HYBRID SUPER-RELAXED PROXIMAL POINT ALGORITHMS AND GENERAL NONLINEAR VARIATIONAL INCLUSION PROBLEMS

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Abstract: First a general framework for a hybrid super-relaxed proximal point algorithm based on the notion of $H$-maximal monotonicity is introduced, and then the convergence analysis for solving a general class of nonlinear inclusion problems is explored. The framework developed in this communication is quite suitable to generalize first-order evolution equations based on the generalized nonlinear Yosida regularization/approximation and beyond. Furthermore, the obtained results can also be applied to generalize Douglas-Rachford splitting methods for finding the zero of the sum of two generalized maximal monotone mappings.

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1. Introduction

Let $X$ be a real Hilbert space with the norm $\| \cdot \|$ and the inner product $\langle \cdot , \cdot \rangle$. We consider a general class of nonlinear variational inclusion problems of the form: find a solution to

$$0 \in M(x),$$

where $M : X \rightarrow 2^X$ is a set-valued mapping on $X$. Among the advances [1]-[32] on proximal point algorithms relating to solvability of variational inclusion problems...
problems of the form (1), we are primarily concerned with the recent work of Xu [32], Agarwal and Verma [1], Eckstein and Bertsekas [3], where the application of the relaxed proximal point algorithm is not limited only to $M$, but goes beyond $M_\rho = \rho^{-1}(I - J^M_\rho)$, the Yosida regularization of $M$ for $J^M_\rho = (I + \rho M)^{-1}$. It further follows that using the right choice of relaxation parameters, it yields the classical proximal point algorithm for $M$ itself. As a matter of fact, for large enough $\rho > 0$, the map $M_\rho$ could be locally monotone while $M$ is not, and as a result, local maximal monotonicity would ensure the local convergence for the proximal point algorithm. Verma [16] considered the generalized version of the relaxed proximal point algorithm studied by Eckstein and Bertsekas [3] on the weak convergence, based on the work of Rockafellar [12] on the proximal point algorithm, and has applied to approximate a solution to inclusion problem of the form (1). Indeed, Eckstein and Bertsekas [3] further examined a generalized alternating direction method of multipliers on convex programming, and they applied the obtained results to the Douglas-Rachford splitting method for finding the zero of the sum of two monotone operators, and as a result, showed that it was a special case of the proximal point algorithm. Eckstein [Doctoral Dissertation, MIT Report LIDS-TH-1877, 1989] has shown using the generalized Douglas-Rachford splitting with relaxation factors $\alpha_k$ other than 1 (especially greater than 1) converge faster than the regular Douglas-Rachford splitting. This was the case for a highly parallel algorithm for linear programming, while for a choice of $\alpha_k = 1.5$ for all $k$ appeared to converge to a given accuracy about 15% faster than the choice $\alpha_k = 1$ for all $k$. It turned out that the inclusion of over-relaxation factors offers some practical significance. As a matter of fact, this could have been the beginning of the general over-relaxed proximal point algorithms. Recently the author [28] used a more general over-relaxed proximal point algorithm, namely, the super-relaxed proximal point algorithms to approximate the solution of an inclusion problem of the form (1).

In this communication, our main goal is to introduce a hybrid super-relaxed proximal point algorithm based on the notion of $H$-maximal monotonicity, a generalization to the theory of set-valued maximal monotone mappings by Fang and Huang [4] for solving general inclusion problems using generalized resolvent operator technique in Hilbert space settings. This concept has greatly impacted the general theory of maximal monotone set-valued mappings in a Hilbert space as well as in uniformly smooth Banach space settings. The generalized resolvent operator techniques in general can also effectively be applied to several other fields of interest, for instance, equilibria problems in economics, optimization and control theory, operations research, mathematical finance, management and decision sciences, and mathematical programming. For more literature, we
HYBRID SUPER-RELAXED PROXIMAL POINT...

recommend the reader [1]-[36].

2. Generalized Maximal Monotonicity

In this section we discuss some results based on the basic properties of H-maximal monotonicity (also referred to as H-monotonicity in literature) and its variant forms. Let \( M : X \to 2^X \) be a multivalued mapping on \( X \). We shall denote both the map \( M \) and its graph by \( M \), that is, the set \( \{(x, y) : y \in M(x)\} \). This is equivalent to stating that a mapping is any subset \( M \) of \( X \times X \), and \( M(x) = \{y : (x, y) \in M\} \). If \( M \) is single-valued, we shall still use \( M(x) \) to represent the unique \( y \) such that \( (x, y) \in M \) rather than the singleton set \( \{y\} \). This interpretation shall much depend on the context. The domain of a map \( M \) is defined (as its projection onto the first argument) by

\[
D(M) = \{x \in X : \exists y \in X : (x, y) \in M\} = \{x \in X : M(x) \neq \emptyset\}.
\]

\( D(M) = X \), shall denote the full domain of \( M \), and the range of \( M \) is defined by

\[
R(M) = \{y \in X : \exists x \in X : (x, y) \in M\}.
\]

The inverse \( M^{-1} \) of \( M \) is \( \{(y, x) : (x, y) \in M\} \). For a real number \( \rho \) and a mapping \( M \), let \( \rho M = \{(x, \rho y) : (x, y) \in M\} \). If \( L \) and \( M \) are any mappings, we define

\[
L + M = \{(x, y + z) : (x, y) \in L, (x, z) \in M\}.
\]

**Definition 2.1.** Let \( M : X \to 2^X \) be a multivalued mapping on \( X \), and let \( A : X \to X \) be a single-valued mapping on \( X \). Then:

(i) \( M \) is \((r)\)-strongly monotone if there exists a positive constant \( r \) such that

\[
\langle u^* - v^*, u - v \rangle \geq r\|u - v\|^2 \ \forall (u, u^*), (v, v^*) \in M.
\]

(ii) \( M \) is \((m)\)-relaxed monotone if there exists a positive constant \( m \) such that

\[
\langle u^* - v^*, u - v \rangle \geq (1 - m)\|u - v\|^2 \ \forall (u, u^*), (v, v^*) \in M.
\]

(iii) \( M \) is \((c)\)-cocoercive if there is a positive constant \( c \) such that

\[
\langle u^* - v^*, u - v \rangle \geq c\|u^* - v^*\|^2 \ \forall (u, u^*), (v, v^*) \in M.
\]

(iv) \( A \) is \((r)\)-strongly monotone if there exists a positive constant \( r \) such that

\[
\langle A(u) - A(v), u - v \rangle \geq r\|u - v\|^2 \ \forall (u, u^*), (v, v^*) \in M.
\]
(v) $A$ is $(m)$-relaxed monotone if there exists a positive constant $m$ such that
\[
\langle A(u) - A(v), u - v \rangle \geq (-m)\|u - v\|^2 \quad \forall (u, u^*), (v, v^*) \in M.
\]
(vi) $A$ is $(c)$-cocoercive if there is a positive constant $c$ such that
\[
\langle A(u) - A(v), u - v \rangle \geq c\|A(u) - A(v)\|^2 \quad \forall (u, u^*), (v, v^*) \in M.
\]

**Definition 2.2.** (see [4]) Let $H : X \to X$ be $(r)$-strongly monotone. The map $M : X \to 2^X$ is said to be $H$-maximal monotone if:

(i) $M$ is monotone,
(ii) $R(H + \rho M) = X$ for $\rho > 0$.

**Definition 2.3.** Let $H : X \to X$ be an $(r)$-strongly monotone mapping and let $M : X \to 2^X$ be an $H$-maximal monotone mapping. Then the generalized resolvent operator $J_{\rho,H}^M : X \to X$ is defined by
\[
J_{\rho,H}^M(u) = (H + \rho M)^{-1}(u).
\]

**Proposition 2.2.** Let $H : X \to X$ be an $(r)$-strongly monotone mapping and let $M : X \to 2^X$ be an $H$-maximal monotone mapping. Then the operator $(H + \rho M)^{-1}$ is single-valued for $r > 0$.

**Proposition 2.3.** (see [4]) Let $H : X \to X$ be an $(r)$-strongly monotone mapping, and let $M : X \to 2^X$ be an $H$-maximal monotone mapping. Then resolvent $J_{\rho,H}^M = (H + \rho M)^{-1}$ is $(\frac{1}{r})$-Lipschitz continuous.

### 3. Hybrid Super-Relaxed Proximal Point Algorithm

This section deals with an introduction of a hybrid super-relaxed proximal point algorithm and its applications to approximation solvability of the inclusion problem (1) based on the $H$-maximal monotonicity.

**Theorem 3.1.** Let $X$ be a real Hilbert space, let $H : X \to X$ be $(r)$-strongly monotone, and let $M : X \to 2^X$ be $H$-maximal monotone. Then the following statements are equivalent:

(i) An element $u \in X$ is a solution to (1).
(ii) For an $u \in X$, we have
\[
u = J_{\rho,H}^M(H(u)),
\]
where
\[
J_{\rho,H}^M(u) = (H + \rho M)^{-1}(u).
\]
Definition 3.1. A mapping \( M^{-1} \), the inverse of \( M : X \to 2^X \) is \((c)\)-Lipschitz continuous at 0 if for \( c \geq 0 \) there exists a unique solution \( z^* \) to \( 0 \in M(z) \) (equivalently \( M^{-1}(0) = \{z^*\} \)) such that

\[
\|z - z^*\| \leq c\|w - 0\| \text{ for } z \in M^{-1}(w) \text{ and } \|w\| \leq t \,(t > 0).
\]

Lemma 3.1. (see [25]) Let \( \{M^k\}, M : X \to 2^X \) be \( H \)-maximal \((m)\)-relaxed monotone, and let \( H : X \to X \) be \((r)\)-strongly monotone and \((s)\)-Lipschitz continuous. Then \( M^k \xrightarrow{H} M \) for \( k = 0, 1, 2, \cdots \), if and only if,

\[
J_{\rho_k,H}^k(u) \to J_{\rho_k,H}^M(u) \quad \forall u \in X,
\]

where \( J_{\rho_k,H}^k = \left(H + \rho_k M^k\right)^{-1} \) and \( J_{\rho_k,H}^M = \left(H + \rho M\right)^{-1}. \)

Proposition 3.1. Let \( X \) be a real Hilbert space, let \( H : X \to X \) be \((r)\)-strongly monotone, and let \( M : X \to 2^X \) be \( H \)-maximal monotone. Let us set \( J_k = H - H o J_{\rho_k,H}^M o H. \) Suppose that \( H o J_{\rho_k,H}^M \) is \((\gamma)\)-cocoercive, that is,

\[
\langle H(J_{\rho_k,H}^M(H(u))) - H(J_{\rho_k,H}^M(H(v))), H(u) - H(v)\rangle \\
\geq \gamma\|H(J_{\rho_k,H}^M(H(u))) - H(J_{\rho_k,H}^M(H(v)))\|^2 \quad \text{for } \gamma > \frac{1}{2}.
\]

Then we have

\[
(2\gamma - 1)\|H(J_{\rho_k,H}^M(H(u))) - H(J_{\rho_k,H}^M(H(v)))\|^2 + \|J_k(u) - J_k(v)\|^2 \\
\leq \|H(u) - H(v)\|^2.
\]

Proof. For any \( u, v \in X \), we have

\[
\|H(u) - H(v)\|^2 \\
= \|H(J_{\rho_k,H}^M(H(u))) - H(J_{\rho_k,H}^M(H(v))) + J_k(u) - J_k(v)\|^2 \\
= \|H(J_{\rho_k,H}^M(H(u))) - H(J_{\rho_k,H}^M(H(v)))\|^2 + \|J_k(u) - J_k(v)\|^2 \\
+ 2\langle H(J_{\rho_k,H}^M(H(u))) - H(J_{\rho_k,H}^M(H(v))), J_k(u) - J_k(v)\rangle \\
= \|H(J_{\rho_k,H}^M(H(u))) - H(J_{\rho_k,H}^M(H(v)))\|^2 + \|J_k(u) - J_k(v)\|^2 \\
+ 2\langle H(u) - H(v), H(J_{\rho_k,H}^M(H(u))) - H(J_{\rho_k,H}^M(H(v)))\rangle \\
- 2\langle H(J_{\rho_k,H}^M(H(u))) - H(J_{\rho_k,H}^M(H(v))), H(J_{\rho_k,H}^M(H(u))) - H(J_{\rho_k,H}^M(H(v)))\rangle \\
\geq \|H(J_{\rho_k,H}^M(H(u))) - H(J_{\rho_k,H}^M(H(v)))\|^2 + \|J_k(u) - J_k(v)\|^2 \\
+ 2\langle H(J_{\rho_k,H}^M(H(u))) - H(J_{\rho_k,H}^M(H(v))), J_k(u) - J_k(v)\rangle \\
- \|H(J_{\rho_k,H}^M(H(u))) - H(J_{\rho_k,H}^M(H(v)))\|^2 \\
\geq (2\gamma - 1)\|H(J_{\rho_k,H}^M(H(u))) - H(J_{\rho_k,H}^M(H(v)))\|^2 + \|J_k(u) - J_k(v)\|^2,
\]
where $\gamma > \frac{1}{2}$.

For $\gamma = 1$ in Proposition 3.1, we have

**Proposition 3.2.** Let us set

$$J_k = H - H o J^M_{\rho_k, H o H}.$$  

In addition, if

$$\langle H(J^M_{\rho_k, H}(H(u))) - H(J^M_{\rho_k, H}(H(v))), H(u) - H(v) \rangle \geq \|H(J^M_{\rho_k, H}(H(u))) - H(J^M_{\rho_k, H}(H(v)))\|^2,$$  

then

$$\|H(J^M_{\rho_k, H}(H(u))) - H(J^M_{\rho_k, H}(H(v)))\|^2 + \|J_k(u) - J_k(v)\|^2 \leq \|H(u) - H(v)\|^2.$$  

When $H = I$ and $\gamma = 1$ in Proposition 3.1, we have (see Proposition 1(c) in [13]).

**Proposition 3.3.** Let us set

$$J_k = I - J^M_{\rho_k}.$$  

Then

$$\|J^M_{\rho_k}(u) - J^M_{\rho_k}(v)\|^2 + \|J_k(u) - J_k(v)\|^2 \leq \|u - v\|^2,$$  

where $J^M_{\rho_k} = (I + \rho_k M)^{-1}$ is the classical resolvent.

**Theorem 3.2.** Let $X$ be a real Hilbert space, let $H : X \to X$ be $(r)$-strongly monotone and nonexpansive, and let $\{M_k\}, M : X \to 2^X$ be $H$-maximal monotone such that $M_k \stackrel{H}{\rightarrow} M$ for $k = 0, 1, 2, \cdots$. For an arbitrarily chosen initial point $x^0$, suppose that the sequence $\{x^k\}$ is generated by the hybrid super-relaxed proximal point algorithm

$$H(x^{k+1}) = (1 - \alpha_k)H(x^k) + \alpha_k y^k \ \forall \ k \geq 0,$$  

and $y^k$ satisfies

$$\|y^k - H(M^k_{\rho_k, H}(H(x^k)))\| \leq \delta_k \|y^k - H(x^k)\|,$$  

where $M^k_{\rho_k, H} = (I + \rho_k M)^{-1}$, and

$$\{\delta_k\}, \{\alpha_k\}, \{\rho_k\} \subseteq [0, \infty).$$  

are scalar sequences. Suppose that $\{x^k\}$ is bounded in the sense that there exists at least one solution to (1), and that $M^{-1}$ is $(c)$-Lipschitz continuous at 0. Suppose, in addition, for $\gamma > 1$, that
Then in light of Proposition 3.1 and Theorem 3.1, we conclude that
\[ \lim_{k \to \infty} \| \frac{1}{2} \gamma k \alpha \| H(J_{\rho_k}^M H(x^k)) - H(J_{\rho_k}^M H(x^*)) \| \leq \frac{2}{\alpha} \gamma \| H(J_{\rho_k}^M H(x^k)) - H(J_{\rho_k}^M H(x^*)) \| \leq \left( \sqrt{1 - \alpha (2(1 - \gamma d^2) - (1 - (2 \gamma - 1)d^2))} \right) \leq 1, \]
where \( \alpha_k^2 + 2 \alpha \alpha_k (1 - \alpha_k)\gamma > 0, \) for \( \alpha_k \geq 1, \sum_{k=0}^{\infty} \delta_k < \infty, \delta_k \to 0, \alpha = \limsup_{k \to \infty} \alpha_k, \rho = \limsup_{k \to \infty} \rho_k \) and
\[ d = \limsup_{k \to \infty} d_k = \limsup_{k \to \infty} \sqrt{\frac{c^2}{(2\gamma - 1)r^2c^2 + \rho_k^2}} < 1. \]

Proof. Suppose that \( x^* \) is a zero of \( M \) for \( x^* \in X. \) From Theorem 3.1, it follows that \( x^* \) is a fixed point of \( J_{\rho_k}^M \circ H. \) First, we express
\[ H(z^{k+1}) = (1 - \alpha_k) H(x^k) + \alpha_k H(J_{\rho_k}^M H(x^k)). \]
Since \( x^* \) is a solution to (1), we have in light of Theorem 3.1 that
\[ H(x^* ) = (1 - \alpha_k) H(x^*) + \alpha_k H(J_{\rho_k}^M H(x^*)). \]
Now, applying Lemma 3.1, we find the estimate
\[ \| H(x^{k+1}) - H(x^*) \| = \| (1 - \alpha_k) H(x^k) + \alpha_k H(J_{\rho_k}^M H(x^k)) - (1 - \alpha_k) H(x^*) - \alpha_k H(J_{\rho_k}^M H(x^*)) \| \]
\[ = \| (1 - \alpha_k) (H(x^k) - H(x^*)) + \alpha_k (H(J_{\rho_k}^M H(x^k)) - H(x^*)) + \alpha_k H(J_{\rho_k}^M H(x^k)) - H(x^*) \| \]
\[ \leq \| (1 - \alpha_k) (H(x^k) - H(x^*)) + \alpha_k H(J_{\rho_k}^M H(x^k)) - H(x^*) \| + \| H(J_{\rho_k}^M H(x^k)) - H(x^*) \| \]
\[ \leq \| (1 - \alpha_k) (H(x^k) - H(x^*)) + \alpha_k H(J_{\rho_k}^M H(x^k)) - H(x^*) \| + 2 \| H(J_{\rho_k}^M H(x^k)) - H(x^*) \|, \]
\[ (10) \]
where
\[ f_k = \| H(J_{\rho_k}^M H(x^k)) - H(J_{\rho_k}^M H(x^*)) \| \to 0, \]
\[ g_k = \| H(J_{\rho_k}^M H(x^k)) - H(x^*) \| \to 0. \]
Then in light of Proposition 3.1 and Theorem 3.1, we conclude that
\[ \| J_{\rho_k}^M H(x^k) - x^* \| \leq d_k \| H(x^k) - H(x^*) \|, \]
\[ (13) \]
where \( d_k = \sqrt{\frac{c^2}{(2\gamma - 1)r^2c^2 + \rho_k^2}} < 1. \]
To prove (14), set $J_k = H - HoJ^M_{p k,H} o H$ and then apply Proposition 3.1 as follows:

Since $\rho_k^{-1}J_k(x^k) \in M(J^M_{p k,H}(H(x^k)))$, it follows that

$$J^M_{p k,H}(H(x^k)) \in M^{-1}(\rho_k^{-1}J_k(x^k)).$$

On applying the Lipschitz condition (2) by taking $w = \rho_k^{-1}J_k(x^k)$ and $z = J^M_{p k,H}(H(x^k))$, we get

$$\|J^M_{p k,H}(H(x^k)) - x^*\| \leq c\|\rho_k^{-1}J_k(x^k)\| \forall k \geq k'.$$

Next, applying Proposition 3.1, we derive

$$(2 \gamma - 1)\|H(J^M_{p k,H}(H(x^k))) - H(x^*)\|^2 + \|J_k(x^k)\|^2 \leq \|H(x^k) - H(x^*)\|^2, \quad (15)$$

and as result of (15), we have

$$(2 \gamma - 1)\|H(J^M_{p k,H}(H(x^k))) - H(x^*)\|^2 \leq \|H(x^k) - H(x^*)\|^2 - \|J_k(x^k)\|^2$$

$$\leq \|H(x^k) - H(x^*)\|^2 - \left(\frac{\rho_k}{c}\right)^2\|J^M_{p k,H}(H(x^k)) - x^*\|^2.$$  

Since $H$ is $(\tau)$-strongly monotone (and hence $\|H(u) - H(v)\| \geq \tau\|u - v\|$ for all $u, v \in X$), we have

$$\|J^M_{p k,H}(H(x^k)) - x^*\|^2 \leq \frac{c^2}{(2 \gamma - 1)\tau^2} + \frac{\rho_k}{c}\|H(x^k) - H(x^*)\|^2. \quad (16)$$

Now we continue the proof from (11) and using the nonexpansiveness of $H$, we estimate $(\alpha_k \geq 1)$

$$\|(1 - \alpha_k)(H(x^k) - H(x^*)) + \alpha_k(H(J^M_{p k,H}(H(x^k))) - H(J^M_{p k,H}(H(x^*))))\|^2$$

$$= (1 - \alpha_k)^2\|H(x^k) - H(x^*)\|^2$$

$$+ 2\alpha_k(1 - \alpha_k)\gamma\|H(x^k) - H(x^*), H(J^M_{p k,H}(H(x^k))) - H(J^M_{p k,H}(H(x^*)))\|^2$$

$$+ \alpha_k^2\|H(J^M_{p k,H}(H(x^k))) - H(J^M_{p k,H}(H(x^*)))\|^2$$

$$\leq (1 - \alpha_k)^2\|H(x^k) - H(x^*)\|^2$$

$$+ 2\alpha_k(1 - \alpha_k)\gamma\|H(J^M_{p k,H}(H(x^k))) - H(J^M_{p k,H}(H(x^*)))\|^2$$

$$+ \alpha_k^2\|H(J^M_{p k,H}(H(x^k))) - H(J^M_{p k,H}(H(x^*)))\|^2$$

$$= (1 - \alpha_k)^2\|H(x^k) - H(x^*)\|^2$$

$$+ \alpha_k^2 + 2\alpha_k(1 - \alpha_k)\gamma\|H(J^M_{p k,H}(H(x^k))) - H(J^M_{p k,H}(H(x^*)))\|^2$$

$$\leq (1 - \alpha_k)^2\|H(x^k) - H(x^*)\|^2$$

$$+ \alpha_k^2 + 2\alpha_k(1 - \alpha_k)\gamma\|J^M_{p k,H}(H(x^k)) - J^M_{p k,H}(H(x^*))\|^2$$

$$\leq (1 - \alpha_k)^2\|H(x^k) - H(x^*)\|^2$$

$$+ \alpha_k^2 + 2\alpha_k(1 - \alpha_k)\gamma\|H(x^k) - H(x^*)\|^2$$
where \( \alpha^2_k + 2\alpha_k(1 - \alpha_k)\gamma > 0 \) for \( \alpha_k \geq 1, \gamma > 1 \) and \( d_k = \sqrt{\frac{c^2}{(2\gamma-1)c^2+\rho_k^2}} < 1 \).

Thus, we have
\[
\|H(z^{k+1}) - H(x^*)\| \leq \theta_k \|H(x^k) - H(x^*)\| + \alpha_k f_k + \alpha_k g_k,
\]
where
\[
\theta_k = \sqrt{[(1 - \alpha_k)^2 + (\alpha^2_k + 2\alpha_k(1 - \alpha_k)\gamma)d_k^2]} < 1,
\]
and \( \alpha^2_k + 2\alpha_k(1 - \alpha_k)\gamma > 0 \) for \( \alpha_k \geq 1, \gamma > 1 \).

On the other hand, we have
\[
\|H(x^{k+1}) - H(z^{k+1})\| = \|\alpha_k(y_k - H(J^{M_k}_{\rho_k}H(x^k)))\| \leq \alpha_k \delta_k \|y_k - H(x^k)\|. \tag{18}
\]

Since using (8), \( H(x^{k+1}) - H(x^k) = \alpha_k(y_k - H(x^k)) \) and applying (18), we estimate
\[
\|H(x^{k+1}) - H(x^*)\| \leq \|H(z^{k+1}) - H(x^*)\| + \|H(x^{k+1}) - H(z^{k+1})\|
\leq \|H(z^{k+1}) - H(x^*)\| + \alpha_k \delta_k \|y_k - H(x^k)\|
= \|H(z^{k+1}) - H(x^*)\| + \delta_k \|H(x^{k+1}) - H(x^*)\| + \delta_k \|H(x^k) - H(x^*)\|
\leq \theta_k \|H(x^k) - H(x^*)\| + \delta_k \|H(x^{k+1}) - H(x^*)\| + \delta_k \|H(x^k) - H(x^*)\|
\leq \theta_k \|H(x^k) - H(x^*)\| + \delta_k \|H(x^{k+1}) - H(x^*)\| + \delta_k \|H(x^k) - H(x^*)\| + \alpha_k f_k + \alpha_k g_k. \tag{19}
\]

It follows that
\[
\|H(x^{k+1}) - H(x^*)\| \leq \theta_k + \delta_k \|H(x^k) - H(x^*)\| + \frac{\alpha_k}{1 - \delta_k} f_k + \frac{\alpha_k}{1 - \delta_k} g_k, \tag{20}
\]
where
\[
\theta_k = \sqrt{[(1 - \alpha_k)^2 + (\alpha^2_k + 2\alpha_k(1 - \alpha_k)\gamma)d_k^2]} < 1,
\]
and \( \alpha^2_k + 2\alpha_k(1 - \alpha_k)\gamma > 0 \) for \( \alpha_k \geq 1, \gamma > 1 \) and \( d_k = \sqrt{\frac{c^2}{(2\gamma-1)c^2+\rho_k^2}} < 1 \).

Finally, since \( H \) is \( (r) \)-strongly monotone (and hence \( (r) \)-expanding), and \( \delta_k \to 0 \), it follows from (20) that \( \{x^k\} \) converges linearly to \( x^* \) for
\[
\limsup \frac{\theta_k + \delta_k}{1 - \delta_k} = \limsup \theta_k = \sqrt{[(1 - \alpha)^2 + (\alpha^2 + 2\alpha(1 - \alpha)\gamma)d^2]} < 1,
\]
where \( d = \limsup d_k = \limsup \sqrt{\frac{c^2}{(2\gamma-1)c^2+\rho_k^2}} < 1. \)

When \( \gamma = 1 \) in Theorem 3.2, we have

**Theorem 3.3.** Let \( X \) be a real Hilbert space, let \( H : X \to X \) be \( (r)\)-
are scalar sequences. Suppose that \{y^k\} and \{y\} are generated by the hybrid super-relaxed proximal point algorithm
\[ H(x^{k+1}) = (1 - \alpha_k)H(x^k) + \alpha_k y^k \quad \forall k \geq 0, \quad (21) \]
and \( y^k \) satisfies
\[ \|y^k - H(J_{\rho_k}^M(H(x^k)))\| \leq \delta_k \|y^k - H(x^k)\|, \quad (22) \]
where \( J_{\rho_k}^M = (H + \rho_k M^k)^{-1} \), and
\[ \{\delta_k\}, \{\alpha_k\}, \{\rho_k\} \subseteq [0, \infty) \]
are scalar sequences. Suppose that \{\|x^k\|\} is bounded and that \( M^{-1} \) is \( c \)-Lipschitz continuous at 0.

If, in addition,
\[ \langle H(x^k) - H(x^*), H(J_{\rho_k}^M(H(x^k))) - H(J_{\rho_k}^M(H(x^*))) \rangle \geq \|H(J_{\rho_k}^M(H(x^k))) - H(J_{\rho_k}^M(H(x^*)))\|^2, \quad (23) \]
then the sequence \{\|x^k\|\} converges linearly to a unique solution \( x^* \) of (1) with the convergence rate
\[ \sqrt{1 - \alpha \{2(1 - d^2) - (1 - d^2)\alpha\}} < 1, \]
where \( \alpha^2 + 2\alpha(1 - \alpha) > 0 \), for \( \alpha_k \geq 1 \), \( \sum_{k=0}^{\infty} \delta_k < \infty \), \( \delta_k \to 0 \), and \( \alpha = \limsup_{k \to \infty} \alpha_k \), \( \rho = \limsup_{k \to \infty} \rho_k \) and \( d = \limsup d_k = \limsup \sqrt{\frac{\|J_{\rho_k}^M(x^k)\|^2}{\|x^k\|^2 + \rho_k^2}} < 1 \).

Corollary 3.1. Let \( X \) be a real Hilbert space, and let \{\|x^k\|\}, \( M : X \to 2^X \) be maximal monotone such that \( M^k \stackrel{H}{\to} M \) for \( k = 0, 1, 2, \ldots \). For an arbitrarily chosen initial point \( x^0 \), suppose that the sequence \{\|x^k\|\} is generated by the relaxed proximal point algorithm
\[ x^{k+1} = (1 - \alpha_k)x^k + \alpha_k y^k \quad \forall k \geq 0, \quad (24) \]
and \( y^k \) satisfies
\[ \|y^k - J_{\rho_k}^M(x^k)\| \leq \delta_k \|y^k - x^k\|, \quad (25) \]
where \( J_{\rho_k}^M = (I + \rho_k M^k)^{-1} \), and
\[ \{\delta_k\}, \{\alpha_k\}, \{\rho_k\} \subseteq [0, \infty) \]
are scalar sequences. Suppose that \{\|x^k\|\} is bounded and that \( M^{-1} \) is \( c \)-Lipschitz continuous at 0.

Then the sequence \{\|x^k\|\} converges linearly to a unique solution \( x^* \) of (1)
with the convergence rate

\[ \sqrt{1 - \alpha \{ 2(1 - d^2) - (1 - d^2) \alpha \}} < 1, \]

where \( \alpha^2 + 2\alpha(1 - \alpha) > 0 \), for \( \alpha_k \geq 1 \), \( \sum_{k=0}^{\infty} \delta_k < \infty \), \( \delta_k \to 0 \), and \( \alpha = \limsup_{k \to \infty} \alpha_k \), \( \rho = \limsup_{k \to \infty} \rho_k \) and \( d = \limsup d_k = \limsup \sqrt{\frac{c^2}{\gamma^2 + \rho^2_k}} < 1. \)

**Theorem 3.4.** Let \( X \) be a real Hilbert space, let \( H : X \to X \) be \((r)\)-strongly monotone and nonexpansive, and let \( \{M_k\}, M : X \to 2^X \) be \( H \)-maximal monotone such that \( M^k \rightharpoonup M \) for \( k = 0, 1, 2, \cdots \). For an arbitrarily chosen initial point \( x^0 \), suppose that the sequence \( \{x^k\} \) is generated by the hybrid super-relaxed proximal point algorithm

\[ H(x^{k+1}) = (1 - \alpha_k)H(x^k) + \alpha_k y^k \quad \forall k \geq 0, \quad (26) \]

and \( y^k \) satisfies

\[ \|y^k - H(J_{\rho_k H}^M(H(x^k)))\| \leq \delta_k \|y^k - H(x^k)\|, \quad (27) \]

where \( J_{\rho_k H}^M = (H + \rho_k M^k)^{-1} \), and

\[ \{\delta_k\}, \{\alpha_k\}, \{\rho_k\} \subseteq [0, \infty) \]

are scalar sequences. If, in addition, for \( \gamma > 1 \),

\[ \langle H(u) - H(v), H(J_{\rho_k H}^M(H(u))) - H(J_{\rho_k H}^M(H(v))) \rangle \geq \gamma \|H(J_{\rho_k H}^M(H(u))) - H(J_{\rho_k H}^M(H(v)))\|^2 \quad \forall u, v \in X, \quad (28) \]

then the sequence \( \{x^k\} \) converges linearly to a unique solution \( x^* \) of (1) with the convergence rate

\[ \sqrt{1 - \alpha \{ 2(1 - \gamma d^2) - (1 - d^2) \gamma (\frac{1}{\rho} + 1) \alpha \}} < 1, \]

where \( \alpha_k^2 + 2\alpha_k(1 - \alpha_k)\gamma > 0 \), for \( \alpha_k \geq 1 \), \( \sum_{k=0}^{\infty} \delta_k < \infty \), \( \delta_k \to 0 \), \( \alpha = \limsup_{k \to \infty} \alpha_k \), and \( \rho = \limsup_{k \to \infty} \rho_k \).

**Proof.** Suppose that \( x^* \) is a zero of \( M \) for \( x^* \in X \). From Theorem 3.1, it follows that \( x^* \) is a fixed point of \( J_{\rho_k H}^M \circ H \). First, we express

\[ H(x^{k+1}) = (1 - \alpha_k)H(x^k) + \alpha_k H(J_{\rho_k H}^M(H(x^k))). \]

Since \( x^* \) is a solution to (1), we have in light of Theorem 3.1 that

\[ H(x^*) = (1 - \alpha_k)H(x^*) + \alpha_k H(J_{\rho_k H}^M(H(x^*)). \]

Now, applying Lemma 3.1, we find the estimate

\[ \|H(x^{k+1}) - H(x^*)\| = \|(1 - \alpha_k)H(x^k) + \alpha_k H(J_{\rho_k H}^M(H(x^k)))\|
\]

\[ - \|(1 - \alpha_k)(H(x^k) - H(x^*)) + \alpha_k(H(J_{\rho_k H}^M(H(x^k))) - H(J_{\rho_k H}^M(H(x^*)')))\|\]

\[ = \|(1 - \alpha_k)(H(x^k) - H(x^*)) + \alpha_k(H(J_{\rho_k H}^M(H(x^k))) - H(J_{\rho_k H}^M(H(x^*)')))\|\]
\[ \leq \|(1 - \alpha_k)(H(x^k) - H(x^*)) + \alpha_k(H(J_{\rho_k}^M(H(x^k))) - H(J_{\rho_k}^M(H(x^*))))\| \\
+ \|\alpha_k(H(J_{\rho_k}^M(H(x^k))) - H(J_{\rho_k}^M(H(x^*))))\|, \quad (29) \]

where

\[ f_k = \|H(J_{\rho_k}^M(H(x^k))) - H(J_{\rho_k}^M(H(x^*)))\| \rightarrow 0. \quad (30) \]

Now using the nonexpansiveness of $H$ and Proposition 2.3, we estimate ($\alpha_k \geq 1$)

\[ \|(1 - \alpha_k)(H(x^k) - H(x^*)) + \alpha_k(H(J_{\rho_k}^M(H(x^k))) - H(J_{\rho_k}^M(H(x^*))))\|^2 \]
\[ = (1 - \alpha_k)^2\|H(x^k) - H(x^*)\|^2 \\
+ 2\alpha_k(1 - \alpha_k)(H(x^k) - H(x^*), H(J_{\rho_k}^M(H(x^k))) - H(J_{\rho_k}^M(H(x^*)))) \\
+ \alpha_k^2\|H(J_{\rho_k}^M(H(x^k))) - H(J_{\rho_k}^M(H(x^*))))\|^2 \\
\leq (1 - \alpha_k)^2\|H(x^k) - H(x^*)\|^2 \\
+ 2\alpha_k(1 - \alpha_k)\gamma\|H(J_{\rho_k}^M(H(x^k))) - H(J_{\rho_k}^M(H(x^*)))\|^2 \\
+ \alpha_k^2\|H(J_{\rho_k}^M(H(x^k))) - H(J_{\rho_k}^M(H(x^*))))\|^2 \\
= (1 - \alpha_k)^2\|H(x^k) - H(x^*)\|^2 \\
+ [\alpha_k^2 + 2\alpha_k(1 - \alpha_k)\gamma]\|H(J_{\rho_k}^M(H(x^k))) - H(J_{\rho_k}^M(H(x^*)))\|^2 \\
\leq (1 - \alpha_k)^2\|H(x^k) - H(x^*)\|^2 \\
+ [\alpha_k^2 + 2\alpha_k(1 - \alpha_k)\gamma]\frac{1}{\rho^2}\|H(x^k) - H(x^*)\|^2 \\
= [(1 - \alpha_k)^2 + (\alpha_k^2 + 2\alpha_k(1 - \alpha_k)\gamma)\frac{1}{\rho^2}]\|H(x^k) - H(x^*)\|^2, \]

where $\alpha_k^2 + 2\alpha_k(1 - \alpha_k)\gamma > 0$ for $\alpha_k \geq 1$, $\gamma > 1$.

Thus, we have

\[ \|H(z^{k+1}) - H(x^*)\| \leq \theta_k\|H(x^k) - H(x^*)\| + \alpha_k f_k, \quad (31) \]

where

\[ \theta_k = \sqrt{[(1 - \alpha_k)^2 + (\alpha_k^2 + 2\alpha_k(1 - \alpha_k)\gamma)\frac{1}{\rho^2}]} < 1, \]

and $\alpha_k^2 + 2\alpha_k(1 - \alpha_k)\gamma > 0$ for $\alpha_k \geq 1$, and $\gamma > 1$.

On the other hand, we have

\[ \|H(x^{k+1}) - H(z^{k+1})\| = \|\alpha_k(y^k - H(J_{\rho_k}^M(H(x^k))))\| \leq \alpha_k \delta_k\|y^k - H(x^k)\|. \quad (32) \]
Since using (26), \(H(x^{k+1}) - H(x^k) = \alpha_k(y^k - H(x^k))\) and applying (32), we estimate
\[
\|H(x^{k+1}) - H(x^*)\| \leq \|H(z^{k+1}) - H(x^*)\| + \|H(x^{k+1}) - H(z^{k+1})\|
\leq \|H(z^{k+1}) - H(x^*)\| + \alpha_k \delta_k \|y^k - H(x^k)\|
= \|H(z^{k+1}) - H(x^*)\| + \delta_k \|H(x^{k+1}) - H(x^k)\|,
\]
and even if we drop the nonexpansiveness of \(H\) altogether without using Proposition 2.3. We further observe that even if we achieve linear convergence such that the nonexpansiveness of \(H\) can be avoided altogether without using Proposition 2.3. We further observe that even if we drop the nonexpansiveness of \(H\), and the Lipschitz continuity of \(M^{-1}\) at zero, the linear convergence seems to be achievable.

**Theorem 3.5.** Let \(X\) be a real Hilbert space, let \(H : X \to X\) be \((r)\)-strongly monotone, and let \(\{M^k\}, M : X \to 2^X\) be \(H\)-maximal monotone such that \(M^k \rightharpoonup M\) for \(k = 0, 1, 2, \ldots\). For an arbitrarily chosen initial point \(x^0\), suppose that the sequence \(\{x^k\}\) is generated by the hybrid super-relaxed proximal point algorithm
\[
H(x^{k+1}) = (1 - \alpha_k)H(x^k) + \alpha_k y^k \quad \forall k \geq 0,
\]
and \(y^k\) satisfies
\[
\|y^k - H(J_{\rho_k, H}(H(x^k)))\| \leq \delta_k \|y^k - H(x^k)\|,
\]
Finally, since \(H\) is \((r)\)-strongly monotone (and hence \((r)\)-expanding), \(\delta_k \to 0\), it follows from (33) that \(\{x^k\}\) converges linearly to \(x^*\) for
\[
\limsup \frac{\theta_k + \delta_k}{1 - \delta_k} = \limsup \theta_k = \sqrt{\frac{(1 - \alpha_k)^2 + (\alpha_k^2 + 2 \alpha_k (1 - \alpha_k) \gamma) \frac{1}{r^2}}{1}} < 1. \quad \square
\]
where \( J_{\rho_k,H}^M = (H + \rho_k M)^{-1} \), and
\[
\{\delta_k\}, \{\alpha_k\}, \{\rho_k\} \subseteq [0, \infty)
\]
are scalar sequences. Suppose that \( \{x^k\} \) is bounded in the sense that there is at least one solution to (1). Suppose, in addition, for \( \gamma > 1 \), that
\[
(H(u) - H(v), H(J_{\rho_k,H}^M(H(u))) - H(J_{\rho_k,H}^M(H(v)))) 
\geq \gamma \|H(J_{\rho_k,H}^M(H(u))) - H(J_{\rho_k,H}^M(H(v)))\|^2 \quad \forall u, v \in X,
\]
that is, \( HJ_{\rho_k,H}^M \) is \( (\gamma) \)-cocoercive. Then the sequence \( \{x^k\} \) converges linearly to a unique solution \( x^* \) of (1) with the convergence rate
\[
\sqrt{1 - \alpha(2(1 - \gamma D^2) - (1 - (2\gamma - 1)D^2)\alpha)} < 1,
\]
where \( \alpha_k^2 + 2\alpha_k(1 - \alpha_k)\gamma > 0 \), for \( \alpha_k \geq 1 \), \( \sum_{k=0}^{\infty} \delta_k < \infty \), \( \delta_k \to 0 \), \( \alpha = \limsup_{k \to \infty} \alpha_k \), \( \rho = \limsup_{k \to \infty} \rho_k \) and \( D = \frac{\gamma}{\gamma - 1} < 1 \).

**Proof.** Suppose that \( x^* \) is a zero of \( M \) for \( x^* \in X \). From Theorem 3.1, it follows that \( x^* \) is a fixed point of \( J_{\rho_k,H}^M H \). First, we express
\[
H(z^{k+1}) = (1 - \alpha_k)H(x^k) + \alpha_k H(J_{\rho_k,H}^M(H(x^k))).
\]
Since \( x^* \) is a solution to (1), we have in light of Theorem 3.1 that
\[
H(x^*) = (1 - \alpha_k)H(x^k) + \alpha_k H(J_{\rho_k,H}^M(H(x^k))).
\]
Now, applying Lemma 3.1, we find the estimate
\[
\|H(z^{k+1}) - H(x^*)\| = \|(1 - \alpha_k)H(x^k) + \alpha_k H(J_{\rho_k,H}^M(H(x^k)))\|
\]
\[
\begin{align*}
&\leq \|\alpha_k(H(J_{\rho_k,H}^M(H(x^k))) - H(J_{\rho_k,H}^M(H(x^*))))\| \\
&\leq \|\alpha_k(H(J_{\rho_k,H}^M(H(x^k))) - H(J_{\rho_k,H}^M(H(x^*))))\| \\
&\leq \|\alpha_k(H(J_{\rho_k,H}^M(H(x^k))) - H(J_{\rho_k,H}^M(H(x^*))))\| \\
&\leq \|\alpha_k(H(J_{\rho_k,H}^M(H(x^k))) - H(J_{\rho_k,H}^M(H(x^*))))\| \\
&\leq \|\alpha_k(H(J_{\rho_k,H}^M(H(x^k))) - H(J_{\rho_k,H}^M(H(x^*))))\| \\
&\leq \|\alpha_k(H(J_{\rho_k,H}^M(H(x^k))) - H(J_{\rho_k,H}^M(H(x^*))))\|
\end{align*}
\]
where
\[
f_k = \|H(J_{\rho_k,H}^M(H(x^k))) - H(J_{\rho_k,H}^M(H(x^*)))\| \to 0,
\]
and
\[
g_k = \|H(J_{\rho_k,H}^M(H(x^k))) - H(J_{\rho_k,H}^M(H(x^*)))\| \to 0.
\]
Then using (36), we have
\[
\|H(J_{\rho_k,H}^M(H(x^k))) - H(J_{\rho_k,H}^M(H(x^*)))\| \leq D\|H(x^k) - H(x^*)\|,
\]
(40)
where $D = \frac{1}{\gamma} < 1$ for $\gamma > 1$.

Now we continue the proof from (37), we estimate $(\alpha_k \geq 1)$

\[
\| (1 - \alpha_k)(H(x^k) - H(x^*)) + \alpha_k(H(J^M_{\rho_k, H}(H(x^k))) - H(J^M_{\rho_k, H}(H(x^*)))) \|^2
\]

\[
= (1 - \alpha_k)^2\|H(x^k) - H(x^*)\|^2 + 2\alpha_k(1 - \alpha_k)\|H(x^k) - H(x^*), H(J^M_{\rho_k, H}(H(x^k))) - H(J^M_{\rho_k, H}(H(x^*))) \|^2
\]

\[
+ \alpha_k^2\|H(J^M_{\rho_k, H}(H(x^k))) - H(J^M_{\rho_k, H}(H(x^*)))\|^2
\]

\[
\leq (1 - \alpha_k)^2\|H(x^k) - H(x^*)\|^2 + 2\alpha_k(1 - \alpha_k)\|H(x^k) - H(x^*), H(x^*)\|^2
\]

\[
+ \alpha_k^2\|H(x^k) - H(x^*)\|^2
\]

\[
\leq (1 - \alpha_k)^2\|H(x^k) - H(x^*)\|^2 + 2\alpha_k(1 - \alpha_k)\gamma\|H(J^M_{\rho_k, H}(H(x^k))) - H(J^M_{\rho_k, H}(H(x^*)))\|^2
\]

\[
\leq (1 - \alpha_k)^2\|H(x^k) - H(x^*)\|^2 + \gamma\|H(x^k) - H(x^*)\|^2
\]

\[
= [(1 - \alpha_k)^2 + (\alpha_k^2 + 2\alpha_k(1 - \alpha_k)\gamma)D^2]\|H(x^k) - H(x^*)\|^2,
\]

where $\alpha_k^2 + 2\alpha_k(1 - \alpha_k)\gamma > 0$ for $\alpha_k \geq 1$, $\gamma > \frac{1}{2}$ and $D = \frac{1}{\gamma} < 1$ for $\gamma > 1$.

Thus, we have

\[
\|H(z^{k+1}) - H(x^*)\| \leq \theta_k\|H(x^k) - H(x^*)\| + \alpha_kf_k + \alpha_kg_k,
\]

where

\[
\theta_k = \sqrt{(1 - \alpha_k)^2 + (\alpha_k^2 + 2\alpha_k(1 - \alpha_k)\gamma)D^2} < 1,
\]

and $\alpha_k^2 + 2\alpha_k(1 - \alpha_k)\gamma > 0$ for $\alpha_k \geq 1$, and $\gamma > 1$.

On the other hand, we have

\[
\|H(x^{k+1}) - H(z^{k+1})\|
\]

\[
= \|\alpha_k(y^k - H(J^M_{\rho_k, H}(H(x^k))))\| \leq \alpha_k\delta_k\|y^k - H(x^*)\|.
\]

Since using (34), $H(x^{k+1}) - H(x^k) = \alpha_k(y^k - H(x^k))$ and applying (41), we estimate

\[
\|H(x^{k+1}) - H(x^*)\| \leq \|H(z^{k+1}) - H(x^*)\| + \|H(x^{k+1}) - H(z^{k+1})\|
\]

\[
\leq \|H(z^{k+1}) - H(x^*)\| + \alpha_k\delta_k\|y^k - H(x^*)\|
\]

\[
= \|H(z^{k+1}) - H(x^*)\| + \delta_k\|H(x^{k+1}) - H(x^*)\|
\]

\[
\leq \theta_k\|H(x^k) - H(x^*)\| + \delta_k\|H(x^{k+1}) - H(x^*)\| + \delta_k\|H(x^k) - H(x^*)\|
\]

\[
+ \alpha_kf_k + \alpha_kg_k.
\]
It follows that
\[ \|H(x^{k+1}) - H(x^*)\| \leq \frac{\theta_k + \delta_k}{1 - \delta_k} \|H(x^k) - H(x^*)\| + \frac{\alpha_k}{1 - \delta_k} f_k + \frac{\alpha_k}{1 - \delta_k} g_k, \quad (43) \]
where
\[ \theta_k = \sqrt{[(1 - \alpha_k)^2 + (\alpha_k^2 + 2\alpha_k(1 - \alpha_k)\gamma)D^2]} < 1, \]
and \( \alpha_k^2 + 2\alpha_k(1 - \alpha_k)\gamma > 0 \) for \( \alpha_k \geq 1, \gamma > 1 \) and \( D = \frac{1}{\gamma} < 1 \).

Finally, since \( H \) is \((r)\)-strongly monotone (and, hence \((r)\)-expanding, i.e., \( \|H(u) - H(v)\| \geq r\|u - v\| \) for all \( u, v \in X \)), and \( \delta \to 0 \), it follows from (43) that \( \{x^k\} \) converges linearly to \( x^* \) for
\[ \limsup \frac{\theta_k + \delta_k}{1 - \delta_k} = \limsup \theta_k = \sqrt{[(1 - \alpha)^2 + (\alpha^2 + 2\alpha(1 - \alpha)\gamma)D^2]} < 1, \]
where \( D = \frac{1}{\gamma} < 1 \).

4. Concluding Remark

Note that the solution set for (1) coincides with that of the Yosida inclusion
\[ 0 \in M_{\rho}(x) \quad \text{for} \quad \rho > 0, \quad (44) \]
where \( M_{\rho} = \rho^{-1}(I - J_{\rho}^M) \) for \( J_{\rho}^M = (I + \rho M)^{-1} \), the Yosida regularization of \( M \), and \( \rho \) is a parameter, since \( M_{\rho}^{-1}(0) = M^{-1}(0) + \rho 0 \). The advantage with \( M^{-1} \) is that it may turn out to be locally monotone, while \( M \) may not.

Based on our construction, \( J_k = H - HoJ_{\rho,H}^M oH \), in Proposition 3.1, we introduce the generalized (nonlinear) Yosida regularization of \( M \) as
\[ M_{\rho} = \rho^{-1}(H - HoJ_{\rho,H}^M oH) \quad \text{for} \quad \rho > 0, \quad (45) \]
where \( J_{\rho,H}^M = (H + \rho M)^{-1} \), that reduces to the Yosida regularization for \( H = I \) (identity).

On the other hand, in a forthcoming communication, we have applied the generalized nonlinear Yosida regularization/approximation to the solvability of a class of first-order evolution inclusions. The theory is based on nonlinear maximal accretive/maximal monotone mappings and corresponding nonlinear nonexpansive semigroups on Hilbert spaces as well as on Banach spaces.

References


