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RANKS AND BORDER RANKS: SOME INVARIANTS TO BOUND THEM FOR VERONESE EMBEDDINGS

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Abstract: Let $X_{m,d} \subset \mathbb{P}^{n_{m,d}}$, $n_{m,d} := \binom{m+d}{m} - 1$, be the Veronese embedding of order d of $\mathbb{P}^m$. For any $P \in \mathbb{P}^{n_{m,d}}$ the $X_{m,d}$-rank $r_{X_{m,d}}(P)$ of P is the minimal cardilaty of $S \subset X_{m,d}$ such that $P \in \langle S \rangle$. Fix a zero-dimensional scheme $Z \subset X_{m,d}$. Here we give a way to find upper bounds for all $r_{X_{m,d}}(P)$, $p \in \langle Z \rangle$ in terms of geometrical, numerical or cohomological properties of Z.

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1. Introduction

For all integers $m \geq 1$ set $n_{m,d} := \binom{m+d}{m}$ and $Y_m := \mathbb{P}^m$. Let $j_{m,d} : Y_m \hookrightarrow \mathbb{P}^{n_{m,d}}$ denote the Veronese embedding of Y_m by the complete linear system $|\mathcal{O}_{Y_m}(d)|$. Set $X_{m,d} := j_{m,d}(Y_m)$. For any zero-dimensional scheme $Z \subset Y_m$ let $z(Z)$ denote the minimal integer k such that there is a reduced degree hypersurface W of Y_m containing Z. We often use $j_{m,d}$ to to define the integer $z(Z)$ when Z is seen as a subscheme of $X_{m,d}$. Let $k(Z)$ be the minimal integer k such that $h^1(\mathcal{I}_Z(k)) = 0$. We also introduce the following integer $w_d(Z)$, which is essential if $m \geq 3$. Fix a zero-dimensional scheme $Z \subset Y_m$. Let $w_d(Z)$ be the minimal integer $\dim(\langle T \rangle)$, where $T \subset X_{m,d}$ is a connected curve containing $j_{m,d}(Z)$. We hope that these integers will be useful, but we are only able to give

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a few easy remarks concerning them and their use. Take any $P \in \langle j_{m,d}(Z)\rangle$. Fix T computing $w_d(Z)$ and a take a general hyperplane V of $W := \langle T\rangle$. The proof of Proposition 1 below gives $r_{X_{m,d}}(P) \le w_d(Z)$.

Example 1. Take $m = 2$. Fix an integer $x \ge 4$, a line $L \subset Y_2$, a set $S \subset L$ such that $\sharp(S) = x$ and any finite set $S' \subset L \setminus S$. For each $Q \in S$ fix a length 2 zero-dimensional scheme v_Q with Q as its reduction and not contained in L (i.e. a tangent vector at Q not tangent to L). Set $Z := \cup_{Q\in S} v_Q$ and $Z_1 := Z \cup S'$. Thus length$(Z) = 2k$ and length$(Z_1) = 2k + \sharp(S')$. The double line $2L$ contains Z and Z_1, but it is not reduced. Let $W' \subset Y_2$ be any smooth degree k plane curve containg S. Set $W := L \cup W'$. Thus W is a reduced plane curve of degree $k+1$ containing Z_1 and henece Z. It is easy to check that $z(Z) = z(Z_1) = k+1$.

Remark 1. Let $Z \subset Y_m$ be a zero-dimensional scheme. Fix an integer k such that $h^1(Y_m, \mathcal{I}_Z(k)) = 0$. Castelnuovo-Mumford's Lemma says that the homogeneous ideal of Z is generated by forms of degree at most $k+1$. Thus $z(Z) \le k+1$. Hence $z(Z) \le k(Z)+1$.

Proposition 1. *Fix a zero-dimensional scheme $W \subset Y_m$. Set $Z := j_{m,d}(W)$ and $k := z(W)$. Assume $k < d$. Then $r_{X_{m,d}}(P) \le \binom{m+d}{m} - \binom{m+d-k}{m} - m + 1$ for all $P \in \langle Z\rangle$.*

Proof. Fix any $P \in \langle Z\rangle$. If $P \in X_{m,d}$, then $r_{X_{m,d}}(P) = 1$. Hence we may assume $P \notin X_{m,d}$. Let $T \subset Y_m$ be a reduced hypersurface of degree k containing W. Set $U := j_{m,d}(T)$. By assumption U is a reduced and connected complete intersection of dimensioin $m-1$ and $P \in \langle U\rangle$. Since T is a degree k hypersurface and $k < d$, the restriction map $H^0(Y_{m,d}, \mathcal{O}_{Y_{m,d}}(d)) \to H^0(T, \mathcal{O}_T(d))$ is surjective and its kernel has dimension $\binom{m+d}{m} - \binom{m+d-k}{m}$. Thus $\dim(\langle U\rangle) = \binom{m+d}{m} - \binom{m+d-k}{m} - 1$. Let $V \subset \langle U\rangle$ be a general linear subspace of codimension $m-1$ containing P. Since $P \notin U$, Bertini's Theorem says that $V \cap U$ is a reduced set of points. To get $r_{X_{m,d}}(P) \le \dim(V)+1$ (and hence prove Proposition 1) it is sufficient to prove that $U \cap V$ spans V. This is not quite obvious, because U is not assumed to be irreducible. Let $W \subset \langle U\rangle$ be a general linear subspace of codimension $m-2$ containing P. We may see V as a general hyperplane of V. Bertini's Theorem gives that $U \cap W$ is a reduced curve. Since $U \cap W$ (seen as a subcurve of Y_m) is a complete intersection, it is connected. Hence $h^1(W, \mathcal{I}_{W\cap U}) = 0$. Thus the exact sequence of coherent sheaves on W:

$$0 \to \mathcal{I}_{W\cap U} \to \mathcal{I}_{W\cap U}(1) \to \mathcal{I}_{V\cap U}(1) \to 0 \tag{1}$$

gives the surjectivity of the restriction map $\rho : H^0(W, \mathcal{I}_{W\cap U}(1)) \to H^0(V \cap U, \mathcal{I}_{V\cap U}(1))$. Since $W \cap U$ spans U, we have $H^0(W, \mathcal{I}_{W\cap U}(1)) = 0$. Hence the surjectivity of ρ gives $H^0(V \cap U, \mathcal{I}_{V\cap U}(1)) = 0$. Thus $V \cap U$ spans U. $\square$

Lemma 1. *Let $Z \subset Y_m$, $m \geq 2$, be a zero-dimensional scheme. Set $z := \text{length}(Z)$. Then $k(Z) \leq z - 1$ and equality holds if and only if there is a line $L \subset Y_m$ such that $Z \subset L$.*

Proof. Use [1], Lemma 4.1. □

Corollary 1. *If $k(Z) \geq z - 1$, then either Z is reduced and $b_{X_{m,d}}(P) = r_{X_{m,d}}(P) = z$ or Z is not reduced, $b_{X_{m,d}}(P) = z$ and $r_{X_{m,d}}(P) = d + 2 - z$.*

Proof. Apply Lemma 1 and the description of the ranks with respect to rational normal curves ([3], [6], Theorem 4.1). □

To get more we need to distinguish several cases.

Lemma 2. *Fix positive integers z, d, x such that $d \geq x^2$. Let $Z \subset Y_2$ be a length z zero-dimensional scheme. Assume $k(Z) \geq x - 2 + d/x$. Then either $z \equiv 0 \pmod{x}$, $k(Z) = x - 2 + d/x$, $z = d(x - 2 + k(Z))/x$ and Z is the complete intersection of a curve of degree x and a curve of degree $z - 2 + k(Z)$ or there is an integer $t \in \{1, \dots, x\}$ and a plane curve T of degree t such that $t(k(Z) - 1 + (5 - t)/2) \geq \text{length}(T) \geq t(k(Z) - t + 2)$.*

Proof. The integer $k(Z) - 1$ is the integer τ of [5], Corollary 2. Apply it with $z := d$ and $s := x$. □

Lemma 3. *Let $Z \subset Y_m$, $m \geq 2$, be a zero-dimensional scheme. Set $z := \text{length}(Z)$ and assume $k(Z) \geq (z - m + 2)/3$. Then either there is a line $D \subset Y - m$ such that $\text{length}(Z \cap D) = k(Z) + 1$ or there is a smooth conic C such that $\text{length}(C \cap Z) = 2 \cdot k(Z)$.*

Proof. If $m = 2$, then apply the case $x = 2$ of Lemma 2. Now assume $m > 2$ and that the lemma is true for Y_{m-1}. Let $H \subset Y_m$ be a hyperplane such that $\text{length}(Z \cap H)$ is maximal. Set $Z' := Z \cap H$ and $z' := \text{length}(Z')$. If $Z' = Z$, then we easily conclude by the inductive assumption. Hence we may assume $Z' \neq Z$, i.e. that Z spans Y_m. The maximality of z' implies $z' \geq m$. Consider the exact sequence of the residual scheme

$$0 \to \mathcal{I}_{\text{Res}_H(Z)}(t-1) \to \mathcal{I}_Z(t) \to \mathcal{I}_{Z',H}(t) \to 0. \quad (2)$$

Since $\text{Res}_H(Z)$ is zero-dimensional, $h^2(Y_m, \mathcal{I}_{\text{Res}_H(Z)}(t-1)) = 0$ for all t. Hence (2) gives $k(Z') \leq k(Z)$. From (2) we get $k(Z) \geq \max\{k(Z'), k(\text{Res}_H(Z)) + 1\}$. Thus either $h^1(H, \mathcal{I}_{Z',H}(k(Z)-1) > 0$ and $k(Z') = k(Z)$ or $h^1(\mathcal{I}_{\text{Res}_H(Z)}(k(Z)-2)) > 0$. First assume $h^1(H, \mathcal{I}_{Z',H}(k(Z)-1) > 0$ and $k(Z') = k(Z)$. If $z' - m + 3 \leq 3 \cdot k(Z)$, then we may use the inductive assumption to get a line D or a smooth conic C as in the statement and contained in H. Thus we may assume

$z'-m+2 \geq 3\cdot k(Z)$. Since $z-m+2 \leq 3\cdot k(Z)$ and $z' < z$, we get a contradiction. Now assume $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(k(Z)-2)) > 0$. The maximality of Z gives $z' \geq m$ with equality if and only if Z is in linearly general position. Hence by Remark 2 below we may assume $z' \geq m+1$. Since length$(\mathrm{Res}_H(Z)) = z - z'$, we get the existence of a line L' or a conic C' for the scheme $\mathrm{Res}_H(Z)$. However, L' or C' must be contained in a hyperplane. The maximality of z' gives $z' \geq z/2$ and hence L' or C' must be contained in H. Since $k(Z') \leq k(Z)$, we came to the situation analyzed before. □

To simplify the next statement we introduce the following short-hands related to the cohomology groups of line bundles on complete intersections of $m-1$ degree t hypersurfaces of Y_m. Fix the integer $m \geq 2$. For all integers u set $\alpha_t(m,u) := \binom{m+u}{m}$ if $u \geq 0$ and $\alpha_t(m,u) := 0$ if $u < 0$. Then for all integers x such that $1 \leq x \leq m-1$ define recursively the integer $\alpha(x,u)$ by the formula $\alpha_t(x,u) := \alpha_t(x+1,u) - \alpha_t(x+1,u-t)$. Each integer $\alpha(1,u)$ is an alternating sum of certain binomial couefficients $\binom{m+y}{m}$, except that we need to use the convention $\binom{m+y}{m} := 0$ if $y < 0$.

Proposition 2. *Assume* $\mathrm{char}(\mathbb{K}) = 0$ *and set* $t := k(Z)+1$. *Assume* $t < d$. *Then* $w_d(Z) \leq \alpha_t(1,d)$.

Proof. By Castenuovo-Mumford's Lemma the homogeneous ideal of Z in Y_m is generated by forms of degree at most t. Hence Bertini's Theorem shows that the complete intersection C of $m-1$ general hypersurfaces of degree t containing Z is smooth outside Z. Since a complete intersection has no embedded component, C must be reduced. As any complete intersection C is connected. Since C is a complete intersection, the restriction map $H^0(Y_m, \mathcal{O}_{Y_m}(d)) \to H^0(C, \mathcal{O}_C(d))$ is surjective. Hence $\dim(\langle j_{m,d}(C)\rangle) = h^0(C, \mathcal{O}_C(d)) - 1$. The well-known cohomology of line bundles on a complete intersection concludes the proof. □

Proposition 3. *Assume* $\mathrm{char}(\mathbb{K}) = 0$. *Fix integers* $s \geq 1$ *and* $z_i \geq 1$, $1 \leq i \leq s$, *and set* $z := z_1 + \cdots + z_s$. *Let* t *be the minimal positive integer such that* $\binom{m+t}{m} \geq z$. *Let* $Z \subset Y_m$ *be a general curvilinear subscheme with* s *connected components with lengths* $z_1, \dots, z_s$. *Then* $k(Z) = t$. *If* $\binom{m+t}{m} \geq z+m-1$, *then* Z *is contained in a smooth and conneced complete intersection of* $m-1$ *hypersurfaces of degree* t. *We have* $w_d(Z) \leq \alpha_t(1,d)$ *for all integers* $d > t$.

Proof. The equality $k(Z) = t$ is just [2] and the cohomology of line bundles on Y_m. We also get $h^0(\mathcal{I}_Z(t) = \binom{m+t}{m} - z$. Assume $\binom{m+t}{m} \geq z+m-1$. A general intersection of $m-1$ general hypersurfaces of degree t is smooth and connected. Since the set Φ of all curvilinear subschemes of Y_m with s connected components with lengths $z_1, \dots, z_s$ is irreducible, for a general $Z \in \Phi$ we get that the intersection of $m-1$ general elements of $|\mathcal{I}_Z(t)|$ is smooth and connected. The last assertion follows from Proposition 2. □

We say that zero-dimensional scheme $Z \subset Y_m$ is in *linearly general position* if length$(Z \cap V) \le \dim(V) + 1$ for every proper linear subspace $V \subsetneq Y_m$.

Remark 2. Assume that $Z \subset Y_m$ is in linearly general position. If $z := \text{length}(Z) \le m$, then $z(Z) = 1$. If $z > m$, then $z(Z) \le \lceil (z-1)/m \rceil$ ([4], Theorem 3.2). Set $t := 1 + \lceil (z-1)/m \rceil$. Proposition 3 gives $w_d(Z) \le \alpha_t(1, d)$.

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