International Journal of Pure and Applied Mathematics
Volume 61 No. 4 2010, 359-368

ON THE STRUCTURE OF THE BASIS INVERSE FOR THE TRANSPORTATION PROBLEM

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Abstract: The structure of the basis inverse for the transportation problem with inequality constraints is analyzed. The results derived from this highly structured network problem can provide insights about the characterization of other specially structured network problems, or more general problems containing network components, and have an impact on algorithmic development.

AMS Subject Classification: 05C75
Key Words: transportation basis inverse, spanning forest, bipartite graph, network problems

1. Introduction

The transportation problem is a specially structured minimum cost network flow problem that belongs to the larger class of linear programming problems. The solution to the problem yields an optimal distribution of goods from multiple source nodes to multiple destination nodes. Background information is given in Ahuja et al [1] or Dantzig [4], for example.

Many algorithms have been developed to solve the transportation problem. A network-based implementation of the simplex method, known as the network simplex method, was first developed by Dantzig; this approach and its variations have proven to be extremely efficient. Other classes of algorithms deal directly with the network structure. An extensive discussion of the various types of

Received: February 18, 2010

algorithms is given in [1]. It is well known that employing these algorithms for solving transportation problems is a much more efficient approach than using general linear programming software. However, computational problems can arise due to degeneracy. Consequently, there has been much research regarding criteria for selecting variables to enter the basis and also regarding effective starting solutions. A critical discussion of some of the computational issues can be found in [5].

Given the nature of current algorithms, it is typically unnecessary, from a computational perspective, to explicitly consider the basis matrix. However, understanding the basis inverse can influence pricing issues, basis updating, or approaches to degeneracy. For example, the basis inverse is explicitly employed when studying various postoptimal analysis methods for the transportation problem [2]. The knowledge could also provide insights for the study of algorithms for other specially structured problems. For example, general linear programming problems that contain a network component, or generalized network problems where there are gains or losses on the network, could benefit.

We consider the following transportation problem P1:

$$\min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \tag{1}$$

P1

$$\text{s.t.} \qquad \sum_{j=1}^{n} x_{ij} \leq a_i, \; i = 1, \ldots, m, \tag{2}$$

$$\sum_{i=1}^{m} x_{ij} \geq b_j, \; j = i, \ldots, n, \tag{3}$$

$$x_{ij} \geq 0, \; i = 1, \ldots, m, \; j = 1, \ldots, n. \tag{4}$$

The constraint matrix is referred to as the node-arc incidence matrix. It is assumed that $a_i > 0$ for all i, $b_j > 0$ for all j, and c_{ij} for all i and j are unrestricted in sign. We assume that a feasible solution to P1 exists.

P1 is defined on a bipartite directed network. The underlying graph is $G = (V, A)$ where $V = (V_1, V_2)$. V_1 contains the indices corresponding to m supply nodes u_i, $i = 1, \ldots, m$ and V_2 contains the indices corresponding to n demand nodes v_j, $j = 1, \ldots, n$. The set A consists of directed arcs (u_i, v_j) in the graph. In the network for P1, node u_i, $i \in V_1$ has a_i units of resource available and node v_j, $j \in V_2$ requires b_j units of resource. For each arc (u_i, v_j) there exists a unit cost c_{ij} associated with the flow x_{ij}. The objective is to find the minimum cost flow through the network.

Note that although P1 is defined on a complete bipartite network, the results hold for transportation problems defined on incomplete bipartite networks. We assume, without loss of generality, that the network is connected since, if it were not, a separate network problem could be solved for each connected component.

In Section 2 we review network definitions, and describe and characterize the problem we study. In Section 3 the structure of the basis inverse for transportation problems is analyzed.

2. Preliminaries

We begin by reviewing relevant network definitions, and then define the equality form of the transportation problem we study. Next we discuss the structure of the optimal basis network.

2.1. Network Definitions

A *chain* is a sequence of arcs connecting two nodes without any repetition of nodes. A *cycle* is a chain with an additional arc connecting the two end nodes. A *tree* is a connected graph that contains no cycle. A *rooted tree* is a tree with a specially designated node called a root. When referring to a rooted tree, we assume it is pictured vertically in the plane with the root as the lowest node and the tree extending upward. All immediate successors of a particular node are at the same vertical level. A non-root node with no successors is called a *leaf node*. A *rooted spanning tree* contains every node in the network. A *rooted spanning forest* is a collection of one or more rooted trees that, taken together, contain every node in the network. See Ahuja et al [1, pp. 26-29] and Bazaraa et al [3, p. 422] for additional background.

2.2. Equality Form of P1

When slack and surplus variables are added to P1, the following problem P2 results:

$$\min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \tag{5}$$

P2

$$\text{s.t.} \quad \sum_{j=1}^{n} x_{ij} + S_i = a_i, \; i = 1, \ldots, m, \tag{6}$$

$$\sum_{i=1}^{m} x_{ij} - T_j = b_j, \; j = 1, \ldots, n, \tag{7}$$

$$x_{ij} \geq 0, \; i = 1, \ldots, m, \; j = 1, \ldots, n, \tag{8}$$

$$S_i \geq 0, \; i = 1, \ldots, m, \tag{9}$$

$$T_j \geq 0, \; j = 1, \ldots, n. \tag{10}$$

2.3. Optimal Basis

The following discussion is limited to network flow problems containing slack or surplus variables. It is well known that there is a one-to-one correspondence between bases of minimum cost flow problems and rooted spanning trees or forests (see Ahuja et al [1, pp. 442-443], for example). By convention, a tree in a basis network will be rooted at the node whose corresponding slack or surplus variable is in the basis. Henceforth, we say that the slack or surplus variable associated with the root node belongs to that tree. The associated slack or surplus variable is represented as an arc that extends below the root node, and is not connected to any other node.

Since the constraints of P1 are inequalities, the network corresponding to the optimal basis for P2 will be a rooted spanning forest possibly containing more than one tree. When the basis network is a rooted spanning forest consisting of τ rooted trees, we denote the trees as $\mathcal{T}^t$, $t = 1, \ldots, \tau$. Henceforth, for ease of exposition, when we refer to a forest, it is understood that it is a rooted spanning forest such that each tree contains the arc corresponding to the associated slack or surplus variable.

Denote the optimal basis for P2 as B. The rows and columns of the node-arc incidence matrix of any basis from P2 can be rearranged to lower triangular format. When B is in lower triangular format, it is denoted $\hat{B}$. If the associated basis network contains more than one tree, $\hat{B}$ will be block triangular.

In an optimal basic solution to P2, it is possible that $S_i = a_i$ for some i, say, $i = k$. Then $x_{kj} = 0$ for all j, and it may be that none of these x_{kj} is basic. In this case, the corresponding rooted tree would consist of a single supply node u_k. However, a tree consisting of a single demand node v_j is impossible, since we assume $b_j > 0$, $j = 1, \ldots, n$, and $T_j = -b_j < 0$ violates nonnegativity requirements.

Example. The following example clarifies definitions and concepts. Assume that there are 5 supply nodes and 8 demand nodes, and that the basic variables in the optimal solution to P2 are: x_{11}, x_{12}, x_{17}, x_{26}, x_{28}, x_{33}, x_{34}, x_{35}, x_{41}, x_{44}, x_{52}, S_4, T_8. Since both S_4 and T_8 are in the optimal basis, the corresponding optimal basis network is a forest containing two trees. $\mathcal{T}^1$ is rooted at u_4 and $\mathcal{T}^2$ is rooted at v_8. The basis network is presented in Figure 1.

Basis matrices B, and $\hat{B}$ are given in Figure 2. In B, the columns correspond in order to the variables: x_{11}, x_{12}, x_{17}, x_{26}, x_{28}, x_{33}, x_{34}, x_{35}, x_{41}, x_{44}, x_{52}, S_4, T_8. The 13 rows correspond in order to the nodes: $u_1, \ldots, u_5$, $v_1, \ldots, v_8$.

In $\hat{B}$ the rows and columns correspond to the trees. The columns correspond in order to the variables: x_{17}, x_{52}, x_{12}, x_{11}, x_{41}, x_{33}, x_{35}, x_{34}, x_{44}, S_4, x_{26}, x_{28}, T_8. The rows correspond in order to the nodes: v_7, u_5, v_2, u_1, v_1, v_3, v_5, u_3, v_4, u_4, v_6, u_2, v_8.

3. Structure of the Basis Inverse for Transportation Problems

The following four propositions provide characteristics of the basis inverse for transportation problems. Note, it is well known that the elements of B^{-1} are only $+1$, -1, or 0 (see, for example, [1, Section 11.12] or [3, Section 10.2]).

Proposition 3.1. *Let α be any row of B^{-1}. Then α can be partitioned into two components $\alpha = (\alpha^1, \alpha^2)$ corresponding to V_1 and V_2 such that one of the two following statements holds:*

i) $\alpha_k^1 = +1$ *or* 0 *for* $k = 1, ..., m$ *and* $\alpha_k^2 = -1$ *or* 0 *for* $k = 1, ..., n$,

or

ii) $\alpha_k^1 = -1$ *or* 0 *for* $k = 1, ..., m$ *and* $\alpha_k^2 = +1$ *or* 0 *for* $k = 1, ..., n$.

Proof. Let p_{ij} be the column vector corresponding to variable x_{ij}. If α_i^1 and α_j^2 are of the same sign, αp_{ij} will equal ± 2, which is impossible since αp_{ij} can only equal 0, $+1$, or -1 (see [6, p. 280] or [3, p. 481]). Thus, either it is the case that α_i^1 and α_j^2 are of opposite sign or both zero and $\alpha p_{ij} = 0$; otherwise, it is the case that $\alpha_i^1 + \alpha_j^2 = \pm 1$ and $\alpha p_{ij} = \pm 1$. A similar argument applies to all slack and surplus variables. □

Proposition 3.2. *Let α be a row in B^{-1} corresponding to either a slack or surplus variable associated with a tree in the basis, say tree T^k, then* $\alpha_i^1 = 1$

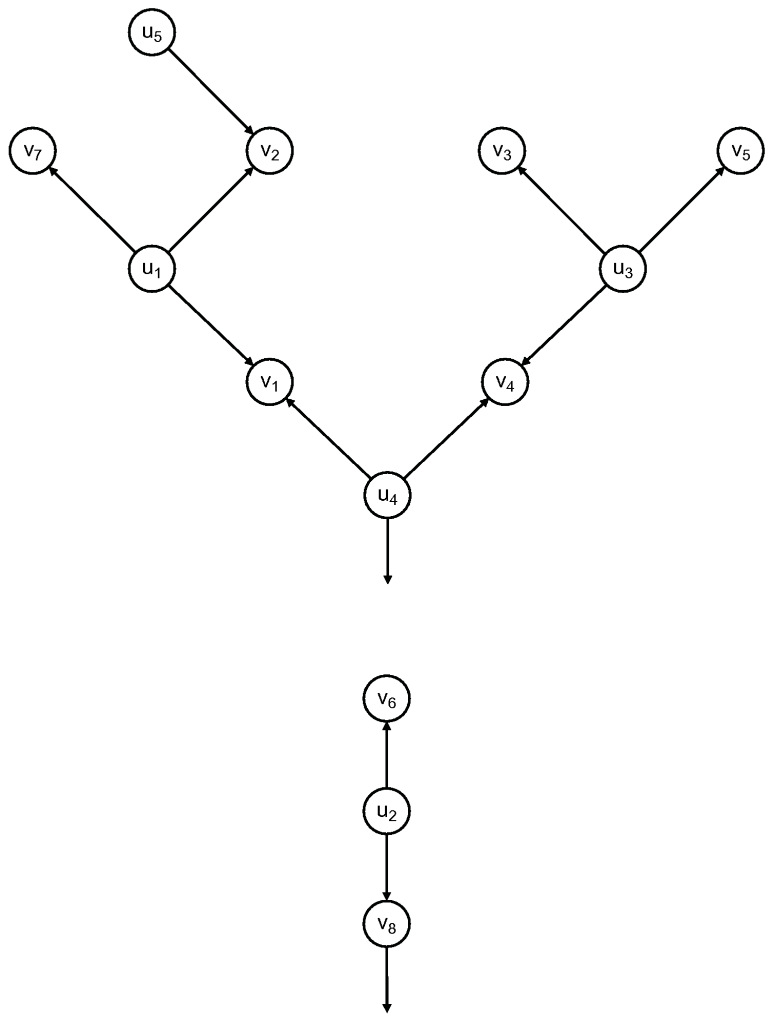

Figure 1: Rooted spanning forest containing trees $\mathcal{T}^1$ and $\mathcal{T}^2$

for all nodes $u_i \in T^k$, *and* $\alpha_j^2 = -1$ *for all nodes* $v_j \in T^k$ *where* $i \in V_1$ *and* $j \in V_2$.

Proof. Without loss of generality assume α corresponds to a slack variable S_i. Since $\alpha S_i = 1$ it follows that $\alpha_i^1 = 1$. Let p_{ij}^B be the column vector corresponding to the basic variable x_{ij}. Then necessarily $\alpha p_{ij}^B = 0$ for all p_{ij}^B. Consequently, α_i^1 and α_j^2, where u_i and v_j are adjacent nodes in tree T^k, are

$$B = \begin{pmatrix}
1 & 1 & 1 & & & & & & & & & & \\
 & & & 1 & 1 & & & & & & & & \\
 & & & & & 1 & 1 & 1 & & & & & \\
 & & & & & & & & 1 & 1 & & 1 & \\
 & & & & & & & & & & 1 & & \\
1 & & & & & & & & 1 & & & & \\
 & 1 & & & & & & & & & 1 & & \\
 & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & & 1 & & & \\
 & & & & & & & 1 & & & & & \\
 & & & 1 & & & & & & & & & \\
 & & 1 & & & & & & & & & & \\
 & & & & 1 & & & & & & & & -1
\end{pmatrix}$$

$$\hat{B} = \begin{pmatrix}
1 & & & & & & & & & & & & \\
 & 1 & & & & & & & & & & & \\
 & 1 & 1 & & & & & & & & & & \\
1 & & 1 & 1 & & & & & & & & & \\
 & & & 1 & 1 & & & & & & & & \\
 & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & 1 & 1 & & & & & \\
 & & & & & & & 1 & 1 & & & & \\
 & & & & 1 & & & & 1 & 1 & & & \\
 & & & & & & & & & & 1 & & \\
 & & & & & & & & & & 1 & 1 & \\
 & & & & & & & & & & & 1 & -1
\end{pmatrix}$$

Figure 2: Basis matrices B and $\hat{B}$

of opposite sign. Since we have a bipartite graph and the tree is connected, Proposition 3.2 follows. □

Proposition 3.3. *When the basis contains t surplus variables:*

i) Each of the t rows of B^{-1} corresponding to the surplus variables has at least one element $\alpha_j^2 = -1$ and at least one element $\alpha_k^1 = +1$.

ii) Of the remaining $m+n-t$, rows there exist m rows such that α^1 has at least one element $\alpha_k^1 = +1$ and $n-t$ rows such that α^2 has at least one element

$\alpha_j^2 = +1$.

Proof. i) Arbitrarily select one of the t rows of B^{-1} corresponding to a surplus variable T_j. Denote this row as α. It follows from Proposition 3.2 that $\alpha_j^2 = -1$. Since a tree consisting of a single demand node v_j is not possible, it follows from Proposition 3.2 that there will be at least one element $\alpha_k^1 = +1$.

ii) Since $\hat{B}^{-1}\hat{B} = I$, it is evident that the $m+n-t$ remaining rows in $\hat{B}^{-1}$ contain a $+1$ on the diagonal. The $m+n-t$ columns of $\hat{B}^{-1}$ associated with these diagonal elements correspond to m nodes in V_1 and $n-t$ nodes in V_2, although not partitioned. □

Proposition 3.4. *Let $\alpha = (\alpha^1, \alpha^2)$ be any row of B^{-1}. If either $\alpha^1 = 0$ or $\alpha^2 = 0$, then α is a unit vector.*

Proof. Without loss of generality, assume $\alpha^2 = 0$. Then, if α is not a unit vector, by Proposition 3.3 *(ii)* and Proposition 3.1, α will contain at least two $+1$'s corresponding to, say, nodes u_i, u_s, $i, s \in V_1$. Recalling that when the basis is a forest $\hat{B}^{-1}$ is block triangular, we assume that row α corresponds to an arc in tree T^k; hence, nodes u_i and u_s are in tree T^k. Since u_i and u_s are connected by a unique path, there exists at least one node $v_j \in V_2$ on the path between them. Assume nodes v_j and v_t, $j, t \in V_2$, are adjacent to nodes u_i and u_s respectively, where possibly $j = t$. Then x_{ij} and x_{st} are basic variables; hence, both $\alpha p_{ij}^B = 1$ and $\alpha p_{st}^B = 1$, which is impossible since $B^{-1}B = I$. □

From Proposition 3.4 we have the following:

Remark 3.1. When α is a unit vector the node to which the $+1$ coefficient corresponds has only one edge incident to it; that is, the node must be a leaf node.

The inverse $\hat{B}^{-1}$ corresponding to the basis B for the example in Section 2.4 is shown in Figure 3. Note that the rows and columns of $\hat{B}^{-1}$ correspond to the trees. The rows correspond in order to the variables: x_{17}, x_{52}, x_{12}, x_{11}, x_{41}, x_{33}, x_{35}, x_{34}, x_{44}, S_4, x_{26}, x_{28}, T_8. The columns correspond in order to the nodes: v_7, u_5, v_2, u_1, v_1, v_3, v_5, u_3, v_4, u_4, v_6, u_2, v_8.

4. Conclusion

This analysis of the structure of the basis inverse for the inequality form of the transportation problem provides insights regarding this form of the transportation problem. It could be particularly useful for the analysis of algorithms for

$$\hat{B}^{-1} = \begin{pmatrix}
1 &&&&&&&&&&&& \\
& 1 &&&&&&&&&&& \\
& -1 & 1 &&&&&&&&&& \\
-1 & 1 & -1 & 1 &&&&&&&&& \\
1 & -1 & 1 & -1 & 1 &&&&&&&& \\
&&&&& 1 &&&&&&& \\
&&&&&& 1 &&&&&& \\
&&&&& -1 & -1 & 1 &&&&& \\
&&&&& 1 & 1 & -1 & 1 &&&& \\
-1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 &&& \\
&&&&&&&&&& 1 && \\
&&&&&&&&&& -1 & 1 & \\
&&&&&&&&&& -1 & 1 & -1
\end{pmatrix}$$

Figure 3: Basis inverse $\hat{B}^{-1}$

network and other specially structured problems containing bipartite network components.

Acknowledgments

Partial financial support under PSC-CUNY Research Awards #669357 and #624260031.

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