International Journal of Pure and Applied Mathematics
Volume 59 No. 1 2010, 77-91

# OSCILLATION FOR A NONLINEAR SECOND-ORDER DIFFERENCE EQUATION IN ARCHIMEDEAN SPACE

Lili Zhao[1 §], Yuchun Hua[2]
[1,2]Department of Mathematics
Yunnan University
Kunming, Yunnan, 650091, P.R. CHINA
[1]e-mail: llzhao@ynu.edu.cn

**Abstract:** In Archimedean space $(X, \prec)$ we study the oscillation of solutions of the second-order difference equation of the form

$$\Delta\big[p(k)\Psi(y(k))\Delta y(k)\big] + q(k)h(y(k))g(\Delta y(k-r(k)))\Delta y(k) + f(k, y(k), y(k-r_1(k)), \ldots, y(k-r_n(k))) = 0,$$

where $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\mathbb{N}(k_0) = \{k_0, k_0+1, \ldots\}$, $k_0 \in \mathbb{N}$, $p, q : \mathbb{N}(k_0) \to \mathbb{R}$, $\Psi, h, g : X \to \mathbb{R}$, $f : \mathbb{N}(k_0) \times X^{n+1} \to X$, $r, r_i : \mathbb{N}(k_0) \to \mathbb{N}$, $i = 1, 2, \ldots, n$. Our results are new and complement of previously known results. Also, three examples are given to show the effectiveness of the proposed method and results.

**AMS Subject Classification:** 26A33
**Key Words:** order relation, second-order difference equation, oscillation, Archimedean space

## 1. Introduction and Preliminaries

In the past decades, much research has been done on oscillatory and asymptotic behavior of second-order linear and nonlinear difference equations. For example, one can see [7], [5], [6], [3], [8], [1] and the references cited therein. However, most papers study oscillatory solutions of second-order linear and nonlinear difference equations in $\mathbb{R}$, and few papers [4] have been published on oscillation of solutions of difference equations in Archimedean space.

Received: January 11, 2010 

§Correspondence author

Recently, Yaşar Bolat [1] has studied the oscillation of solutions of the nonlinear second-order difference equation of the form

$$\Delta\big[p(k)\Psi(y(k))\Delta y(k)\big] + q(k)h(y(k))g(\Delta y(k-r(k)))\Delta y(k) + f(k, y(k), y(k-r_1(k)), \dots, y(k-r_n(k))) = 0, \tag{1.1}$$

where $\Delta$ is the difference operator defined by $\Delta y(k) = y(k+1) - y(k-1)$, $\Delta^2 y(k) = \Delta(\Delta y(k))$, $\mathbb{N} = \{0, 1, 2, \dots\}$, $\mathbb{N}(k_0) = \{k_0, k_0+1, \dots\}$, $k_0 \in \mathbb{N}$, $p, q : \mathbb{N}(k_0) \to \mathbb{R}$, $\Psi, h, g : \mathbb{R} \to \mathbb{R}$, $f : \mathbb{N}(k_0) \times \mathbb{R}^{n+1} \to \mathbb{R}$, $r, r_i : \mathbb{N}(k_0) \to \mathbb{N}$, $i = 1, 2, \dots, n$.

In this paper, we shall investigate the oscillation of solutions of the second-order nonlinear difference equations in Archimedean space of the form

$$\Delta\big[p(k)\Psi(y(k))\Delta y(k)\big] + q(k)h(y(k))g(\Delta y(k-r(k)))\Delta y(k) + f(k, y(k), y(k-r_1(k)), \dots, y(k-r_n(k))) = 0, \tag{1.2}$$

where $\Delta$ is the difference operator defined by $\Delta y(k) = y(k+1) - y(k-1)$, $\Delta^2 y(k) = \Delta(\Delta y(k))$, $\mathbb{N} = \{0, 1, 2, \dots\}$, $\mathbb{N}(k_0) = \{k_0, k_0+1, \dots\}$, $k_0 \in \mathbb{N}$, $p, q : \mathbb{N}(k_0) \to \mathbb{R}$, $\Psi, h, g : X \to \mathbb{R}$, $f : \mathbb{N}(k_0) \times X^{n+1} \to X$, $r, r_i : \mathbb{N}(k_0) \to \mathbb{N}$, $i = 1, 2, \dots, n$, $X$ is an Archimedean space. Since $(\mathbb{R}, <)$ is also an Archimedean space, equation (1.1) is a special case of equation (1.2).

**Definition 1.1.** A real vector space $X$ is said to be an ordered vector space if an order relation has been endowed such that the following conditions are satisfied:

(i) if $x_1, x_2 \in X$ and $x_1 \prec x_2$, then $x_1 + x \prec x_2 + x$ for all $x \in X$;

(ii) if $x_1, x_2 \in X$ and $x_1 \prec x_2$, then $ax_1 \prec ax_2$ for all $a > 0$.

Let $x, y, z_j, u_j \in X$, $j = 1, 2, \dots, n$, and $s \in \mathbb{R}$. Then the following lemmas can be proved easily. So we shall omit their proofs.

**Lemma 1.1.** *If $x \prec y$, $s$ is a positive real number, then $sx \prec sy$.*

**Lemma 1.2.** *If $x \prec y$, $s$ is an negative real number, then $sy \prec sx$.*

**Lemma 1.3.** *If $z_j \prec u_j, j = 1, 2, \dots, n$, then $\sum_{j=1}^n z_j \prec \sum_{j=1}^n u_j$.*

**Definition 1.2.** $(X, \prec)$ is an ordered vector space, $x \in (X, \prec)$ is said to be positive if $0 \prec x$, and negative if $x \prec 0$, where 0 is the zero element of $X$.

**Definition 1.3.** (see [2]) An ordered vector space $X$ is said to be an Archimedean space if for any element $x$, which is not negative, the set $\{ax : a > 0\}$ is not bounded from above.

In the Archimedean space the following lemma can be proved rather easily. So we shall omit its proof.

**Lemma 1.4.** (see [2]) *In Archimedean space* $(X, \prec)$, *for any element* $x$, *which is not positive (negative), and any sequence* $\{e(k)\}$ *of positive real numbers such that* $\lim_{k\longrightarrow\infty} e(k) = \infty$, *then the set* $\{e(k)x : k \in \mathbb{N}\}$ *is not bounded from below (above).*

Throughout this paper, we always assume that:

$(H_1)$ $p(k+1) > p(k) > 0$, $q(k) > 0$, $k \in \mathbb{N}(k_0)$;

$(H_2)$ there exist $\theta_1 > 0$ such that if $x_1, x_2 \in X, \theta_1 < \Psi(x_1) < \Psi(x_2) < 1$ , when $x_1 \prec x_2$;

$(H_3)$ there exist $\theta_2 > 0$ such that if $x \in X, 0 < h(x) < \theta_2$, and if $k \in \mathbb{N}(k_0), x \in X, q(k)h(x) < p(k)\Psi(x)$;

$(H_4)$ if $x \in X, 0 < g(x) < 1$;

$(H_5)$ $\lim_{k\to\infty}(k - r_i(k)) = \infty$, $r_i : \mathbb{N}(k_0) \to \mathbb{N}$, and exist $m_0, m_i \in \mathbb{N}$ such that $r(k) \leq m_0$, $r_i(k) \leq m_i$ for $k \in \mathbb{N}(k_0)$, $i = 1, 2, \ldots, n$;

$(H_6)$ for $k \in \mathbb{N}(k_0), f(k, u_0, u_1, \ldots, u_n) \prec 0$ if $u_i \prec 0, i = 0, 1, 2, \ldots, n$ and $0 \prec f(k, u_0, u_1, \ldots, u_n)$ if $0 \prec u_i, i = 0, 1, 2, \ldots, n$;

$(H_7)$ $\lim_{k\to\infty} \sum_{t=k_0+1}^{k-1} \frac{1}{p(t)} \prod_{j=k_0}^{t-1} \left[1 - \frac{\theta_2}{\theta_1}\frac{q(j)}{p(j)}\right] = +\infty$;

$(H_8)$ if $\overline{k} < k, v_i \prec u_i$, $i = 0, 1, 2, \ldots, n$, then
$$f(\overline{k}, v_0, v_1, v_2, \ldots, v_n) \prec f(k, u_0, u_1, u_2, \ldots, u_n).$$

Let $m = \max_{0\leq i\leq n} m_i$, and let $N_0$ be a fixed nonnegative integer. By a solution of (1.1), we mean a vector sequence $\{y(k)\}$ which is defined for all $k \geq N_0 - m$ and satisfies (1.1) for $k \geq N_0$.

**Definition 1.4.** A solution $\{y(k)\}$ of equation (1.1) is said to be eventually positive if $0 \prec y(k)$, for all large $k$, and eventually negative if $y(k) \prec 0$, for all large $k$. It is said to be oscillatory if it is neither eventually positive nor eventually negative.

In this work, we shall be concerned only with the nontrivial solutions of (1.1).

## 2. Main Results

In this section, we will give some sufficient conditions for oscillation of all solutions of (1.1).

**Theorem 2.1.** *Assume that* $(H_1)$*-*$(H_7)$ *hold. If* $\{y(k)\}$ *is a non-oscillatory solution of equation (1.1), then* $y(k) \prec 0, \Delta y(k) \prec 0$, *or* $0 \prec y(k), 0 \prec \Delta y(k)$ *for all large* $k$.

*Proof.* Let $\{y(k)\}$ be a non-oscillatory solution of equation (1.1). Without loss of generality, assumed that $\{y(k)\}$ is eventually positive (the proof is similar when $\{y(k)\}$ is eventually negative). Then we can find a $k_1 \in \mathbb{N}(k_0)$ such that $0 \prec y(k)$ for $k \geq k_1$. Since $\lim_{k\to\infty}(k - r_i(k)) = +\infty, i = 1, 2, \ldots, n$, we can find a $k_2 \geq k_1$ such that $0 \prec y(k - r_i(k))$ for $i = 1, 2, \ldots, n$, and $k \geq k_2$. We claim that $0 \prec \Delta y(k)$, eventually. Let us suppose the contrary, that is, we can find a $k_3 \geq k_2$ such that $\Delta y(k) \prec 0$ for $k \geq k_3$. Rewrite (1.1) as

$$\begin{aligned} & -\Delta\big[p(k)\Psi(y(k))\Delta y(k)\big] - q(k)h(y(k))g(\Delta y(k - r(k))\Delta y(k) \\ = \;& f(k, y(k), y(k - r_1(k)), \ldots, y(k - r_n(k))). \end{aligned}$$

Hence by condition $(H_6)$, for $k \geq k_3$, we have

$$\Delta\big[p(k)\Psi(y(k))\Delta y(k)\big] + q(k)h(y(k))g(\Delta y(k - r(k))\Delta y(k) \prec 0.$$

That is

$$\Delta\omega(k) + \frac{q(k)h(y(k))g(\Delta y(k - r(k)))}{p(k)\Psi(y(k))}\omega(k) \prec 0, \tag{2.1}$$

where $\omega(k) = p(k)\Psi(y(k))\Delta y(k)$. It follows from (2.1) for $k \geq k_3$ that

$$\omega(k-1)\left[1 - \frac{q(k-1)h(y(k-1))g(\Delta y(k-1-r(k-1)))}{p(k-1)\Psi(y(k-1))}\right] \succ \omega(k). \tag{2.2}$$

So the difference inequality (2.2) has a solution of the form

$$\omega(k_3)\prod_{t=k_3}^{k-1}\left[1 - \frac{q(t)h(y(t))g(\Delta y(t - r(t)))}{p(t)\Psi(y(t))}\right] \succ \omega(k). \tag{2.3}$$

Dividing both sides of (2.3) by $p(k)\Psi(y(k))$ we obtain

$$\begin{aligned} \Delta y(k) &\prec \frac{1}{p(k)\Psi(y(k))}\omega(k_3)\prod_{t=k_3}^{k-1}\left[1 - \frac{q(t)h(y(t))g(\Delta y(t - r(t)))}{p(t)\Psi(y(t))}\right] \\ &\prec \frac{1}{p(k)}\prod_{t=k_3}^{k-1}\left[1 - \frac{\theta_2}{\theta_1}\frac{q(t)}{p(t)}\right]\omega(k_3). \end{aligned} \tag{2.4}$$

Summing up (2.4) from $k_3+1$ to $k-1$, we can get

$$\sum_{t=k_3+1}^{k-1} \Delta y(t) \prec \sum_{t=k_3+1}^{k-1} \frac{1}{p(t)} \prod_{j=k_3}^{t-1} \left[1 - \frac{\theta_2}{\theta_1}\frac{q(j)}{p(j)}\right] \omega(k_3).$$

Noticing that $\omega(k) = p(k)\Psi(y(k))\Delta y(k)$, we have

$$\sum_{t=k_3+1}^{k-1} \Delta y(t) \prec \sum_{t=k_3+1}^{k-1} \frac{1}{p(t)} \prod_{j=k_3}^{t-1} \left[1 - \frac{\theta_2}{\theta_1}\frac{q(j)}{p(j)}\right] p(k_3)\Psi(y(k_3))\Delta y(k_3). \tag{2.5}$$

From condition $(H_7)$ and Lemma 1.4, we find that the set

$$\left\{\sum_{t=k_3+1}^{k-1} \frac{1}{p(t)} \prod_{j=k_3}^{t-1} \left[1 - \frac{\theta_2}{\theta_1}\frac{q(j)}{p(j)}\right] p(k_3)\Psi(y(k_3))\Delta y(k_3)\right\}_{k=2+k_3}^{\infty}$$

is not bounded from below. Therefore from (2.5) it follows that the set $\{\sum_{t=k_3+1}^{k-1} \Delta y(t)\}_{k=2+k_3}^{\infty}$ is not bounded from below. However, if $k \geq k_3+2$, we can obtain

$$-y(k_3+1) \prec y(k) - y(k_3+1) = \sum_{t=k_3+1}^{k-1} \Delta y(t) \prec 0.$$

That is the set $\{\sum_{t=k_3+1}^{k-1} \Delta y(t)\}_{k=2+k_3}^{\infty}$ is bounded from below by $-y(k_3+1)$. This is a contradiction. The proof is complete. $\square$

From Theorem 2.1 one can get

**Theorem 2.2.** *Assume that $(H_1)$-$(H_7)$ hold. Let $\{y(k)\}$ be a solution of equation (1.1), if $\{\Delta y(k)\}$ is oscillatory then $\{y(k)\}$ is also oscillatory.*

*Proof.* If $\{y(k)\}$ is non-oscillatory, then $\{y(k)\}$ is eventually positive or eventually negative. From Theorem 2.1, we can obtain that $\{\Delta y(k)\}$ is eventually positive or eventually negative, this is a contradiction. The proof is complete. $\square$

**Theorem 2.3.** *Assume that $(H_1)$-$(H_8)$ are satisfied. Then every solution of equation (1.1) is oscillatory.*

*Proof.* Let $\{y(k)\}$ be a non-oscillatory solution of equation (1.1). Without loss of generality, assumed that $\{y(k)\}$ is eventually positive. Then we can find a $k_1 \in \mathbb{N}(k_0)$ such that $0 \prec y(k)$ for $k \geq k_1$. It follows from $\lim_{k\to\infty}(k - r_i(k)) = \infty, i = 1, 2, \ldots, n$ that there exists a $k_2 \geq k_1$ such that $k - r_i(k) \geq k_1$ for $k \geq k_2, i = 1, 2, \ldots, n$. That is $0 \prec y(k - r_i(k))$ for $k \geq k_2, i = 1, 2, \ldots, n$. According to Theorem 2.1 there exists a $k_3 \geq k_2$ such that $0 \prec \Delta y(k)$ for $k \geq k_3$. Rewrite (1.1) as

$$\Delta\big[p(k)\Psi(y(k))\Delta y(k)\big] + f(k, y(k), y(k - r_1(k)), \ldots, y(k - r_n(k)))$$

$$= \quad -q(k)h(y(k))g(\Delta y(k-r(k))\Delta y(k).$$

From the above equation, and conditions $(H_1)$ and $(H_3)$, $(H_4)$ it follows that, for $k \geq k_3$,

$$\Delta\big[p(k)\Psi(y(k))\Delta y(k)\big] + f(k, y(k), y(k-r_1(k)), \ldots, y(k-r_n(k))) \prec 0. \quad (2.6)$$

Since $0 \prec \Delta y(k)$ for $k \geq k_3$, we know that $\{y(k)\}$ is increasing for $k \geq k_3$. It follows from $\lim_{k\to\infty}(k - r_i(k)) = \infty, i = 1, 2, \ldots, n$, that there exists a $k_4 \geq k_3$ such that $k - r_i(k) \geq k_3$ for $k \geq k_4, i = 1, 2, \ldots, n$. Now we know that

$$\{f(k, y(k), y(k-r_1(k)), \ldots, y(k-r_n(k)))\}$$

is increasing when $k \geq k_4$. Summing up (2.6) from $k_4$ to $k-1$, we obtain

$$\begin{aligned} & p(k)\Psi(y(k))\Delta y(k) - p(k_4)\Psi(y(k_4))\Delta y(k_4) \\ \prec \quad & -\sum_{t=k_4}^{k-1} f(k, y(k), y(k-r_1(k)), \ldots, y(k-r_n(k))), \\ \prec \quad & -(k-k_4)f(k_4, y(k_4), y(k_4-r_1(k_4)), \ldots, y(k_4-r_n(k_4))). \end{aligned} \quad (2.7)$$

From Lemma 1.4, we can get the set

$$\{-(k-k_4)f(k_4, y(k_4), y(k_4-r_1(k_4)), \ldots, y(k_4-r_n(k_4)))\}_{k=1+k_4}^{\infty}$$

is not bounded from below. Therefore by (2.7) we can have the set

$$\Big\{-\sum_{t=k_4}^{k-1} f(k, y(k), y(k-r_1(k)), \ldots, y(k-r_n(k)))\Big\}_{k=1+k_4}^{\infty}$$

is also not bounded from below. However if $k \geq k_4+1$, we can obtain

$$\begin{aligned} -p(k_4)\Psi(y(k_4))\Delta y(k_4) \quad \prec \quad & p(k)\Psi(y(k))\Delta y(k) - p(k_4)\Psi(y(k_4))\Delta y(k_4) \\ \prec \quad & -\sum_{t=k_4}^{k-1} f(k, y(k), y(k-r_1(k)), \ldots, y(k-r_n(k))). \end{aligned}$$

That is the set

$$\Big\{-\sum_{t=k_4}^{k-1} f(k, y(k), y(k-r_1(k)), \ldots, y(k-r_n(k)))\Big\}_{k=1+k_4}^{\infty}$$

is bounded from below by $-p(k_4)\Psi(y(k_4))\Delta y(k_4)$, this is a contradiction. The proof is complete. □

**Theorem 2.4.** *Assume that $(H_1)$-$(H_5)$ and $(H_7)$ hold. Moreover, the following condition is satisfied: if $u_0 \succ (\prec) 0$*

$$f(k, u_0, u_1, u_2, \ldots, u_n) \succeq (\preceq) a(k+1)u_0 + f_1(k, u_0, u_1, u_2, \ldots, u_n),$$

*where $f_1 : \mathbb{N}(k_0) \times X^{n+1} \to X$ and $f_1(k, u_0, u_1, u_2, \ldots, u_n) \succ 0$ if $u_i \succ 0$ for all $j = 0, 1, 2, \ldots, n$ and $f_1(k, u_0, u_1, u_2, \ldots, u_n) \prec 0$, if $u_i \prec 0$ for all $j =$*

$0, 1, 2, \ldots, n$, *further*

$$\lim_{k\to\infty} \frac{a(k)}{p(k)} = s > 0.$$

*Then all solutions of equation (1.1) are oscillatory.*

*Proof.* Let $\{y(k)\}$ is an non-oscillatory solution of equation (1.1). Without loss of generality, assumed that $\{y(k)\}$ is eventually positive. Then we can find a $k_1 \in \mathbb{N}(k_0)$, such that $0 \prec y(k)$ for $k \geq k_1$. For $\lim_{k\to\infty}(k - r_i(k)) = +\infty, i = 1, 2, \ldots, n$, then we can find a $k_2 \geq k_1$, such that $0 \prec y(k - r_i(k))$ for $i = 1, 2, \ldots, n$, and $k \geq k_2$. From Theorem 2.1 there exist a $k_3 \geq k_2$ such that $0 \prec \Delta y(k)$, for $k \geq k_3$. We can rewrite from (1.1) as

$$\begin{aligned} &p(k+1)\Psi(y(k+1))\Delta^2 y(k) + a(k+1)y(k) \\ &= -q(k)h(y(k))g(\Delta y(k-r(k))\Delta y(k) - f_1(k, y(k), y(k-r_1(k)), \ldots, y(k-r_n(k))) \\ &\qquad - \Delta[p(k)\Psi(y(k))]\Delta y(k). \end{aligned}$$

Then from the above equation, and condition $(H_1)$-$(H_2)$, we can obtain

$$p(k+1)\Psi(y(k+1))\Delta^2 y(k) + a(k+1)y(k) \prec 0.$$

That is

$$\begin{aligned} \Delta^2 y(k) &\prec -\frac{a(k+1)}{p(k+1)\Psi(y(k+1))} y(k) \\ &\prec -\frac{a(k+1)}{p(k+1)} y(k), \end{aligned} \tag{2.8}$$

for $k \geq k_3$. Since $\lim_{k\to\infty} \frac{a(k)}{p(k)} = s$, exist $k_4 \geq k_3$ such that if $k \geq k_4$,

$$a(k) > \frac{s}{2} p(k).$$

Summing (2.8) from $k_4$ to $k-1$, we can get

$$\begin{aligned} \Delta[y(k) - y(k_4)] &= \Delta y(k) - \Delta y(k_4) \\ &\prec -\sum_{i=k_4}^{k-1} \frac{a(i+1)}{p(i+1)} y(i) \\ &\prec -\frac{s}{2}(k - k_4) y(k_4). \end{aligned} \tag{2.9}$$

Summing (2.9) from $k_4 + 1$ to $k-1$ again, we have

$$y(k) - y(k_4+1) \prec -\frac{s}{2} \sum_{i=k_4+1}^{k-1} (i - k_4) y(k_4). \tag{2.10}$$

From Lemma 1.4 and equation (2.10), we can get the set

$$\left\{-\frac{s}{2}\sum_{i=k_4+1}^{k-1}(i-k_4)y(k_4)\right\}_{k=2+k_4}^{\infty}$$

is not bounded from below. Then by (2.10) we can obtain that the set $\{y(k)\}_{k=2+k_4}^{\infty}$ is not bounded from below. On the other hand, $\{y(k)\}$ is eventually positive, so we can get $\{y(k)\}_{k=2+k_4}^{\infty}$ is bounded from below by 0. This is a contradiction. The proof is complete. □

**Corollary 2.1.** *Suppose that $(H_1)$-$(H_5)$ and $(H_7)$ hold. Furthermore, assume that*

$$f(k,u_0,u_1,\dots,u_n)\succ(\prec)p(k+1)u_0,$$

*for $u_0\succ(\prec)0$, then every solution of equation (1.1) is oscillatory.*

*Proof.* The proof follows from Theorem 2.4 if we take $a(k)=p(k)$, and

$$f_1(k,u_0,u_1,\dots,u_n)=f(k,u_0,u_1,\dots,u_n)-p(k+1)u_0.$$ □

**Theorem 2.5.** *Assume that $(H_1)$-$(H_8)$ hold. Suppose that there exist a positive real sequence $\{d(k)\}$ and a positive real number $A$ such that*

$$\Delta d(k-1)p(k)\le -A \text{ and } \lim_{k\longrightarrow\infty}\sum_{t=1+k_0}^{k-1}d(t)=+\infty.$$

*Then every solution of equation (1.1) is oscillatory.*

*Proof.* Let $\{y(k)\}$ be a non-oscillatory solution of equation (1.1). Without loss of generality, assumed that $\{y(k)\}$ is eventually positive. Then we can find a $k_1\in\mathbb{N}(k_0)$, such that $0\prec y(k)$ for $k\ge k_1$. Since $\lim_{k\to\infty}(k-r_i(k))=+\infty, i=1,2,\dots,n$, we can find a $k_2\ge k_1$, such that $k-r_i(k)\ge k_1$ for $k\ge k_2, i=1,2,\dots,n$. That is $0\prec y(k-r_i(k))$ for $k\ge k_2, i=1,2,\dots,n$. From Theorem 2.1 there exist a $k_3\ge k_2$ such that $0\prec\Delta y(k)$, for $k\ge k_3$. We can rewrite from (1.1) as

$$\Delta\big[p(k)\Psi(y(k))\Delta y(k)\big]+f(k,y(k),y(k-r_1(k)),\dots,y(k-r_n(k)))$$
$$=-q(k)h(y(k))g(\Delta y(k-r(k))\Delta y(k).$$

From the above equation and conditions $(H_1)$ and $(H_3)$-$(H_4)$, we can know that if $k\ge k_3$,

$$\Delta\big[p(k)\Psi(y(k))\Delta y(k)\big]+f(k,y(k),y(k-r_1(k)),\dots,y(k-r_n(k)))\prec 0. \quad (2.11)$$

Since $0\prec\Delta y(k)$ for $k\ge k_3$, we can obtain that $\{y(k)\}$ is increasing, for $k\ge k_3$. It follows from $\lim_{k\to\infty}(k-r_i(k))=\infty, i=1,2,\dots,n$, that there exists a

$k_4 \geq k_3$ such that $k - r_i(k) \geq k_3$ for $k \geq k_4, i = 1, 2, \ldots, n$. Now we know that

$$\Big\{ f(k, y(k), y(k - r_1(k)), \ldots, y(k - r_n(k))) \Big\}$$

is increasing when $k \geq k_4$.

If we multiply (2.11) by $d(k)$, and latter sum it from $k_4$ to $k - 1$, we can obtain

$$d(k-1)p(k)\Psi(y(k))\Delta y(k) - \sum_{t=k_4}^{k-1} \Delta d(t-1)p(t)\Psi(y(t))\Delta y(t)$$
$$\prec s_0 - \sum_{t=k_4}^{k-1} d(t) f(t, y(t), y(t - r_1(t)), \ldots, y(t - r_n(t))), \qquad (2.12)$$

where $s_0 = d(k_4 - 1)p(k_4)\Psi(y(k_4))\Delta y(k_4)$. However

$$\begin{aligned} \sum_{t=k_4}^{k-1} \Delta d(t-1)p(t)\Psi(y(t))\Delta y(t) &\prec -A \sum_{t=k_4}^{k-1} \Psi(y(t))\Delta y(t) \\ &= -A\theta_1 y(k) + A\theta_1 y(k_4), \\ &\prec A\theta_1 y(k_4). \end{aligned}$$

That is

$$-A\theta_1 y(k_4) \prec -\sum_{t=k_4}^{k-1} \Delta d(t-1)p(t)\Psi(y(t))\Delta y(t). \qquad (2.13)$$

And

$$-\sum_{t=k_4}^{k-1} d(t) f(t, y(t), y(t - r_1(t)), \ldots, y(t - r_n(t)))$$
$$\prec -\sum_{t=k_4}^{k-1} d(t)\big(f(k_4, y(k_4), y(k_4 - r_1(k_4)), \ldots, y(k_4 - r_n(k_4))\big)). \qquad (2.14)$$

By Lemma 1.4, and

$$\lim_{k \longrightarrow \infty} \sum_{t=1+k_0}^{k-1} d(t) = \infty,$$

we know that the set

$$\Big\{ -\sum_{t=k_4}^{k-1} d(t) f(k_4, y(k_4), y(k_4 - r_1(k_4)), \ldots, y(k_4 - r_n(k_4))) \Big\}_{k=1+k_4}^{\infty}$$

is not bounded from below. From (2.14) we can get the set

$$\left\{ -\sum_{t=k_4}^{k-1} d(t)f(t, y(t), y(t-r_1(t)), \ldots, y(t-r_n(t))) \right\}_{k=1+k_4}^{\infty}$$

is not bounded from below.

From (2.12), (2.13) and (2.14) we can get the set

$$\left\{ d(k-1)p(k)\Psi(y(k))\Delta y(k) \right\}_{k=1+k_4}^{\infty}$$

is not bounded from below. On the other hand, $\{\Delta y(k)\}$ is eventually positive and $\{d(k)\}$, $\{p(k)\}$, $\Psi(x)$ are positive, so we can obtain that

$$\left\{ d(k-1)p(k)\Psi(y(k))\Delta y(k) \right\}_{k=1+k_4}^{\infty}$$

is bounded from below by 0. This a contradiction. The proof is compete. □

**Corollary 2.2.** *Assume that $(H_1 - H_8)$ hold. In addition, if condition*

$$\lim_{k\to\infty} \sum_{l=1+k_0}^{k-1} \sum_{t=l}^{\infty} \frac{1}{p(t+1)} = +\infty$$

*is satisfied. Then every solution of equation (1.1) is oscillatory.*

*Proof.* The proof follows from Theorem 2.5 if we take

$$d(k) = \sum_{t=k}^{\infty} \frac{1}{p(t+1)}.$$

□

## 3. Example

In this section, we gave three examples to illustrate the validity of our results.

**Example 3.1.** Consider a difference equation of the from

$$\Delta\left[k\frac{1+|\sin(y(k))|}{2+|\sin(y(k))|}\Delta y(k)\right] + \frac{2}{3}\frac{1}{2+|\sin(y(k))|}\frac{|\sin(\Delta y(k))|}{1+|\sin(\Delta y(k))|}\Delta y(k)$$
$$+ (k+2)y(k) + \frac{k+1}{\pi}|y(k)|y(k) + y(k-1) + y(k-2) + 4(k+1)y(k-3)$$
$$= 0, \quad (3.1)$$

where

$$p(k) = k,\ q(k) = \frac{2}{3},\ \Psi(x) = \frac{1+|\sin x|}{2+|\sin x|},\ h(x) = \frac{1}{2+|\sin x|},\ g(x) = \frac{|\sin x|}{|\sin x|+1},$$

$$r(k) = 0,\ r_1(k) = 1,\ r_2(k) = 2,\ r_3(k) = 3,$$

$$f(k, u_0, u_1, u_2, u_3) = (k+2)u_0 + \frac{k+1}{\pi}|u_0|u_0 + u_1 + u_2 + 4(k+1)u_3.$$

Obviously, we can have

$$\frac{1}{2} = \theta_1 \leq \Psi(x) = \frac{1+|\sin x|}{2+|\sin x|} = 1 - \frac{1}{2+|\sin x|} < 1,$$

$$0 < h(x) = \frac{1}{2+|\sin x|} \leq \frac{1}{2} = \theta_2,$$

$$q(k)h(x) = \frac{\frac{2}{3}}{2+|\sin x|} < \frac{k(1+|\sin x|)}{2+|\sin x|} = p(k)\Psi(x),$$

$$0 \leq g(x) = \frac{|\sin x|}{|\sin x|+1} = 1 - \frac{1}{|\sin x|+1} < 1,$$

$$\lim_{k\to\infty} k = \infty, \quad \lim_{k\to\infty}(k-1) = \infty, \quad \lim_{k\to\infty}(k-2) = \infty,$$

$$\lim_{k\to\infty}(k-3) = \infty, \quad m_0 = 0, \quad m_1 = 1, \quad m_2 = 2, \quad m_3 = 3,$$

$$\lim_{k\to\infty}\sum_{t=2}^{k-1}\frac{1}{p(t)}\prod_{j=1}^{t-1}[1 - \frac{\theta_2}{\theta_1}\frac{q(j)}{p(j)}] = \lim_{k\to\infty}\sum_{t=2}^{k-1}\frac{1}{t}\prod_{j=1}^{t-1}[1 - \frac{2}{3j}] = +\infty,$$

if $k \in \mathbb{N}$, and $0 \prec (\succ) u_i, i = 0, 1, 2, \ldots, n$,

$$f(k, u_0, u_1, u_2, u_3) = (k+2)u_0 + \frac{k+1}{\pi}|u_0|u_0 + u_1 + u_2 + 4(k+1)u_3 \succ (\prec) 0,$$

if $\overline{k} < k, v_i \prec u_i,\ i = 0, 1, 2, 3,$

$$\begin{aligned} f(\overline{k}, v_0, v_1, v_2, v_3) &= (\overline{k}+2)v_0 + \frac{k+1}{\pi}|v_0|v_0 + v_1 + v_2 + 4(k+1)v_3 \\ &\prec (k+2)u_0 + \frac{k+1}{\pi}|u_0|u_0 + u_1 + u_2 + 4(k+1)u_3 \\ &= f(k, u_0, u_1, u_2, u_3). \end{aligned}$$

Therefore, the conditions $(H_1)$-$(H_8)$ hold. That is, all conditions of Theorem 2.3 are satisfied.

If we let $a(k) = k+1$, and $f_1(k, u_0, u_1, u_2, u_3) = u_1 + u_2 + 4(k+1)u_3$, then $\lim_{k\to\infty}\frac{a(k)}{p(k)} = 1$, and $f_1(k, u_0, u_1, u_2, u_3) \succ 0$, if every $u_i \succ 0$, $f_1(k, u_0, u_1, u_2, u_3) \prec 0$, if every $u_i \prec 0$, obviously, every condition of Theorem 2.4 hold.

Hence, every solution of equation (3.1) is oscillatory by Theorem 2.3 or Theorem 2.4. One of the solutions of equation (3.1) is

$$y(k) = (-1)^k \pi.$$

**Example 3.2.** Consider a difference equation of the from

$$\Delta\left[\frac{2^{k+1}}{9\pi}\frac{1+|\sin(y(k))|}{2+1|\sin(y(k))|}\Delta y(k)\right] + \frac{2^{k+2}}{9\pi}\frac{1}{4+2|\sin(y(k))|}\frac{|\cos(\Delta y(k))|}{1+|\cos(\Delta y(k))|}\Delta y(k) + 16\frac{2^k}{3\pi}y(k) + \frac{25}{3}\frac{1}{\pi^2}|y(k)|y(k-1) = 0, \quad (3.2)$$

where

$$p(k) = \frac{2^{k+1}}{9\pi}, \quad q(k) = \frac{2^{k+2}}{9\pi}, \quad \Psi(x) = \frac{1+|\sin x|}{2+|\sin x|},$$

$$h(x) = \frac{1}{4+2|\sin x|},\ g(x) = \frac{|\cos x|}{|\cos x|+1},$$

$$r(k) = r_1(k) = 0,\ r_2(k) = 1,\ f(k,u_0,u_1,u_2) = 16\frac{2^k}{3\pi}u_0 + \frac{25}{3}\frac{1}{\pi^2}|u_1|u_2.$$

By calculating, we can get

$$\frac{1}{2} = \theta_1 \le \Psi(x) = \frac{1+|\sin x|}{2+|\sin x|} = 1 - \frac{1}{2+|\sin x|} < 1,$$

$$0 < h(x) = \frac{1}{4+2|\sin x|} \le \frac{1}{4} = \theta_2,$$

$$q(k)h(x) = \frac{2^{k+1}}{(2+|\sin x|)9\pi} < \frac{2^{k+1}(1+|\sin x|)}{(2+|\sin x|)9\pi} = p(k)\Psi(x),$$

$$0 \le g(x) = \frac{|\cos x|}{1+|\cos x|} = 1 - \frac{1}{1+|\cos x|} < 1,$$

$$\lim_{k\to\infty} k = \infty,\ \lim_{k\to\infty}(k-1) = \infty,\ m_0 = m_1 = 0,\ m_2 = 1,$$

$$\lim_{k\to\infty}\sum_{t=5}^{k-1}\frac{1}{p(t)}\prod_{j=4}^{t-1}[1-\frac{\theta_2}{\theta_1}\frac{q(j)}{p(j)}] = \lim_{k\to\infty}\sum_{t=5}^{k-1}\frac{2^{t+1}}{9\pi}\prod_{j=4}^{t-1}[1-\frac{1}{2}] = \lim_{k\to\infty}\sum_{t=5}^{k-1}\frac{32}{9\pi} = +\infty,$$

if $k \in \mathbb{N}(4)$, and $0 \prec (\succ) u_i, i = 0,1,2,$

$$f(k,u_0,u_1,u_2) = 16\frac{2^k}{3\pi}u_0 + \frac{25}{3}\frac{1}{\pi^2}|u_1|u_2 \succ (\prec) 0,$$

if $\overline{k} < k$, $v_i \prec u_i$, $i = 0, 1, 2$,

$$\begin{aligned} f(\overline{k}, v_0, v_1, v_2) &= 16\frac{2^{\overline{k}}}{3\pi}v_0 + \frac{25}{3\pi^2}|v_1|v_2 \\ &\prec 16\frac{2^k}{3\pi}u_0 + \frac{2}{3\pi^2}|u_1|u_2 \\ &= f(k, u_0, u_1, u_2). \end{aligned}$$

Hence, $(H_1)$-$(H_8)$ hold. It is clear that if we take

$$a(k) = 16\frac{2^{k-1}}{3\pi},\ f_1(k, u_0, u_1, u_2) = \frac{25}{3\pi^2}|u_1|u_2,$$

we can get $\lim_{k\to\infty}\frac{a(k)}{p(k)} = 12$, $f_1(k, u_0, u_1, u_2) \succ 0$, if every $u_i \succ 0$, $f_1(k, u_0, u_1, u_2) \prec 0$, if every $u_i \prec 0$, all conditions of Theorem 2.4 are satisfied.

If we let $d(k) = \frac{1}{k+1}$, then

$$\Delta d(k-1)p(k) = \frac{-2^k}{9\pi(k(k+1))} \le \frac{-4}{45\pi},\ \lim_{k\to\infty}\sum_{t=4}^{k-1} d(k) = +\infty,$$

obviously every condition of Theorem 2.5 holds. Therefore, we can know that every solution of equation (3.2) is oscillatory by using Theorem 2.4 or Theorem 2.5. One of the solutions of equation (3.2) is

$$y(k) = (-2)^k\pi.$$

**Example 3.3.** Consider a difference equation of the from

$$\Delta\left[\frac{4^k}{9\pi}\frac{1+|\sin(y(k))|}{2+1|\sin(y(k))|}\Delta y(k)\right] + \frac{4^k}{9\pi}\frac{1}{4+2|\sin(y(k))|}\frac{|\cos(\Delta y(k))|}{1+|\cos(\Delta y(k))|}\Delta y(k)$$
$$+11\frac{4^{k-1}}{6\pi}y(k) + \frac{46}{3\pi^3}y(k)y(k-1)y(k-2) = 0, \tag{3.3}$$

where

$$p(k) = \frac{4^k}{9\pi},\quad q(k) = \frac{4^k}{9\pi},\quad \Psi(x) = \frac{1+|\sin x|}{2+|\sin x|},$$
$$h(x) = \frac{1}{4+2|\sin x|},\ g(x) = \frac{|\cos x|}{|\cos x|+1},$$

$r(k) = 0$, $r_1(k) = 1$, $r_2(k) = 2$, $f(k, u_0, u_1, u_2) = 11\dfrac{4^{k-1}}{6\pi}u_0 + \dfrac{46}{3\pi^3}u_0u_1u_2$.

$(H_1)$-$(H_5)$ and $(H_7)$ are satisfied. If we take

$$a(k) = 11\frac{4^{k-2}}{6\pi},\quad f_1(k, u_0, u_1, u_2) = \frac{46}{3\pi^3}u_0u_1u_2,$$

every condition of Theorem 2.4 holds. From this theorem, every bounded and unbounded solution of equation (3.3) is oscillatory, since

$$f(1,-1,-2,-3) - f(1,-1,-3,-3) = \frac{-46}{\pi^3} < 0,$$

$f(k, u_0, u_1, u_2)$ is not increasing for $u_i (i = 0, 1, 2)$. In [8], all theorems except Theorem 2.3, only present sufficient condition for oscillation of all bounded solutions of equation (1.1), although Theorem 2.3 presents sufficient condition for oscillation of all solutions of equation (1.1), it asks $f(k, u_0, u_1, \dots, u_n)$ is not decreasing for $u_i (i = 0, 1, 2, \dots, n)$, so we can not estimate any unbounded solution of equation (3.3) is not oscillatory without knowledge every unbounded solution of equation (3.3) by using Theorems of [5]. One of an unbounded solutions of (3.3) is

$$y(k) = (-2)^k \pi.$$

Conclusions of this paper are complement of [8].

## Acknowledgments

This work is supported by the National Natural Sciences Foundation of People's Republic of China under Grant 10971183.

## References

[1] Y. Bolat, Oscillation criteria for nonlinear second-order difference equations with a nonlinear dampted term, *Appl. Math. Lett.*, **18**, No. 3 (2005), 329-338.

[2] R. Cristescu, *Ordered Vector Space and Linear Operator*, Abacus Press, Tunbridge Wells, Kent (1976).

[3] W.T. Li, Oscillation theorems for second-order nonlinear difference equations, *Math. Comput. Modell.*, **31**, No-s: 6-7 (2000), 71-79.

[4] Y.K. Li, L.L. Zhao, Oscillation of a second-order difference equation with a nonlinear damped term in archimedean space, *Commun. Math. Anal.*, **5** (2008), 36-43.

[5] S.H. Saker, Oscillation of second-order peturbed non-linear difference equations, *Appl. Math. Comput.*, **144**, No-s: 2-3 (2003), 305-324.

[6] S.H. Saker, New oscillation criteria for second-order nonlinear neutral delay difference equations, *Appl. Math. Comput.*, **142**, No. 1 (2003), 99-111.

[7] Y.G. Sun, Oscillation and non-oscillation for second-order linear difference equations, *Appl. Math. Comput.*, **170**, No. 12 (2005), 1095-1103.

[8] X.H. Tang, J.S. Yu, D.H. Peng, Oscillation and nonoscillation of neutral difference equations with positive and negative coefficients, *Comput. Math. Appl.*, **39**, No. 78 (2000), 169-181.