j+1)^2} = 0.355066 < \infty,$$

$$\sum_{j=0}^{\infty} q(j+1) = \sum_{j=1}^{\infty} \frac{1}{(j+1)^3} = 0.202057 < \infty,$$

$$\sum_{j=1}^{\infty} \frac{1}{a(j)} = \sum_{j=1}^{\infty} j = \infty.$$

Hence the conditions (C_1)-(C_4) hold for equation (27). Therefore every non-oscillatory solution $\{x(n)\}$ of (27) such that $\lim_{n\to\infty} \inf |x(n)| > 0$ fulfills (10) and (11). In fact one such solution is $\{x(n)\} = \{n\}$. Also numerical simulation of the solutions of equation (27) using *MATLAB* is provided.

Example 2. In the following non-linear difference equation

$$\Delta^2 x(n) + \frac{x(n+1)[1+x^2(n+1)]}{n^2+1} = \frac{1}{n^2+1}, \quad n \geq 0, \tag{28}$$

we have $a(n) = 1$, $\phi(x) = x$, $f(x) = x(1+x^2)$, $q(n+1) = r(n) = \frac{1}{n^2+1}$ and $\psi(x) = x$. We see that

$$\lim_{|x|\to\infty} |f(x)| = \infty,$$

$$\sum_{j=n_0}^{\infty} q(j+1) = \sum_{j=n_0}^{\infty} |r(j)| = \sum_{j=n_0}^{\infty} \frac{1}{j^2+1} = 2.07667 < \infty \text{ for } n_0 = 0,$$

$$\sum_{j=n_0}^{\infty} \psi\left(\frac{1}{a(j)}\right) = \infty.$$

Thus (C_1), (C_2), (C_3), (C_4) and (22) hold. On the other hand

$$h_0(n) = \sum_{j=n}^{\infty} [q(j+1) - \ell r(j)] = (1-\ell) \sum_{j=n}^{\infty} \frac{1}{j^2+1} > 0 \text{ for } \ \ell \in (0,1), \quad n \geq 1,$$

and also

$$h_1(n) = \sum_{j=n}^{\infty} h_0(j) \psi\left(\frac{h_0(j)}{a(j)}\right) = \infty$$

for $n \geq 1$.

We also provide numerical simulation of solutions for the difference equation (28) using *MATLAB*. It can be seen from Figure 2 that the solutions are

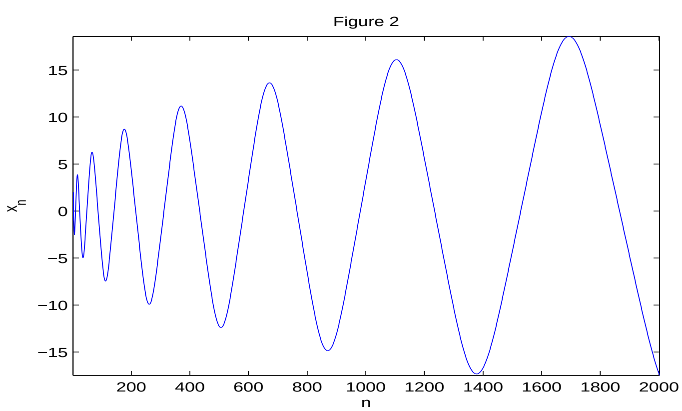

Figure 2: Graph of the solution of (28)

oscillatory. We take $\ell = 0.5 \in (0,1)$ and tabulate the values of $h_0(n)$ for $n = 0, 1, 2, \ldots, 10$ (see Table 1). We conclude that $h_0(n) > 0$ for $n \geq 1$. Hence

n	$\sum_{j=n}^{\infty} \frac{1}{j^2+1}$	$h_0(n)$
0	2.07667	1.038335
1	1.07667	0.538335
2	0.576674	0.288337
3	0.376674	0.188337
4	0.276674	0.138337
5	0.217851	0.108926
6	0.179389	0.089695
7	0.152362	0.076181
8	0.132362	0.066181
9	0.116977	0.058489
10	0.104782	0.052391

Table 1:

it follows from Theorem 4 that every solution $x(n)$ of (28) is either oscillatory or $\lim_{n\to\infty} \inf |x(n)| = 0$.

Example 3. We consider the following nonlinear difference equation

$$\Delta \left[\Delta x(n)\right]^{\frac{1}{3}} + \frac{[x(n+1)]^{\frac{1}{3}}}{n^2+1} = \frac{1}{n^2+1}, \quad n \geq 0. \tag{29}$$

In the above equation $a(n) = 1$, $\phi(x) = x^{\frac{1}{3}}$, $f(x) = x^{\frac{1}{3}}$, $q(n+1) = r(n) = \frac{1}{n^2+1}$ and $\psi(x) = x^3$. We have

$$\lim_{|x|\to\infty} |f(x)| = \infty,$$

$$\sum_{j=n_0}^{\infty} q(j+1) = \sum_{j=n_0}^{\infty} |r(j)| = 2.07667 < \infty \text{ for } n_0 = 0,$$

$$\sum_{j=n_0}^{\infty} \psi\left(\frac{1}{a(j)}\right) = \infty.$$

Thus (C_1), (C_2), (C_3), (C_4) and (22) hold. On the other hand

$$h_0(n) = \sum_{j=n}^{\infty} [q(j+1) - \ell r(j)] = (1-\ell) \sum_{j=n}^{\infty} \frac{1}{j^2+1} > 0$$

for each $\ell \in (0,1)$, $n \geq 1$. And

$$h_1(n) = \sum_{j=n}^{\infty} h_0(j)\psi\left(\frac{h_0(j)}{a(j)}\right) = \infty$$

for $n \geq 1$. Hence it follows from Theorem (4) that every solution $x(n)$ of (29) is either oscillatory or $\lim_{n\to\infty} \inf |x(n)| = 0$. The graph is the numerical simulation

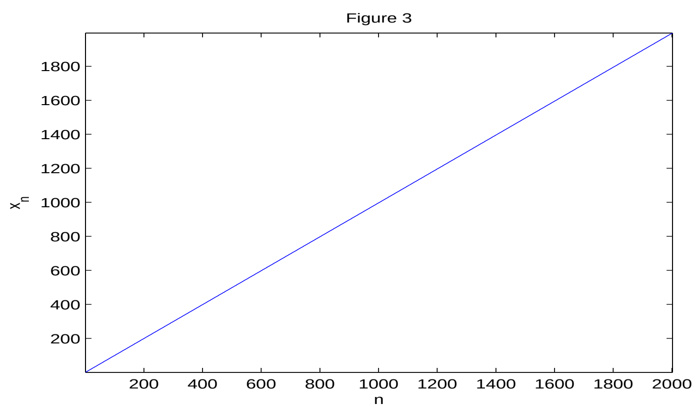

Figure 3: Graph of the solution of (29)

of the solutions of equation (29). We conclude that the difference equation (29) is not oscillatory but the solutions are such that $\lim_{n\to\infty} \inf |x(n)| = 0$.

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