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ON THE POSTULATION OF A GENERAL UNION OF TRIPLE POINTS AND DOUBLE POINTS IN $\mathbb{P}^4$

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Abstract: Fix integers d, x, y such that $d \ge 8$, $x \ge 0$, $y \ge 0$. Let $Z \subset \mathbb{P}^4$ be a general union of x triple points and y double points. Here we prove that either $h^1(\mathbb{P}^4, \mathcal{I}_Z(d)) = 0$ (case $15x + 5y \le \binom{d+4}{4}$) or $h^0(\mathbb{P}^4, \mathcal{I}_Z(d)) = 0$ (case $15x + 5y \ge \binom{d+4}{4}$). When $d = 7$ we are only able to prove the corresponding result assuming $(x, y) \notin \{(22, 0), (21, 3)\}$.

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1. Introduction

In [4] we considered the postulation of general unions of double and triple points in $\mathbb{P}^3$. Here we look at $\mathbb{P}^4$ and prove the following results.

Theorem 1. *Fix non-negative integers d, x, y such that $d \ge 8$. Let $Z \subset \mathbb{P}^4$ be a general union of x triple points and y double points. Then either $h^1(\mathbb{P}^4, \mathcal{I}_Z(d)) = 0$ (case $15x+5y \le \binom{d+4}{4}$) or $h^0(\mathbb{P}^4, \mathcal{I}_Z(d)) = 0$ (case $15x+5y \ge \binom{d+4}{4}$).*

In the case $d = 7$ we are only able to prove the following result.

Proposition 1. *Fix non-negative integers x, y such that $(x, y) \notin \{(22, 0), (21, 3)\}$. Let $Z \subset \mathbb{P}^4$ be a general union of x triple points and y double points.*

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Then either $h^1(\mathbb{P}^4, \mathcal{I}_Z(7)) = 0$ *(case* $15x+5y \le 330$*) or* $h^0(\mathbb{P}^4, \mathcal{I}_Z(7)) = 0$ *(case* $15x+5y \ge 330$*).*

In a previous version of this paper we used a preliminary version of [4]. Since in the final version of [4] a strong restriction occurs ($d \ge 15$) we substitue some of its quotations with either new lemmas or *Macaulay 2* (Remark 2).

By [3] for all integers $n \ge 2$, $m \ge 2$ there is an integer $d(n,m)$ such that for all integers $d \ge d(n,m)$ either $h^1(\mathbb{P}^n, \mathcal{I}_Z(d)) = 0$ or $h^0(\mathbb{P}^n, \mathcal{I}_Z(d)) = 0$, where Z is a general union of a prescribed numbers of fat points with multiplicities $\le m$. Call $\delta(n,m)$ the minimal such integer $d(n,m)$. By the case $m=2$ (i.e. the Alexander-Hirschowitz Theorem (see [1], [2], [10], [7]) it is natural to make the following conjecture.

Conjecture 1. *Is there an integer* $d(m)$ *such that* $\delta(n,m) \le d(m)$ *for all* n,m*?*

It may be easier to prove the existence of easily described functions $\eta(m)$, $\gamma(n)$ such that $\delta(n,m) \le \eta(m) + \gamma(n)$ or degree 1 functions $a_1(n), a_2(n)$ such that $\delta(n,m) \le a_1(n)m + a_2(n)$. We failed to prove a similar result (with perhaps $d \ge n+3m$) in an arbitrary $\mathbb{P}^n$, because we do not know how to analyze infinitely many initial cases. It may be possible with the existing technology to do a few single cases. We stress that our main tool is the Differential Horace Method proved in [3]. In parts (a) and (b) of the proof of Proposition 1 we will discuss why we failed to prove Proposition 1 in the cases $(x,y) = (22,0)$ and $(x,y) = (21,3)$ (we only could arrive at $h^0(\mathbb{P}^4, \mathcal{I}_Z(7)) = h^1(\mathbb{P}^4, \mathcal{I}_Z(7)) \le 1$).

We work over an algebraically closed field $\mathbb{K}$ such that $\mathrm{char}(\mathbb{K}) = 0$.

2. Preliminary Results

For any smooth and connected quasi-projective variety, any integer $m > 0$ and any $P \in A$ let $\{mP, A\}$ denote the infinitesimal neighborhood of order $m-1$ of P in A, i.e. the closed subscheme of A with $(\mathcal{I}_{P,A})^m$ as its ideal sheaf. Hence $\{mP, A\}$ is zero-dimensional, $\{mP, A\}_{red} = \{P\}$ and $\mathrm{length}(\{mP, A\}) = \binom{n+m-1}{n}$, where $n := \dim(A)$. We will often write mP instead of $\{mP, \mathbb{P}^n\}$ (mainly when $n = 4$).

For any closed subscheme Z of any quasi-projective scheme A and every effective Cartier divisor D of A let $\mathrm{Res}_D(Z)$ denote the residual scheme of Z with respect to D, i.e. the closed subscheme of A with $\mathcal{I}_{Z,A} : \mathcal{I}_{D,A}$ as its ideal sheaf. If Z is zero-dimensional, then $\mathrm{length}(Z \cap D) + \mathrm{length}(\mathrm{Res}_D(Z)) = \mathrm{length}(Z)$. If

$P \in D_{reg}$, then $\{mP, A\} \cap D = \{mP, D\}$ and $\mathrm{Res}_D(\{mP, A\}) = \{(m-1)P, A\}$ for any integer $m > 0$, with the convention $\{0P, A\} := \emptyset$.

We will often use the following elementary form of the so-called Horace Lemma.

Lemma 1. *Let $H \subset \mathbb{P}^n$ be a hyperplane and $Z \subset \mathbb{P}^n$ a closed subscheme. Then:*

(a) $h^0(\mathbb{P}^n, \mathcal{I}_Z(d)) \le h^0(\mathbb{P}^n, \mathcal{I}_{\mathrm{Res}_H(Z)}(d-1)) + h^0(H, \mathcal{I}_{Z\cap H}(d))$;

(b) $h^1(\mathbb{P}^n, \mathcal{I}_Z(d)) \le h^1(\mathbb{P}^n, \mathcal{I}_{\mathrm{Res}_H(Z)}(d-1)) + h^1(H, \mathcal{I}_{Z\cap H}(d))$.

Proof. By the very definition of residual scheme with respect to H, there is the following exact sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_H(Z)}(d-1) \to \mathcal{I}_Z(d) \to \mathcal{I}_{Z\cap H}(d) \to 0 \qquad (1)$$

whose long cohomology exact sequence proves the lemma. □

The following result (called the Differential Horace Lemma) is a very particular case of [3], Lemma 2.3 (see in particular Figure 1 at p. 308).

Remark 1. Here we use that $\mathrm{char}(\mathbb{K}) \notin \{2, 3\}$. First we consider double points. Let $H \subset \mathbb{P}^n$ be a hyperplane and $Z \subset \mathbb{P}^n$ a closed subscheme not containing H. Fix a general $P \in H$. Let W be the union of Z with a general double point and M the union of Z with a general triple point. Fix $i \in \{0, 1\}$. To show that $h^i(\mathbb{P}^3, \mathcal{I}_W(t)) = 0$ it is sufficient to show that $h^i(H, \mathcal{I}_{(Z\cap H)\cup\{P\}}(t)) = h^i(\mathbb{P}^n, \mathcal{I}_{\mathrm{Res}_H(Z)\cup\{2P,H\}}(t-1)) = 0$. We will quote this trick as an application of Remark 1 at (P, H) with respect to the sequence $(1, n)$. Now we consider triple points. Notice that $3P \cap H = \{3P, H\}$, $\mathrm{Res}_H(3P) = 2P$ and that $\mathrm{Res}_H(2P) = \{P\}$. Hence $\mathrm{length}(3P \cap H) = (n+1)n/2$, $\mathrm{length}(\mathrm{Res}_H(3P) \cap H) = n$ and $\mathrm{length}(\mathrm{Res}_H((\mathrm{Res}_H(3P)) \cap H) = 1$. To prove $h^i(\mathbb{P}^n, \mathcal{I}_M(t)) = 0$, it is sufficient to do the same in which instead of the triple point we add one of the two following virtual schemes A_1 or A_2. A_1 has the same length as $3P$, but first we insert P on H, then $\{3P, H\}$ and only at the third step $\{2P, H\}$. Hence $\mathrm{length}(A_1 \cap H) = 1$, $\mathrm{length}(\mathrm{Res}_H(A_1) \cap H) = 6$ and $\mathrm{Res}_H(\mathrm{Res}_H(A_1)) = \{2P, A\}$. A_2 has the same length as $3P$, but first we insert $\{2P, H\}$, then $\{3P, H\}$ and only at the third step P. Hence $A_2 \cap H = \{2P, H\}$, $\mathrm{Res}_H(A_2) \cap H = \{3P, H\}$ and $\mathrm{Res}_H(\mathrm{Res}_H(A_2)) = \{P\}$. We will call the use of A_1 (resp. A_2) an application of Remark 1 at (P, H) with respect to the sequence $(1, (n+1)n/2, n)$ (resp. $(n, (n+1)n/2, 1)$). We may apply simultaneously these tricks just described to several double points and several triple points with respect to different general points of H.

Notation 1. Fix a hyperplane $H \subset \mathbb{P}^n$. A double (resp. triple) point of H is a scheme $\{2P, H\}$ (resp. $\{3P, H\}$). A double (resp. triple) point with support on H is a double (resp. triple) point of $\mathbb{P}^n$, say $2P$ (resp. $3P$), for some $P \in H$. Fix $P \in H$. We will say that a scheme $A \subset \mathbb{P}^n$ is a virtual scheme of type $((n+1)n/2, n)$ if it is obtained from a triple point applying Remark 1 at (P, H) with respect to the sequence $(1, (n+1)n/2, n)$; hence $A_{red} = \{P\}$, $\text{length}(A) = (n+2)(n+1)/2 - 1$, $A \cap H = \{3P, H\}$ and $\text{Res}_H(A) = \{2P, H\}$. We will say that a scheme $B \subset \mathbb{P}^n$ is a virtual scheme of type $((n+1)n/2, 1)$ if it is obtained from a triple point applying Remark 1 at (P, H) with respect to the sequence $(n+1, (n+1)n/2, 1)$; hence $B_{red} = \{P\}$, $\text{length}(B) = (n+1)n/2 + 1$, $A \cap H = \{3P, H\}$ and $\text{Res}_H(A) = \{P\}$. Let W be a zero-dimensional scheme such that $W_{red} = \{P\}$. Set $W_0 := W$ and for all integers $i \geq 1$ define inductively W_i by the formula $W_i := \text{Res}_H(W_{i-1})$. Set $a_i := \text{length}(W_i)$. The non-decreasing sequence $(a_0, a_1, \cdots)$ with be called the type of W. Notice that $a_i = 0$ for $i \gg 0$ and $\sum_{i \geq 0} a_i = \text{length}(W)$.

We will use several times the following lemma proved in [6] (see Lemma 7).

Lemma 2. *Fix integers $d > 0$, $z > 0$, $\gamma \geq 0$, a hyperplane $H \subset \mathbb{P}^n$ and a zero dimensional scheme $Y \subset \mathbb{P}^n$. Let X be the union of Y and z general simple points supported on H. If the following conditions*

$$h^0(\mathbb{P}^n, \mathcal{I}_Y(d)) \leq \gamma + z, \quad \text{and} \quad h^0(\mathbb{P}^n, \mathcal{I}_{Res_H(Y)}(d-1)) \leq \gamma, \qquad (2)$$

take place, then $h^0(\mathbb{P}^n, \mathcal{I}_X(d)) \leq \gamma$. Equivalently if the following conditions

$$h^1(\mathbb{P}^n, \mathcal{I}_Y(d)) \leq \max(0, \gamma + \deg(X) - \binom{d+n}{n}) =: \beta, \text{and}$$

$$h^1(\mathbb{P}^n, \mathcal{I}_{Res_H(Y)}(d-1)) \leq \max(0, \gamma + \deg(Res_H(Y)) - \binom{d+n-1}{n}),$$

take place, then $h^1(\mathbb{P}^n, \mathcal{I}_X(d)) \leq \beta$.

We borrow from [5] the following easy lemma (here the characteristic zero assumption is essential).

Lemma 3. *Fix any scheme $W \subset \mathbb{P}^n$ and any integer $t \geq 0$. Then (after fixing W and t) fix a general $P \in \mathbb{P}^n$ and a general tangent vector v such that $v_{red} = \{P\}$. If $h^0(\mathcal{I}_W(t)) \leq 2$, then $h^0(\mathcal{I}_{W \cup v}(t)) = 0$. If $h^0(\mathcal{I}_W(t)) \geq 2$, then $h^0(\mathcal{I}_{W \cup v}(t)) = h^0(\mathcal{I}_W(t)) - 2$ and $h^1(\mathcal{I}_{W \cup v}(t)) = h^1(\mathcal{I}_W(t))$.*

3. Proof of Proposition 1

We need a few lemmas.

Lemma 4. *Fix a plane $E \subset \mathbb{P}^3$. Let $Z \subset \mathbb{P}^3$ be a general union of* 6 *triple points of $\mathbb{P}^3$ and* 6 *double points of $\mathbb{P}^3$ whose support is contained in E. Then $h^i(\mathbb{P}^3, \mathcal{I}_Z(6)) = 0$ for $i = 0, 1$.*

Proof. It is sufficient to do the case $i = 1$. Fix another general plane $N \subset H$. Let Z' be the virtual scheme obtained taking 4 general double points with support on E, 2 general triple points, 4 general triple points with support on N, a double point with support on a general point of the line $N \cap E$, and a virtual scheme obtained applying Remark 1 with respect to the sequence $(1, 2)$, a general point of $N \cap E$ and the divisor N. It is sufficient to prove $h^1(\mathbb{P}^3, \mathcal{I}_{Z'}(6)) = 0$. The scheme $Z' \cap N$ is a general union of 4 triple points of N, one double point of N and a point of the general line $E \cap N$ of N. Hence to prove $h^1(M, \mathcal{I}_{Z' \cap N}(6)) = 0$ it is sufficient to prove $h^1(M, \mathcal{I}_{Z_1}(6)) = 0$, where Z_1 is the union of the unreduced connected components of $Z \cap N$. Since $\sharp((Z_1)_{red}) = 5 \leq 8$, we have $h^1(M, \mathcal{I}_{Z_1}(6)) = 0$, because the Harbourne-Hirschowitz Conjecture is known to be true for up to 8 points and the linear system $\mathcal{L}_2(6, 3^4, 2)$ is in standard form. The virtual residue $Z_2 := \mathrm{Res}_N(Z')$ is a general union of 2 triple points, 4 double points with support on N, 4 double points with support on E, a double point of M supported by a general point of $N \cap E$ and a general point of $M \cap E$. Let Z_3 be the union of the unreduced connected components of Z_2. To prove $h^1(\mathbb{P}^3, \mathcal{I}_{Z'}(5)) = 0$ (and hence to conclude the proof of the lemma), it is sufficient to prove $h^1(\mathbb{P}^3, \mathcal{I}_{Z_3}(5)) = 0$ (apply twice Lemma 2). To prove the latter vanishing we specialize Z_3 to the scheme Z_4 which is a general union of one triple point, one triple point with support on N, 4 double points with support on N, 4 double points with support on E, and a triple point with support on N. We have $h^i(N, \mathcal{I}_{Z_4 \cap N}(5)) = 0$, $i = 0, 1$, because the system $\mathcal{L}_2(5, 3, 2^4)$ is in standard form with the terminology of [12] or [9] and the Harbourne-Hirschowitz Conjecture is true for union of at most 8 multiple points. The scheme $\mathrm{Res}_N(Z_4)$ is a general union of a triple point, 4 double points with support on E, a double point with support on N and 4 general points of N. Let Z_5 be the union of the unreduced components of $\mathrm{Res}_N(Z_4)$. By Lemma 2 we reduce to prove $h^1(\mathbb{P}^3, \mathcal{I}_{Z_5}(4)) = 0$. We fix a general $Q \in E$. We apply Remark 1 with respect to (E, Q) and the sequence $(1, 3, 2)$; we also specialize the double points with support on E to a general double point with support on $N \cap E$; we call Z_6 the corresponding virtual scheme. $E \cap Z_6$ is a general union of 4 double points of E, one tangent vector and a point. Since $h^1(E, \mathcal{I}_A(4)) = 0$, where A is a a general union of double points, we get $h^1(E, \mathcal{I}_{Z_6 \cap E}(6)) = 0$ (Lemma 3). Since $\mathrm{Res}_E(Z_6)$ is a general union of a virtual scheme of type $(3,2)$ (hence a scheme smaller than a triple point), a general point of $E \cap M$ and 4 general points of M, we are done (Lemma 2). □

Lemma 5. *Fix planes $N, R \subset \mathbb{P}^3$ such that $N \neq R$. Let $Z \subset \mathbb{P}^3$ be a general union of 2 triple points, 5 double points whose support is contained in R and 4 double points whose support is contained in N. Then $h^i(\mathbb{P}^3, \mathcal{I}_Z(5)) = 0$, $i = 0, 1$.*

Proof. It is sufficient to do the case $i = 1$. Fix a general plane E and general $(P_1, P_2, P_3) \in E \times E \times E$. We specialize Z to a general virtual union Z' of 5 double points whose support is contained in R, 2 double points with support on N, 2 triple points with support on E, 2 double points with support on $E \cap N$, 3 virtual scheme obtained by applying Remark 1 with respect to the pairs (E, P_i), $i = 1, 2$, and the sequence $(1, 2)$. The scheme $Z' \cap E$ is a general union of 2 triple points, 2 double points and 3 points. Since $\deg(E \cap Z') = 21$, to prove $h^i(E, \mathcal{I}_{Z' \cap E}(5)) = 0$, $i = 0, 1$ it is sufficient to prove $h^1(E, \mathcal{I}_{Z''}(5)) = 0$, where Z'' is the union of the unreduced connected components of $Z' \cap E$. Let D be the line spanned by the support of the 2 triple points of Z''. Since $\deg(Z'' \cap D) = 6$, we have $h^1(E, \mathcal{I}_{Z''}(5)) = h^1(E, \mathcal{I}_{\mathrm{Res}_D(Z'')}(4))$. Since $\mathrm{Res}_D(Z'')$ is a general union of 4 double points, we have $h^1(E, \mathcal{I}_{\mathrm{Res}_D(Z'')}(4)) = 0$. Thus to prove the lemma it is sufficient to prove $h^1(\mathbb{P}^3, \mathcal{I}_{\mathrm{Res}_E(Z')}(4)) = 0$. The scheme $\mathrm{Res}_E(Z')$ is a general union of 5 double points with support on R, 2 double points with support on E, two general points of $E \cap N$, and 3 general double points of E. We conclude by Lemma 2. □

Lemma 6. *Fix $z \in \{0, 2, 4\}$. Let $Z \subset \mathbb{P}^3$ be a general union of z triple points and $21 - 5z/2$ double points. Then $h^i(\mathbb{P}^3, \mathcal{I}_Z(6)) = 0$, $i = 0, 1$.*

Proof. It is sufficient to do the case $i = 1$. Since the case $z = 0$ is trivial by the Alexander-Hirschowitz Theorem, we may assume $z \in \{2, 4\}$. Let $N \subset \mathbb{P}^3$ be a plane. First assume $z = 4$. We specialize Z to a general union Z' of one triple point, 7 double points, 3 triple points with support on N, 3 double points with support on N and a virtual scheme obtained applying Remark 1 with respect to the sequence $(1, 2)$, the hyperplane N and a general point P of the line D of N spanned by the reductions of two of the triple points with support on N. Set $A := Z' \cap N$. We have $\deg(A) = \binom{8}{2}$. Since $\deg(A \cap D) = 7$, to prove $h^i(N, \mathcal{I}_A(6)) = 0$, $i = 1, 2$, it is sufficient to prove $h^i(N, \mathcal{I}_{\mathrm{Res}_D(A)}(5)) = 0$, $i = 0, 1$. The scheme $\mathrm{Res}_D(A)$ is a general union of 5 double points (two of them with support on D) and one triple point. Hence $h^i(N, \mathcal{I}_{\mathrm{Res}_{D(A)}}(5)) = 0$, $i = 0,1$ (e.g. because the Harbourne-Hirschowitz Conjecture is known to be true for up to 8 points and $\mathcal{L}_2(5,3,2^5)$ is in standard form. Thus it is sufficient to prove $h^1(\mathbb{P}^3, \mathcal{I}_{\mathrm{Res}_{N(Z')}}(5)) = 0$. The scheme $B := \mathrm{Res}_N(Z')$ is a general union of one triple point, 7 double points, 3 double points with support on N and 3 general points of N. Hence $h^0(\mathbb{P}^3, \mathcal{I}_{\mathrm{Res}_{N(B)}}(4)) = 0$. Thus to prove $h^1(\mathbb{P}^3, \mathcal{I}_{\mathrm{Res}_{N(Z')}}(5)) = 0$ it is sufficient to prove $h^1(\mathbb{P}^3, I_F(5)) = 0$, where F is the union of the unreduced components of $\mathrm{Res}_N(Z')$. Apply the proof of Lemma 5 with respect to another plane.

If $z = 2$ instead of Z' we take a general union Z'' of 10 double points, 2 triple points with support on N, 3 double points with support on N and a virtual scheme obtained applying Remark 1 with respect to the sequence $(1,2)$, the hyperplane N and a general point P of the line D of N spanned by the reductions of the triple points with support on N. The same proof works, because $\mathrm{Res}_N(Z'')$ has two more reduced components. □

Remark 2. Let $Z \subset \mathbb{P}^3$ be a general union of double and triple points. If Z has critical value $d \geq 7$, then X has maximal rank (if $d \leq 15$, then use [4], if $7 \leq d \leq 14$, then use the *Macaulay 2* (see [11]) computation written down in [6], §4, for quadruple points. In the same way we get $h^i(\mathbb{P}^3, \mathcal{I}_W(6)) = 0$, $i = 0,1$, if W is a general union of 6 triple points and 6 double points.

Lemma 7. *Fix a hyperplane $H \subset \mathbb{P}^4$, general $P_1, P_2, P_3 \in H$, and an integer c such that $0 \leq c \leq 8$. Let $Z \subset \mathbb{P}^4$ be a general union of the 3 length 4 schemes $\{2P_i, H\}$, $1 \leq i \leq 3$, 7 double points with support on H, c triple points and $3(18-c)-2$ double points. Then $h^1(\mathbb{P}^4, \mathcal{I}_Z(7)) = 0$.*

Proof. Fix general $P, Q, Q_1, Q_2 \in H$. First assume $c \geq 7$. We degenerate Z to a general union Z' of the 3 length 4 schemes $\{2P_i, H\}$, $1 \leq i \leq 3$, 7 double points with support on H, 7 triple points with support on H, $c-7$ triple points and $3(18-c)-2$ double points. The case $(d,x,y) = (7,7,7)$ of Remark 2, gives $h^i(H, \mathcal{I}_{Z' \cap H}(7)) = 0$, $i = 0,1$. Let A be the union of the unreduced components of $\mathrm{Res}_H(Z')$. Since $\mathrm{Res}_H(Z') \backslash A$ is a general union of 7 points of H and $\binom{11}{4} - \mathrm{lenght}(Z) = 330 - 12 - 35 - 15 \cdot 18 + 10 = 23$, Lemma 2 gives that it is sufficient to prove $h^1(\mathbb{P}^4, \mathcal{I}_A(6)) = 0$ and $h^0(\mathbb{P}^4, \mathcal{I}_{\mathrm{Res}_{H(A)}}(5)) \leq 23$. We will first check that $h^1(\mathbb{P}^4, \mathcal{I}_A(6)) = 0$. A is a general union of 7 double points with support on H, $c-7$ triple points and $3(18-c)-2$ double points. Write $84 = 10(c-7) + 4u + v$ with u, v non-negative integers and $0 \leq v \leq 3$. Hence $(u,v) = (21,0)$ if $c = 7$, while $(u,v) = (18,2)$ if $c = 8$. We degenerate A to a general union B of $(c-7)$ triple points with support on H, u double points with support on H, v virtual schemes obtained applying Remark 1 at (P,H) and (Q,H) with respect to the sequence $(1,4)$. If $c=7$, then $h^i(H,\mathcal{I}_{H\cap B}(6))=0$, $i=0,1$, by the Alexander-Hirschowitz Theorem. If $c=8$ Lemma 6 gives $h^i(H,\mathcal{I}_{H\cap B}(6))=0$. Since $\mathrm{Res}_H(B)$ contains $u\geq 18$ general points of H, all the unreduced components of $\mathrm{Res}_H(B)$ are double points and at most one such component intersects H, Lemma 2 reduce the equality $h^0(\mathbb{P}^4,\mathcal{I}_{\mathrm{Res}_H(B)}(5))=0$ to two very particular cases of the Alexander-Hirschowitz Theorem. Now we will check that $h^0(\mathbb{P}^4,\mathcal{I}_{\mathrm{Res}_H(A)}(5))\leq 23$. $\mathrm{Res}_H(A)$ is a general union of 7 points of H, $c-7$ triple points and $3(18-c)-2$ double points. Let A' be the union of the unreduced components of $\mathrm{Res}_H(A)$. By Lemma 2 it is sufficient to check $h^0(\mathbb{P}^4,\mathcal{I}_{\mathrm{Res}_H(A)}(5))\leq 30$ and $h^0(\mathbb{P}^4,\mathcal{I}_{A'}(4))\leq 23$. Both inequalities are obvious by the description of A' and Alexander-Hirschowitz Theorem. Now assume $c\leq 6$. Write $120=10c+28+4e+f$ with e,f integers and $0\leq f\leq 3$. Hence $f\in\{0,2\}$. We degenerate Z to a general union of the 3 length 4 schemes $\{2P_i,H\}$, $1\leq i\leq 3$, $7+e$ double points with support on H, c triple points with support on H and f virtual schemes obtained applying Remark 1 at (Q_i,H), $1\leq i\leq f$, with respect to the sequence $(1,4)$. The case $(d,x,y)=(7,c,7+u)$ of Remark 2, gives $h^i(H,\mathcal{I}_{H\cap W}(7))=0$, $i=0,1$. Then we continue using at each step Lemma 1 and Remark 1 with respect to double points. □

Lemma 8. *Fix two planes $H,M\subset\mathbb{P}^3$ such that $H\neq M$. Let $Z\subset\mathbb{P}^3$ be a general union of 5 points of H, 4 points of M, a triple point and 4 double points. Then $h^i(\mathbb{P}^3,\mathcal{I}_Z(4))=0$ for $i=0,1$.*

Proof. We specialize Z to a general union Z' of 3 double points, 5 points of H, one point of $M\cap H$, one triple point with support on H and one double point with support on H. The singular plane curves shows that $h^i(H,\mathcal{I}_{Z'\cap H}(4))=0$ for $i=0,1$. Since $\mathrm{Res}_H(Z')$ is a general union of 3 double points, one double point with support on H, a point of H and 3 general points of M, we obviously have $h^i(\mathbb{P}^3,\mathcal{I}_{\mathrm{Res}_H(Z')}(4))=0$ for $i=0,1$. □

Proof of Proposition 1. Fix hyperplanes $H,M\in\mathbb{P}^4$ such that $H\neq M$. For all non-negative integers x,y set $\epsilon(7,x,y):=\binom{11}{3}-15x-5y$. $\epsilon(7,x,y)=0$ if and only if $0\leq x\leq 22$ and $y=3(22-x)$.

(a) Here we assume $(d,x,y)=(7,22,0)$. This is one of the two missing cases in the statement of Proposition 1. We specialize Z to a general union Z' of 10 triple points and 12 triple points whose support is contained in H. By the case $(d,x,y)=(7,12,0)$ of Remark 2, we have $h^i(H,\mathcal{I}_{Z'\cap H}(7))=0$ for

$i = 0, 1$. Hence it is sufficient to prove $h^1(\mathbb{P}^4, \mathcal{I}_{\mathrm{Res}_{H(Z')}}(6)) = 0$. $\mathrm{Res}_H(Z')$ is a general union of 10 triple points and 12 double points with support on H. We specialize $\mathrm{Res}_H(Z')$ to a scheme Z'' in which 6 of the triple points have support on M and 6 of the double points have support on the plane $H \cap M$. We claim that $h^1(M, \mathcal{I}_{M \cap Z''}(6)) = 0$. We prove the claim as a separate statement, i.e. as Lemma 5. By the claim to prove the missing case it would be sufficient to prove $h^1(\mathbb{P}^4, \mathcal{I}_{\mathrm{Res}_{M(Z'')}}(5)) = 0$. $\mathrm{Res}_M(Z'')$ is a general union of 4 triple points, 6 double points whose support is contained in H, 6 double points whose support in contained in M and 6 points of $H \cap M$. We fix a general hyperplane $R \subset \mathbb{P}^4$. We specialize $\mathrm{Res}_M(Z'')$ to a general union A of 2 triple points, 2 triple points whose support is contained in R, 5 double points whose support is contained $H \cap R$, one double point with support on $H \backslash R \cap H$, 4 double points whose support is contained in $M \cap R$, while the remaining 2 double points have support on $M \backslash M \cap R$ and none of the 6 simple points is contained in R. The scheme $A \cap R$ is a general union inside R of 2 triple points, 5 double points whose support is contained in the plane $H \cap R$ and 4 double points whose support is contained in the plane $M \cap R$. Lemma 5 gives $h^i(R, \mathcal{I}_{A \cap R}(5)) = 0$, $i = 0, 1$. Let B be the union of the unreduced components of $\mathrm{Res}_R(A)$. B is a general union of 2 triple points, 2 double points with support in R, one double point with support on H and 2 double points with support on M. The line spanned by the support of the two triple points shows that $h^1(\mathbb{P}^4, \mathcal{I}_B(4)) > 0$.

(b) Here we assume $(x, y) = (21, 3)$. Hence $\epsilon(7, x, y) = 0$. We make the construction of part (a) with 3 double points instead of one triple point. Here A is a general union of one triple point, 3 double points $A \cap R$ is a general union of one triple point, 3 double points, 2 triple points with support contained in R, 5 double points whose support is contained $H \cap R$, one double point with support on $H \backslash R \cap H$, 4 double points whose support is contained in $M \cap R$, while the remaining 2 double points have support on $M \backslash M \cap R$ and none of the 6 simple points is contained in R. As in part (a) Lemma 5 gives $h^i(R, \mathcal{I}_{A \cap R}(5)) = 0$, $i = 0, 1$. Let B be the union of the unreduced components of $\mathrm{Res}_R(A)$. Now B is a general union of one triple point, 3 double points, 2 double points with support in R, one double point with support on H and 2 double points with support on M. $\mathrm{Res}_R(A) \backslash B$ is a general union of 6 points of $H \cap M$, 5 points of $H \cap R$ and 4 points of $M \cap R$. Write $\mathrm{Res}_R(A) = B \cup S_1 \cup S_2 \cup S_3$ with $S_1 \subset H \cap M$, $S_2 \subset H \cap R$ and $S_3 \subset M \cap R$ and $\sharp(S_i) = 7 - i$. We specialize $B \cup S_1 \cup S_2 \cup S_3$ to a general union C of one double point, $S_1 \cup S_2 \cup S_3$, a triple point with support on R, 4 double points with support on R, one double point with support on H and 2 double points with support on M. Unfortunately, it is easy to show that $h^i(R, \mathcal{I}_{C \cap R}(4)) > 0$ (and indeed $h^i(R, \mathcal{I}_{C \cap R}(4) = 1)$.

$\mathrm{Res}_R(C)$ is a general union of S_1, a double point, a double point with support on H and 4 general points of R. Applying Lemma 2 to the hyperplane R and its 4 general points, we immediatetely conclude that $h^1(\mathcal{I}_Z(7)) = h^0(\mathcal{I}_Z(7)) \le 1$ in the case $(x, y) = (21, 3)$.

(c) Here we assume $(x, y) = (20, 6)$. We just make the same construction as in part (a). Now B is a general union of 6 double points, 2 double points with support in R, one double point with support on H and 2 double points with support on M. $\mathrm{Res}_R(A)\backslash B$ is a general union of 6 points of $H \cap M$, 5 points of $H \cap R$ and 4 points of $M \cap R$. Write $\mathrm{Res}_R(A) = B \cup S_1 \cup S_2 \cup S_3$ with $S_1 \subset H \cap M$, $S_2 \subset H \cap R$ and $S_3 \subset M \cap R$ and $\sharp(S_i) = 7 - i$. We specialize Z to a general union of $S_1 \cup S_2 \cup S_3$, one double point, 2 double points with support on R, 2 double points with support on M and 6 double points with support on H. Obviously, $h^i(H, \mathcal{I}_{Z' \cap H}(4)) = 0$ for $i = 0, 1$. Since $\mathrm{Res}_H(Z')$ contains 6 general points of H, we are done.

(d) All cases with $\epsilon(7, x, y) = 0$ have $y = 66 - 3x$ for some x such that $0 \le x \le 22$. We did the case $x = 21$ in part (b) and the case $x = 20$ in part (c). Here we will check the cases with $12 \le x \le 19$. We specialize Z to a general union Z' of $x - 12$ triple points, y double points, and 12 triple points whose support is contained in H. By Remark 2, we have $h^i(H, \mathcal{I}_{Z' \cap H}(7)) = 0$ for $i = 0, 1$. Hence it is sufficient to prove $h^1(\mathbb{P}^4, \mathcal{I}_{\mathrm{Res}_H(Z')}(6)) = 0$. $\mathrm{Res}_H(Z')$ is a general union of $x - 12$ triple points, $66 - 3x$ double points, and 12 double points with support on H. First assume $18 \le x \le 19$. We specialize $\mathrm{Res}_H(Z')$ to a general union A of $x - 18$ triple points, $60 - 3x$ double points, 12 double points with support on H, 6 triple points with support on M and 6 double points with support on M. The second part of Remark 2 gives $h^i(M, \mathcal{I}_{M \cap A}(6)) = 0$ for $i = 0, 1$. $\mathrm{Res}_M(A)$ is a general union of $x - 18$ triple points, $60 - 3x$ double points, 12 double points with support on H and 6 double points with support on M. We apply several times Lemma 1 and Remark 1 with respect to H. The first time we insert the remaining triple point if $x = 19$. In the first and in the second step we specialize 3 of the double points whose support is contained in M: we impose that their support is contained in $M \cap H$. Even from the first step taking the residue we get 12 general points of H. We apply Lemma 2. At each further step we may apply Lemma 2, because from now on we specialize only double points. Now assume $12 \le x \le 17$ and x even. We specialize $\mathrm{Res}_H(Z')$ to a general union B of $x - 12$ triple points with support on M, $51 - 5x/2$ double points with support on M and 12 double points with support contained in H. By [4], Proposition 4, applied to the pair $(x, y) = (x - 12, 51 - 5x/2)$ we have $h^i(M, \mathcal{I}_{M \cap A}(6)) = 0$ for $i = 0, 1$. Then we continue using H as in the case $x = 18$. Now assume $12 \le x \le 17$ and x odd. Fix general $P_1, P_2 \in M$. We specialize $\mathrm{Res}_H(Z')$ to a general union C of $x-12$ triple points with support on M, $51-5(x-1)/2$ double points with support on M, two virtual scheme obtaining applying Lemma 2 at (P_1, M) and (P_2, M) with respect to the sequence $(1, 4)$. The only difference is that now $\mathrm{Res}_M(C)$ is a general union of some double points, 12 double points with support on H, some double points with support on H and the schemes $\{2P_1, M\}$ and $\{2P_2, M\}$. We may degenerate even the last two schemes sending P_1 and P_2 to general points of $H \cap M$.

(e) Assume $0 \le x \le 11$ and $y = 66-3x$. First assume x even. We specialize Z to a general union of x triple points with support on H, $(60-5x)/2$ double points with support on H and $66-3x-(60-5x)/2$ double points. By Remark 2, we have $h^i(H, \mathcal{I}_{Z'\cap H}(7)) = 0$ for $i = 0, 1$. Then we continue using only H. Now assume x odd. Fix general $P_1, P_2 \in H$. We specialize Z to a general union of x triple points with support contained in H, two virtual schemes obtained applying Lemma 2 at (P_1, H) and (P_2, H) with respect to the sequence $(1, 4)$, $(118-10x)/4$ double points with support on H, and $66-3x-2-(118-10x)/4$ general double points. We use the case $(d, x, y) = (7, x, (118-10x)/4)$ of Remark 2, and then continue using always H. At each step we may apply Lemma 2, as always when we insert double points with respect to one fixed hyperplane H.

(f) Assume $|\epsilon(7, x, y)| > 0$. Notice that for all zero dimensional schemes $W \subseteq A$ and any integer t $h^0(\mathbb{P}^4, \mathcal{I}_W(t)) \le h^0(\mathbb{P}^4, \mathcal{I}_A(t))$. Since Z is affine, the restriction map $H^0(Z, \mathcal{O}_Z(t)) \to H^0(W, \mathcal{O}_W(t))$ is surjective. Hence $h^1(\mathbb{P}^4, \mathcal{I}_W(t)) \le h^1(\mathbb{P}^4, \mathcal{I}_A(t))$. Since $\binom{11}{4} = 330$, $\epsilon(7, x, y) \equiv 0 \pmod 5$ for all x, y and $\epsilon(7, x, 0) \equiv 0 \pmod{15}$ for all x. Since $\epsilon(7, x, y-1) = \epsilon(7, x, y) + 5$, all these cases follows formally from the truth of the ones just done with $\epsilon(7, x', y') = 0$, just increasing or decreasing the scheme Z. □

4. Proof of Theorem 1

Proof of Theorem 1. Fix hyperplanes $H, M \in \mathbb{P}^4$ such that $H \neq M$ and general $P_i \in H$, $i \ge 1$. For all integers d, x, y set $\epsilon(d, x, y) = \binom{d+4}{4} - 15x - 5y$. The integer $|\epsilon(d, x, y)|$ is a good measure of the difficulty to prove Theorem 1 for the data d, x, y. Notice again that for all zero dimensional schemes $W \subseteq A$ and any integer t $h^0(\mathbb{P}^4, \mathcal{I}_W(t)) \le h^0(\mathbb{P}^4, \mathcal{I}_A(t))$. Since Z is affine, the restriction map $H^0(Z, \mathcal{O}_Z(t)) \to H^0(W, \mathcal{O}_W(t))$ is surjective. Hence $h^1(\mathbb{P}^4, \mathcal{I}_W(t)) \le h^1(\mathbb{P}^4, \mathcal{I}_A(t))$. $\epsilon(d, x, y+1) = \epsilon(d, x, y) + 5$, $\epsilon(d, x, y-1) = \epsilon(d, x, y) + 5$ and $\epsilon(d, x, y-1)$ is well-defined if $y > 0$. Hence to prove all cases of Theorem 1 with

$\epsilon(d,x,y)\ge 0$, it is sufficient to prove it for all pairs (x,y) of non-negative integers such that $0\le \epsilon(d,x,y)\le 4$. To prove all cases of Theorem 1 with $\epsilon(d,x,y)\le 0$, it is sufficient to prove it for all pairs (x,y) of non-negative integers such that $-4\le \epsilon(d,x,y)\le 0$ and for the pair $(\widetilde{x},0)$, where $\widetilde{x} := \lceil \binom{d+4}{4}/15\rceil$.

(a) In part (a) we will assume $d=8$. Since $\binom{12}{4}=495$, $\epsilon(8,x,y)\equiv 0 \pmod 5$ for all x,y and $\epsilon(8,x,0)\equiv 0 \pmod{15}$ for all x. Hence to do all cases with $\epsilon(8,x,y)\ge 0$ it is sufficient to do all cases with $\epsilon(8,x,y)=0$, while to do all cases with $\epsilon(8,x,y)<0$ it is sufficient to do all cases with $\epsilon(8,x,y)=0$. We have $\epsilon(8,x,y)=0$ if and only $0\le x\le 33$ and $y=99-3x$.

(a1) Here we assume $(x,y)=(33,0)$. Fix a general $(P,Q)\in H\times H$. Hence $P\notin M$ and $Q\notin M$. We specialize Z to a general union Z' of 15 triple points, 16 triple points with support on H, a virtual scheme obtained using Remark 1 at (P,H) with respect to the sequence $(4,10,1)$ and a virtual scheme obtained using Remark 1 at (Q,H) with respect to the sequence $(1,10,4)$. By the case $(d',x',y')=(8,16,1)$ of Remark 2, we have $h^i(H,\mathcal{I}_{H\cap Z'}(8))=0$ for $i=0,1$. $\mathrm{Res}_H(Z')$ is a general union of 15 triple points, 16 double points with support on H, a length 11 scheme with P as its support and a length 14 scheme with Q as its support. We specialize $\mathrm{Res}_H(Z')$ to a general union Z'' of 3 triple points, 12 triple points with support on M, 16 double points with support on H, the previous length 11 scheme with P as its support and the previous length 15 with Q as its support. Hence $Z''\cap M$ is a general union of 12 triple points of M. By the case $(d',x',y')=(7,12,0)$ of Remark 2, $h^i(M,\mathcal{I}_{M\cap Z''}(7))=0$ for $i=0,1$. Since $\mathrm{Res}_M(Z'')\cap H$ is a general union of 2 triple points of H and 16 double points of H, the case $z=2$ of Lemma 6 gives $h^i(H,\mathcal{I}_{\mathrm{Res}_M(Z'')\cap H}(6))=0$ for $i=0,1$. $\mathrm{Res}_H(\mathrm{Res}_M(Z''))$ is a general union of 3 triple points, 12 double points with support on M, the length 4 scheme $\{2Q,H\}$ and 13 general points of H, one of them being P. Let B the union of the unreduced components of $\mathrm{Res}_H(\mathrm{Res}_M(Z''))$. Since $\mathrm{Res}_H(B)$ contains a general union of 3 triple points and 12 double points with support on M, it is easy to check that $h^1(\mathbb{P}^4,\mathcal{I}_B(5))=0$ and $h^0(\mathbb{P}^4,\mathcal{I}_{\mathrm{Res}_H(B)}(4))=0$. Apply Lemma 2.

(a2) Here we assume $0\le x\le 32$ and $y=99-3x$. If $x\ge 30$, we copy verbatim the proof of part (a1). If $24\le x\le 29$ in the second step instead of 12 triple points with support on M we insert $12-2a$ triple points with support on M and $5a$ double points with support on M for some integer a such that $0\le a\le 6$. If $x\le 23$ we could even apply Lemma 1 and Remark 1 using H without never use M.

(b) Here we check the case $d=9$. Since $715\equiv 10 \pmod{15}$, we see that to do all cases for the degree d it is sufficient to do all cases with $\epsilon(9,x,y)=0$

and the case $(x,y) = (48,0)$. $\epsilon(9,x,y) = 0$ if and only if $0 \le x \le 47$ and $y = 2+3(47-x)$.

(b1) Here we assume $(x,y) = (47,2)$. We degenerate Z to a general union Z' of 2 double points, 25 triple points and 22 triple points whose support is contained in H. By the case $(d',x',y') = (9,22,0)$ of Remark 2, we have $h^i(H,\mathcal{I}_{H\cap Z'}(9)) = 0$ for $i = 0,1$. Fix general $P,Q \in M$. We specialize $\mathrm{Res}_H(Z')$ to a general union Z'' of 2 double points, 22 double points whose support is contained in H, 7 triple points, 16 triple points whose support is contained in M a virtual scheme obtained applying Remark 1 at (P,M) with respect to the sequence $(4,10,1)$ and a virtual scheme obtained applying Remark 1 at (Q,M) with respect to the sequence $(1,10,4)$. Hence $Z'\cap M$ is a general union of P, 16 triple points of M, one double point of M with Q as its support. The case $(d',x',y') = (8,16,1)$ of Remark 2, gives $h^i(M,\mathcal{I}_{Z''\cap M}(8)) = 0$ for $i = 0,1$. $\mathrm{Res}_M(Z'')$ is a general union of a length 11 scheme supported by P, a length 14 scheme supported by Q, 7 triple points, 2 double points, 22 double points with support on H and 16 double points with support on H. Notice that $\binom{10}{3} = 120$. Let B be the degeneration of $\mathrm{Res}_M(Z'')$ in which 2 of the double points with support on M are required to have support on $H\cap M$. $B\cap H$ may be seen as a general union of 24 double points of H. Hence $h^i(H,\mathcal{I}_{H\cap B}(7)) = 0$ for $i = 0,1$ by Alexander-Hirschowitz Theorem (see [10]). Let E be the union of all unreduced components of $\mathrm{Res}_H(B)$. Since $\mathrm{Res}_H(B)\backslash E$ is a general union of 24 points of H, we immediately get $h^1(\mathbb{P}^4,\mathcal{I}_E(6)) = 0$ and $h^0(\mathbb{P}^4,\mathcal{I}_{\mathrm{Res}_{H(E)}}(5)) \le 24$ (insert more components on M).

(b2) Here we assume $(x,y) = (48,0)$. Notice that $\epsilon(9,48,0) = -5$. In the proof below (just for h^0) we end up with one more triple point and 2 less double points; instead of $h^1(\mathbb{P}^4,\mathcal{I}_E(6)) = 0$ we just need $h^1(\mathbb{P}^4,\mathcal{I}_E(6)) \le 5$, while the inequality $h^0(\mathbb{P}^4,\mathcal{I}_{\mathrm{Res}_{H(E)}}(5)) \le 24$ is now easier.

(b3) Here we assume $0 \le x \le 47$ and $y = 2+3(47-x)$. For $x \ge 40$ we copy the first 2 steps with H and M of part (g) and then just use more double points. Now assume $38 \le x \le 39$. In the second step (i.e. the step which uses M) we apply Remark 1 at (P,M) and (P,M) with respect to the sequence $(1,4)$. Now assume $29 \le x \le 37$. As in part (b1) we degenerate Z to a general union Z'' of $y := 2+3(47-x)$ double points, $x-22$ triple points and 22 triple points whose support is contained in H. By the case $(d',x',y') = (9,22,0)$ of Remark 2, we have $h^i(H,\mathcal{I}_{H\cap Z''}(9)) = 0$ for $i = 0,1$. Notice that $\binom{11}{3} - 22\cdot 4 = 7\cdot 10+4+3$. We degenerate $\mathrm{Res}_H(Z'')$ to a general union Z_1 of 22 double points, 7 triple points with support on H, $x-29$ triple points, one double point with support on H, 3 virtual schemes obtained applying Remark 1 at (P_1,H), (P_2,H) and

(P_3, H) with respect to the sequence $(1, 4)$ and $y - 4$ double points. The case $(d, x, y) = (8, 7, 23)$ of Remark 2, gives $h^i(H, \mathcal{I}_{H \cap Z_1}(8)) = 0$ for $i = 0, 1$. Let Z_2 be the union of the unreduced components of $\mathrm{Res}_H(Z_1)$. Since $\mathrm{Res}_H(Z_1)\backslash Z_2$ is a general union of 23 points of H, Lemma 2 shows that it is sufficient to prove $h^1(\mathbb{P}^4, \mathcal{I}_{Z_2}(7)) = 0$ and $h^0(\mathbb{P}^4, \mathcal{I}_{\mathrm{Res}_H(Z_2)}(6)) = 0$. We will first check that $h^1(\mathbb{P}^4, \mathcal{I}_{Z_2}(7)) = 0$. Z_2 is a general union of $x - 29$ triple points, $y - 4$ double points, 7 double points with support on H and the 3 length 4 schemes $\{2P_i, H\}$, $1 \le i \le 3$. Apply Lemma 7. Now we will check that $h^0(\mathbb{P}^4, \mathcal{I}_{\mathrm{Res}_H(Z_2)}(6)) = 0$. $\mathrm{Res}_H(Z_2)$ is a general union of 7 points of H, $x - 29$ triple points and $y - 4$ double points. Use again Lemma 2 and the Alexander-Hirschowitz Theorem for double points. Now assume $22 \le x \le 28$. We take Z' as in part (c). Write $120 = 22 \cdot 4 + 10(x - 22) + 4u + v$ with u, v non-negative integers and $0 \le v \le 3$. Hence $v = 0$ if x is even and $v = 2$ if x is odd. We specialize $\mathrm{Res}_H(Z')$ to a general union T of $22 + u$ double points with support on H, $x - 22$ triple points with support on H, v virtual schemes obtained applying Remark 1 at (P_1, H) and (P_2, H) with respect to the sequence $(1, 4)$ and $y - u - v$ double points. The case $(d, x, y) = (8, x - 22, 22 + u)$ of Remark 2, gives $h^i(H, \mathcal{I}_{H \cap T}(8)) = 0$ for $i = 0, 1$. Since $\mathrm{Res}_H(T)$ containins $22 + u$ general points of H and in later steps we only add double points, we conclude as in the previous case. Now assume $x \le 22$. Write $\binom{12}{3} = 10x + 4u' + v'$ with u', v' integers and $0 \le v' \le 3$. Hence $v' = 0$ if x is even and $v' = 2$ if x is odd. We take as Z' a general union of x triple points with support on H, u' double points with support on H, v' virtual schemes obtained applying Remark 1 at (P_i, H), $1 \le i \le v'$, with respect to the sequence $(1, 4)$ and $y - u' - v'$ double points. The case $(d, x, y) = (9, x, u')$ of Remark 2, gives $h^i(H, \mathcal{I}_{Z' \cap H}(9)) = 0$, $i = 0, 1$. Then we continue adding double points and using Lemma 2.

(c) In this subsection we assume $d = 10$. Since $\binom{14}{4} = 1001 \equiv 11 \pmod{15}$, it is sufficient to do all cases (x, y) with $\epsilon(9, x, y) \in \{1, -4\}$. $\epsilon(9, x, y) = 1$ if and only if $0 \le x \le 66$ and $y = 2 + 3(66 - x)$. $\epsilon(9, x, y) = -4$ if and only if either $x = 67$ or $0 \le x \le 66$ and $y = 3 + 3(66 - x)$.

(c1) Here we assume $(x, y) = (66, 2)$. Recall that $\epsilon(9, 66, 2) = 1$. We degenerate Z to a general union of two virtual schemes obtained applying Remark 1 at (P_1, H) and (P_2, H) with respect to the sequence $(1, 10, 4)$, 36 triple points, 1 double point, 28 triple points with support on H and a double point with support on H. By the case $(d', x', y') = (10, 28, 6)$ of Remark 2, we have $h^i(H, \mathcal{I}_{Z' \cap H}(10)) = 0$ for $i = 0, 1$. The scheme $\mathrm{Res}_H(Z')$ has a reduced connected component whose support is general in H (it is the residue of the double point with support on H). Let B the union of the unreduced components of $\mathrm{Res}_H(Z')$. It is sufficient to prove $h^1(\mathbb{P}^4, \mathcal{I}_B(9)) = 0$ and $h^0(\mathbb{P}^4, \mathcal{I}_{\mathrm{Res}_H(B)}(8)) = 0$. We will first check that $h^1(\mathbb{P}^4, \mathcal{I}_B(9)) = 0$. We specialize B to a general union B' of 14 triple points, 1 double point, a scheme of length 14 with P as its support, a scheme of length 14 with Q as its support and 22 triple points whose support is contained in M. By the case $(d', x', y') = (9, 22, 0)$ of Remark 2, we have $h^i(M, \mathcal{I}_{M\cap B'}(9)) = 0$ for $i = 0, 1$. We specialize $\mathrm{Res}_M(B)$ to a general union E of 3 virtual schemes obtained applying Remark 1 at (P_4, H), (P_5, H) and (P_6, H) with respect to the sequence $(1, 10, 4)$, 8 triple points, 1 double points, a scheme of length 14 with P_1 as its support, a scheme of length 14 with P_2 as its support and 22 double points whose support is contained in M. Hence $E \cap H$ is a general union of 5 triple points of H (two of them having P and Q as their support), 28 double points of H and 3 general points P_4, P_5, P_6. By the case $(d', x', y') = (8, 5, 28)$ of Remark 2, we have $h^i(M, \mathcal{I}_{M\cap E}(8)) = 0$ for $i = 0, 1$. $\mathrm{Res}_H(E)$ has 28 general points of H as connected components. Hence from now on, everything is easier than in part (b1). Now we check that $h^0(\mathbb{P}^4, \mathcal{I}_{\mathrm{Res}_H(B)}(8)) = 0$. $\mathrm{Res}_H(B)$ is a general union of two length 14 schemes supported by P_1 and P_2 and with type $(10, 4)$ with respect to H, 1 double point, 36 triple points and 28 double points with support on H. Let W be the union of the 33 of the triple points of $\mathrm{Res}_H(B)$. $h^0(\mathbb{P}^4, \mathcal{I}_W(8)) = 0$ by the case $d = 8$ of Theorem 1 proved before.

(c2) First assume $0 \le x \le 65$ and $y = 2 + 3(66 - x)$. If $x \ge 50$, then we repeat verbatim the proof of part (c1) until we reduce this case to the proof that $h^1(\mathbb{P}^4, \mathcal{I}_{\mathrm{Res}_{M(B')}}(8)) = 0$. From then on we use more double points to apply Lemma 1 and Remark 1. Now we will take $x \le 49$ and show that we may use only the plane H. First assume $x \ge 31$. We degenerate Z to a general union of two virtual schemes obtained applying Remark 1 at (P_1, H) and (P_2, H) with respect to the sequence $(1, 10, 4)$, $x - 30$ triple points, y double points, 28 triple points with support on H and a double point with support on H. By the case $(d', x', y') = (10, 28, 6)$ of Remark 2, we have $h^i(H, \mathcal{I}_{Z'\cap H}(10)) = 0$ for $i = 0, 1$. The scheme $\mathrm{Res}_H(Z')$ has a simple point whose support is general in H (it is the residue of the double point with support on H). Let B the union of all the unreduced components of $\mathrm{Res}_H(Z')$. It is sufficient to prove $h^1(\mathbb{P}^4, \mathcal{I}_B(9)) = 0$. $B \cap H$ is a general union of one triple point of H and 28 double points of H. Since $\binom{12}{3} = 220$ and $\mathrm{length}(B \cap H) = 122$, the strategy is obvious. By assumption B contains at least one triple point not intersecting H. Let m be the maximal integer such that $m \le x - 30$ and $m \equiv 0$. We specialize B so that m of its triple points and $3(30 - x - m)$ of its double points have support on H. If $x \le 30$, then we may use some of the double points even at the first step; let x' be the maximal integer such that $x' \le x$ and $x' \equiv 28 \pmod 3$; here x' of the triple points and $3(28-x')$ of the double points of the new Z' have support on H.

(d) Here we check the case $d=11$. Since $\binom{15}{4}=1365\equiv 0 \pmod{15}$, it is sufficient to do all cases with $\epsilon(11,x,y)=0$. We have $\epsilon(11,x,y)=0$ if and only if $0\le x\le 91$ and $y=3(91-x)$.

(d1) Here we assume $(d,x,y)=(11,91,0)$. Fix $P\in H$. We specialize Z to a general union Z' of the virtual scheme obtained applying Remark 1 at (P,H) with respect to the sequence $(4,10,1)$, 54 triple points and 36 triple points with support on H. Hence $Z'\cap H$ is a general union of a double point of H supported by P and 36 triple points of H. Applying Remark 2, to the case $(d',x',y')=(11,36,1)$ we get $h^i(H,\mathcal{I}_{Z'\cap H}(11))=0$ for $i=0,1$. Fix general $P_1,P_2\in H$. We specialize $\mathrm{Res}_H(Z')$ to a general union B of a scheme of length 11 supported by P and with length 10 intersection with H, two virtual schemes obtained applying Remark 1 at (P_1,H) and (P_2,H) with respect to the sequence $(1,10,4)$, 41 triple points, 36 double points with support on H and 13 triple points with support on H. Hence $B\cap H$ is a general union of 14 triple points of H (one of them having P as its support), 2 simple points and 36 double points of H. By the case $(d',x',y')=(10,14,36)$ of Remark 2, we have $h^i(H,\mathcal{I}_{B\cap H}(10))=0$ for $i=0,1$. Let E be the union of the unreduced components of $\mathrm{Res}_H(B)$. Since $\mathrm{Res}_H(B)\backslash E$ is a general union of 37 points of H (P and the support of the 36 double points of B intersecting H), Lemma 2 gives that it is sufficient to prove $h^1(\mathbb{P}^4,\mathcal{I}_E(9))=0$ and $h^0(\mathbb{P}^4,\mathcal{I}_{\mathrm{Res}_H(E)}(8))\le 37$. E is a general union of 2 length 14 schemes supported by P_1,P_2 and with length 10 intersection with H, 41 triple points and 13 double points with support on H. We specialize it to a general union E' of 2 length 14 schemes supported by P_1,P_2, 32 triple points with support on H, 9 triple points with support on H and 13 double points with support on H. The case $(d',x',y')=(9,11,13)$ of Remark 2, gives $h^1(H,\mathcal{I}_{H\cap E'}(9))=0$, i.e. $h^0(H,\mathcal{I}_{H\cap E'}(9))=2$. Since $\mathrm{Res}_H(E')$ has as connected components 13 general points of it, it is easy to control the h^1-vanishing of the remaining components. The proof of the inequality $h^0(\mathbb{P}^4,\mathcal{I}_{\mathrm{Res}_H(E)}(8))\le 37$ is easier, because $\binom{12}{3}-\mathrm{length}(E\cap H)-33=220-20-52-33\gg 0$.

(d2) Here we assume $0\le x\le 91$ and $y=3(91-x)$. We leave the details to the reader. Notice that in all cases with $y\ge 2$ we may apply Lemma 2 just at the second step.

(e) Here we check the case $d=12$. Since $\binom{16}{4}=1820\equiv 5 \pmod{15}$, to check all cases it is sufficient to check all cases with $\epsilon(12,x,y)=0$ and the case $(x,y)=(122,0)$. We have $\epsilon(12,x,y)=0$ if and only if $0\le x\le 121$ and

$y = 1 + 5(121 - x)$.

(e1) Here we assume $(x, y) = (121, 1)$. We degenerate Z to a general union Z' of 75 triple points, a virtual scheme obtained applying Remark 1 at (P_1, H) with respect to the sequence $(1, 10, 4)$, a double point with support on H and 45 triple points with support on H. By the case $(d, x, y) = (12, 45, 1)$ of Remark 2, we have $h^i(H, \mathcal{I}_{H \cap Z'}(12)) = 0$ for $i = 0, 1$. Let B the union of all the unreduced components of $\mathrm{Res}_H(Z')$. $\mathrm{Res}_H(Z') \backslash B$ is just a point. We will first prove that $h^1(\mathbb{P}^4, \mathcal{I}_B(11)) = 0$, leaving the h^0-vanishing for the last part of this subsetction. We degenerate B to a general union A of a length 14 scheme supported by P_1 and with type $(10, 4)$ with respect to H, 39 triple points, 44 double points with support on $H \backslash H \cap M$, one double point with support on $H \cap M$ and 36 triple points with support on M. Hence $A \cap M$ is a general union of 36 triple points of M and one double point of M. $h^i(M, \mathcal{I}_{M \cap A}(11)) = 0$ for $i = 0, 1$ (Remark 2). Notice that $\mathrm{Res}_M(A)$ contains a simple point, i.e. the complementary in $\mathrm{Res}_M(A)$ of the support of the double point in $M \cap H$. We degenerate $\mathrm{Res}_M(A)$ to a general union D of a length 14 scheme supported by P_2 and with type $(10, 4)$ with respect to H, 45 double points with support on H, a virtual scheme obtained applying Remark 1 at (P_1, H) with respect to the sequence $(1, 10, 4)$, a general point of $H \cap M$, 35 double points with support on M, one double point with support on $H \cap M$, 29 triple points and 9 triple points with support on H. Hence $D \cap H$ is a general union of a point of H, 10 triple points of H, a point of $H \cap M$ and 46 double points of H with support on M. Since any two points of H are coplanar, $H \cap D$ may be seen as a general union of 10 triple points of H, 46 double points of H and two general points of H. Hence $h^i(M, \mathcal{I}_{M \cap D}(10)) = 0$ for $i = 0, 1$ (Remark 2). At the next step we insert some of the remaining triple points on H and up to 3 of the double points from M into $H \cap M$. We omit the details, because after the first residue we get a scheme containing as components 28 general points of H and hence we may use Lemma 2. Now we will prove that $h^0(\mathbb{P}^4, \mathcal{I}_{\mathrm{Res}_H(B)}(10)) = 0$. $\mathrm{Res}_H(B)$ is a general union of 75 triple points, the length 4 scheme $\{2P_1, H\}$, and 45 general points of H. Just use the case $(d, x, y) = (10, 75, 0)$ proved before.

(e2) Assume $0 \leq x \leq 120$ and $y = 1 + 5(121 - x)$. If $x \geq 110$, then we just use the proof of part (e1). For lower x everything is easier (we only apply Remark 1 with respect to the sequence $(1, 4)$ and at most at 3 points of H at each step). As in (e1) we do also the easier case $(x, y) = (122, 0)$. Since a triple point contains a double point, this case follows formally from the truth of the case $(121, 0)$.

(f) Here we check the case $d = 13$. Since $\binom{17}{4} = 2380 \equiv 10 \pmod{15}$ it is sufficient to do the case $(x, y) = (159, 0)$ and all cases with $\epsilon(13, x, y) = 0$. $\epsilon(13, x, y) = 0$ if and only if $0 \le x \le 158$ and $y = 2 + 3(158 - x)$

(f1) Assume $(x, y) = (158, 2)$. We specialize Z to a general union Z' of 102 triple points, 2 double points and 56 triple points with support on H. By the case $(d, x, y) = (13, 56, 0)$ of Remark 2, we have $h^i(H, \mathcal{I}_{H \cap Z'}(13)) = 0$ for $i = 0, 1$. Fix a general $P \in H$. We specialize $\mathrm{Res}_H(Z')$ to a general union A of 58 triple points, 2 double points, a virtual scheme obtained applying Remark 1 at (P, H) with respect to the sequence $(1, 10, 4)$, 56 double points with support on H and 23 triple points with support on H. Hence $A \cap H$ is a general union of one simple point, 56 double points of H and 23 triple points of H. Hence $h^i(H, \mathcal{I}_{H \cap A}(12)) = 0$ for $i = 0, 1$ (Remark 2). $\mathrm{Res}_H(A)$ is the union of a scheme B and 56 general points of H. Hence it is sufficient to prove $h^1(\mathbb{P}^4, \mathcal{I}_B(11)) = 0$ and $h^0(\mathbb{P}^4, \mathcal{I}_{\mathrm{Res}_{H(B)}}(10)) = 0$. Both are obtained just inserting at each step the maximal possible number of triple points available and then the 2 double points. We never use the differential Horace (and hence we may loose up to 9 conditions at each step), because 56 is large (bigger than 9×6) and just taking the first residue we get as connected components 23 general points of H.

(f2) As in part (f1) we do the case $(x, y) = (159, 0)$, while if $0 \le x \le 157$ and $y = 2 + 3(158 - x)$ the proof is even easier.

(g) Here we consider the case $d = 14$. Since $\binom{18}{4} = 3060 \equiv 0 \pmod{15}$, it is sufficient to prove all cases with $\epsilon(14, x, y) = 0$. $\epsilon(14, x, y) = 0$ if and only if $0 \le x \le 204$ and $y = 3(204 - x)$.

(g1) Assume $(dx, y) = (204, 0)$. Fix general $P_i \in H$, $i \ge 1$. We specialize Z to a general union Z' of 136 triple points and 68 triple points with support on H. By the case $(d, x, y) = (14, 68, 0)$ of Remark 2, we have $h^i(H, \mathcal{I}_{H \cap Z'}(14)) = 0$ for $i = 0, 1$. We specialize $\mathrm{Res}_H(Z')$ to a general union A of 2 virtual schemes obtained applying Remark 1 at (P_1, H) and (P_2, H) with respect to the sequence $(4, 10, 1)$, 68 double points with support on H, 28 triple points with support on H and 106 triple points. By the case $(d, x, y) = (13, 28, 70)$ of Remark 2, we have $h^i(H, \mathcal{I}_{H \cap A}(13)) = 0$ for $i = 0, 1$. Let B the union of all unreduced components of $\mathrm{Res}_H(A)$. $\mathrm{Res}_H(A) \backslash B$ is a general union of 68 points of H. By Lemma 2 it is sufficient to prove $h^1(\mathbb{P}^4, \mathcal{I}_B(12)) = 0$ and $h^0(\mathbb{P}^4, \mathcal{I}_{\mathrm{Res}_{H(B)}}(11)) = 0$. We will first do the h^1-vanishing. B is a general union of two length 11 schemes supported by P_1 and P_2 and with type $(10, 1)$ with respect to H, 28 double points with support on H and 106 triple points. We specialize B to a general union B' of two length 11 schemes supported by P_1 and P_2 and with type $(10, 1)$ with respect to H, 3 virtual schemes obtained applying Remark 1 at (P_3, H), (P_4, H) and (P_5, H) with respect to the sequence $(1,10,4)$, 28 double points with support on H, 32 triple points with support on H and 71 triple points. The case $(d',x',y') = (12,35,28)$ of Remark 2, gives $h^1(H,\mathcal{I}_{H\cap B'}(12)) = 0$. Let C be the union of all unreduced components of $\mathrm{Res}_H(B')$. Since $\mathrm{Res}_H(B')\backslash C$ contains 30 general points of H (two of them being P_1 and P_2) and $\binom{15}{4}$ − length$(\mathrm{Res}_H(B')) = 68$, it is sufficient to prove $h^1(\mathbb{P}^4,\mathcal{I}_B(12)) = 0$ and $h^0(\mathbb{P}^4,\mathcal{I}_{\mathrm{Res}_H(C)}(11)) \le 68$. We will only check the h^1-vanishing, because the h^0-inequality is even easier than the vanishing of $h^0(\mathbb{P}^4,\mathcal{I}_{\mathrm{Res}_H(B)}(11))$ considered below. C is a general union of 3 length 14 schemes supported by P_3,P_4,P_5 and with type $(10,4)$ with respect to H, 32 double points with support on H and 71 triple points. We degenerate C to a general union D of 3 length 14 schemes supported by P_3,P_4,P_5 and with type $(10,4)$ with respect to H, 32 double points with support on H, a virtual scheme obtained applying Remark 1 at (P_6,H) with respect to the sequence $(4,10,1)$, two virtual schemes obtained applying Remark 1 at (P_7,H) and (P_8,H) with respect to the sequence $(1,10,4)$, 20 triple points with support on H and 48 triple points. By the case $(d,x,y) = (11,23,33)$ of Remark 2, we have $h^i(H,\mathcal{I}_{H\cap D}(11)) = 0$ for $i = 0,1$. Let E be the union of the unreduced components of $\mathrm{Res}_H(D)$. Since $\mathrm{Res}_H(D)\backslash E$ is a general union of 32 points of H, it is sufficient to prove $h^1(\mathbb{P}^4,\mathcal{I}_E(10)) = 0$ and $h^0(\mathbb{P}^4,\mathcal{I}_{\mathrm{Res}_H(C)}(9)) \le 98$. We will only do the h^1-vanishing, because the h^0-inequality is easier that the h^0-vanishing done before (just quote the case $d = 9$ of Theorem 1). E is a general union of the 3 length 4 schemes $\{2P_3,H\}$, $\{2P_4,H\}$, $\{2P_5,H\}$, one length 11 scheme supported by P_6 and of type $(10,1)$ with respect to H, two length 14 schemes supported by P_7,P_8 and with type $(10,4)$ with respect to H, 20 double points with support on H and 48 triple points. We degenerate E to a general union F of one length 11 scheme supported by P_6 and of type $(10,1)$ with respect to H, two length 14 schemes supported by P_7,P_8 and with type $(10,4)$ with respect to H, 20 double points with support on H, a virtual scheme obtained applying Remark 1 at (P_9,H) with respect to the sequence $(4,10,1)$, 15 triple points with support on H and 32 triple points. The case $(d,x,y) = (10,17,26)$ of Remark 2, gives $h^i(H,\mathcal{I}_{H\cap F}(10)) = 0$ for $i = 0,1$. Let G be the union of all unreduced components of $\mathrm{Res}_H(F)$. Since $\mathrm{Res}_H(F)\backslash G$ is a general union of 21 points of H (one of them being P_6), it is sufficient to prove $h^1(\mathbb{P}^4,\mathcal{I}_G(9)) = 0$ and $h^0(\mathbb{P}^4,\mathcal{I}_{\mathrm{Res}_H(C)}(8)) \le 130$. We will only do the h^1-vanishing, because the h^0-inequality is easier that the h^0-vanishing done before (just quote the case $d = 8$ of Theorem 1). G is a general union of the two length 4 schemes $\{2P_7,H\}$ and $\{P_8,H\}$, a length 11 scheme supported by P_9 and with type $(10,1)$ with respect to H, 15 double points with support on

H and 32 triple points. We degenerate G to a general union A' the two length 4 schemes $\{2P_7, H\}$ and $\{2P_8, H\}$, a length 11 scheme supported by P_9 and with type $(10,1)$ with respect to H, 15 double points with support on H, two virtual schemes obtained applying Remark 1 at (P_{10}, H) and $\{P_{11}, H\}$ with respect to the sequence $(1,10,4)$, 14 triple points with support on H and 17 triple points. The case $(d,x,y) = (9,15,22)$ of Remark 2, gives $h^i(H, \mathcal{I}_{H\cap A'}(9)) = 0$ for $i = 0,1$. Let C' denote the union of all reduced components of $\mathrm{Res}_H(A')$. Since $\mathrm{Res}_H(A')\backslash C'$ is a general union of 21 general points of H (one of them being P_9) it is sufficient to prove $h^1(\mathbb{P}^4, \mathcal{I}_{C'}(8)) = 0$ and an easier and omitted inequality for h^0. C' is a general union of two length 14 schemes supported by P_{10} and P_{11} and with type $(10,4)$ with respect to H, 14 double points with support on H and 17 triple points. We degenerate C' to a general union D' of two length 14 schemes supported by P_{10} and P_{11} and with type $(10,4)$ with respect to H, 14 double points with support on H, a virtual scheme obtained applying Remark 1 at (P_{12}, H) with respect to the sequence $(1,10,4)$, two virtual schemes obtained applying Remark 1 at (P_{13}, H) and at (P_{14}, H) with respect to the sequence $(4,10,1)$, 8 triple points with support on H and 6 triple points. The case $(d,x,y) = (8,10,16)$ of Remark 2, gives $h^i(H, \mathcal{I}_{H\cap D'}(8)) = 0$ for $i = 0,1$. Let E' be the union of the unreduced components of $\mathrm{Res}_H(D')$. Since $\mathrm{Res}_H(D')\backslash E'$ is a general union of 14 points of H, it is sufficient to prove $h^1(\mathbb{P}^4, \mathcal{I}_{E'}(7)) = 0$ and an easier and omitted inequality for h^0. E' is a general union of the two length 4 schemes $\{2P_{10}, H\}$ and $\{2P_{11}, H\}$, two length 14 scheme supported by P_{12} and with type $(10,4)$ with respect to H, a length 11 scheme supported by P_{12} and of type $(10,1)$ with respect to H, 8 double points with support on H and 6 triple points. We degenerate E' to a general union F' of the two length 4 schemes $\{2P_{10}, H\}$ and $\{2P_{11}, H\}$, two length 14 scheme supported by P_{12} and with type $(10,4)$ with respect to H, a length 11 scheme supported by P_{12} and of type $(10,1)$ with respect to H, 8 double points with support on H, 5 triple points with support on H and 1 triple point. The case $(d,x,y) = (7,8,10)$ of Remark 2, gives $h^i(H, \mathcal{I}_{H\cap F'}(7)) = 0$ for $i = 0,1$. Let G' be the union of all unreduced components of $\mathrm{Res}_H(F')$. Since $\mathrm{Res}_H(F')\backslash G'$ is a general union of 9 points of H (one of them being P_{12}), it is sufficient to prove $h^1(\mathbb{P}^4, \mathcal{I}_{E'}(6)) = 0$ and an omitted h^0-inequality. Now we will check that $h^0(\mathbb{P}^4, \mathcal{I}_{\mathrm{Res}_H(B)}(11)) = 0$. $\mathrm{Res}_H(B)$ is a general union of P_1, P_2, 28 general points of H and 106 triple points. Just use the case $d = 11$ done before. In all h^0 we see that the residual contain no triple point with support on H, no double point with support on H and at most two unreduced schemes intersecting H; if any such scheme exists, then it is of the form $\{2P_i, H\}$ for some i; the reduced components of the residue intersecting H are general points of H.

(g2) Here we assume $0 \le x \le 203$ and $y = 3(204 - x)$. If $x \ge 98$, then we do the first two steps of part (g1). now ssume $68 \le x \le 97$. We make the first step of part (g1). Define the non-negative integers u, v by the relations $\binom{16}{3} = 68 \times 4 + 10(x - 68) + 4u + v$, $0 \le v \le 3$. Hence $v \in \{0, 2\}$. We specialize $\mathrm{Res}_H(Z')$ to a general union T of $68 + u$ double points with support on H, $y - u - v$ double points with support on H and v length 5 virtual schemes obtained applying Remark 1 at (P_i, H), $1 \le i \le v$, with respect to the sequence $(1, 4)$. Now assume $0 \le x \le 67$. Define the non-negative integers w, z by the relations $\binom{17}{3} = 10x + 4w + z$, $0 \le z \le 3$. We degenerate Z to a general union of $y - w - z$ double points, x triple points with support on H and z length 5 virtual schemes obtained applying Remark 1 at (P_i, H), $1 \le i \le z$, with respect to the sequence $(1, 4)$.

(h) Here we will do the case $d = 15$. Since $\binom{19}{4} = 3876$, $\epsilon(15, x, y) \equiv 1 \pmod 5$ for all x, y, z. As usual, it is sufficient to do all cases x, y with either $\epsilon(15, x, y) \in \{1, -4\}$ or with $y = 0$. If $y = 0$, then it is sufficient to check the cases $x = 258$ and $x = 259$. Since a triple points contains a double point and $\epsilon(15, 258, 1) = 1$, the case $(x, y) = (258, 1)$ implies the cases $(258, 0)$ and $(259, 0)$. Hence it is sufficient to do all cases x, y with $\epsilon(15, x, y) \in \{1, -4\}$

(h1) Assume $(x, y) = (258, 1)$. We specialize Z to a general union Z' of one double point, two virtual schemes obtained applying Remark 1 at (P_1, H) and (P_2, H) with respect to the sequence $(1, 10, 4)$, a virtual scheme obtained applying Remark 1 at (P_3, H) with respect to the sequence $(4, 10, 1)$, 174 triple points and 81 triple points with support on H. By the case $(d, x, y) = (15, 81, 1)$ of Remark 2, we have $h^i(H, \mathcal{I}_{H \cap Z'}(15)) = 0$ for $i = 0, 1$. We specialize $\mathrm{Res}_H(Z')$ to a general union A of one double point, 81 double points with support in H, two length 14 schemes supported by P_1 and P_2 and of type $(10, 4)$ with respect to H, a length 11 scheme with P_3 as support and type $(10, 1)$ with respect to H, two virtual schemes obtained applying Remark 1 at (P_4, H) and (P_5, H) with respect to the sequence $(1, 10, 4)$, a virtual scheme obtained applying Remark 1 at (P_6, H) with respect to the sequence $(4, 10, 1)$, 139 triple points and 32 triple points with support on H. By the case $(d, x, y) = (14, 35, 82)$ of Remark 2, we have $h^i(H, \mathcal{I}_{H \cap A}(14)) = 0$ for $i = 0, 1$. $\mathrm{Res}_H(A)$ is a general union of one double point, 139 triple points, 82 points of H (one of them being P_3), the two length 4 schemes $\{2P_1, H\}$ and $\{2P_2, H\}$, two length 14 schemes supported by P_4 and P_5 and of type $(10, 4)$ with respect to H, 32 double points with support on H and a length 11 scheme with P_6 as support and type $(10, 1)$ with respect to H. Let B denote the union of the unreduced components of $\mathrm{Res}_H(A)$. By Lemma 2 it is sufficient to prove $h^1(\mathbb{P}^4, \mathcal{I}_B(13)) = 0$ and $h^0(\mathbb{P}^4, \mathcal{I}_{\mathrm{Res}_H(B)}(12)) = 0$.

We will first do the h^1-vanishing. We degenerate B to a general union C of one double point, a virtual scheme obtained applying Remark 1 at (P_7, H) with respect to the sequence $(4, 10, 1)$, 99 triple points, 39 triple points with support on H, the two length 4 schemes $\{2P_1, H\}$ and $\{2P_2, H\}$, two length 14 schemes supported by P_4 and P_5 and of type $(10, 4)$ with respect to H, 32 double points with support on H and a length 11 scheme with P_6 as support and type $(10, 1)$ with respect to H. The case $(d, x, y) = (13, 42, 35)$ of Remark 2, gives $h^i(H, \mathcal{I}_{H\cap C}(13)) = 0$ for $i = 0, 1$. Let D be the union of the unreduced components of $\mathrm{Res}_H(C)$. Since $\mathrm{Res}_H(C)\backslash D$ is a general union of 33 general points of H (one of them being P_6), it is sufficient to prove $h^1(\mathbb{P}^4, \mathcal{I}_D(12)) = 0$ and $h^0(\mathbb{P}^4, \mathcal{I}_{\mathrm{Res}_{H(D)}}(12)) \leq 82$ (Lemma 2). We will do here the h^1-vanishing and check at the end the h^0-inequality. We degenerate D to a general union E of one double point, a length 11 scheme supported by P_7 and with type $(10, 1)$ with respect to H, a virtual scheme obtained applying Remark 1 to (P_8, H) with respect to the sequence $(1, 10, 4)$, 39 double points with support on H, 27 triple points with support on H and 71 triple points. By Remark 2, applied to the data $(d, x, y) = (12, 29, 42)$ we have $h^i(H, \mathcal{I}_{H\cap C}(12)) = 0$ for $i = 0, 1$. Now we will check that $h^0(\mathbb{P}^4, \mathcal{I}_{\mathrm{Res}_{H(B)}}(12)) = 0$. $\mathrm{Res}_H(B)$ is a general union of 139 triple points, the two length 4 schemes $\{2P_4, H\}$ and $\{2P_5, H\}$, the points P_6 and 32 points of H. Apply the case $d = 12$ of the theorem. Let B' be the union of all unreduced components of $\mathrm{Res}_H(B)$. Now we will check that $h^0(\mathbb{P}^4, \mathcal{I}_{\mathrm{Res}_{H(D)}}(12)) \leq 82$. The scheme $\mathrm{Res}_H(D)$ is a general union of one double point, a length 11 scheme supported by P_7 and with type $(10, 1)$ with respect to H, 119 triple points, and 40 general points of H (one of them being P_6).

(i) Here we will consider the case $d = 16$. Since $\binom{20}{4} = 4845 \equiv 0 \pmod{15}$, $\epsilon(16, x, y) \equiv 0 \pmod 5$ for all x, y and it is sufficient to do all cases with $\epsilon(15, x, y) = 0$, i.e. all cases with $0 \leq x \leq 323$ and $y = 969 - 3x$.

(i1) Here we assume $(x, y) = (323, 0)$. We specialize Z to a general union Z' of a virtual scheme obtained applying Remark 1 at (P_1, H) with respect to the sequence $(1, 10, 4)$, two virtual schemes obtained applying Remark 1 at (P_2, H) and (P_3, H) with respect to the sequence $(4, 10, 1)$, 96 triple points with support on H and 224 triple points. The case $(d, x, y) = (16, 96, 2)$ of Remark 2, gives $h^i(H, \mathcal{I}_{H\cap Z'}(15)) = 0$ for $i = 0, 1$. We degenerate $\mathrm{Res}_H(Z')$ to a general union A of a length 14 scheme supported by P_1 and with type $(10, 4)$ with respect to H, two length 11 schemes supported by P_2 and P_3 and with type $(10, 1)$ with respect to H, 96 double points with support on H, 2 virtual schemes obtained applying Remark 1 at (P_4, H) and (P_5, H) with respect to the sequence $(1, 10, 4)$, 40 triple points with support on H and 182 triple points. The case $(d, x, y) = (15, 43, 96)$ of Remark 2, gives $h^i(H, \mathcal{I}_{H\cap A}(14)) = 0$ for $i = 0, 1$. Let B denote the union of the unreduced components of $\mathrm{Res}_H(A)$. Since $\mathrm{Res}_H(A)\backslash B$ is a general union of 98 points of H (two of them being P_2 and P_3), it is sufficient to prove $h^1(\mathbb{P}^4, \mathcal{I}_B(14)) = 0$ and $h^0(\mathbb{P}^4, \mathcal{I}_{\mathrm{Res}_H(B)}(13)) = 0$. We will first do the h^1-vanishing. We degenerate B to a general union C of the length 4 scheme $\{2P_1, H\}$, two length 14 schemes supported by P_4 and P_5 and of type $(10, 4)$ with respect to H, 40 double points with support on H, a virtual scheme obtained applying Remark 1 at (P_6, H) with respect to the sequence $(4, 10, 1)$, two virtual schemes obtained applying Remark 1 at (P_7, H) and (P_8, H) with respect to the sequence $(1, 10, 4)$, 49 triple points with support on H and 130 triple points. The case $(d, x, y) = (14, 43, 96)$ of Remark 2, gives $h^i(H, \mathcal{I}_{H\cap C}(14)) = 0$ for $i = 0, 1$. Let D be the union of all unreduced components of $\mathrm{Res}_H(C)$. Since $\mathrm{Res}_H(C)\backslash D$ is a general union of 40 points of H, it is sufficient to prove $h^1(\mathbb{P}^4, \mathcal{I}_D(13)) = 0$ and $h^0(\mathbb{P}^4, \mathcal{I}_{\mathrm{Res}_H(D)}(12)) \leq 96$. We will only check the vanishing of the h^1, because the h^0-inequality is even easier than the proof that $h^0(\mathbb{P}^4, \mathcal{I}_{\mathrm{Res}_H(B)}(13)) = 0$ given below. We degenerate D to a general union E of the two length 4 schemes $\{2P_4, H\}$ and $\{2P_5, H\}$, a length 11 scheme supported by P_6 and with type $(10, 1)$ with respect to H, two length 14 schemes supported by P_7 and P_8, 49 double points with support on H, 2 virtual schemes obtained applying Remark 1 at (P_9, H) and (P_{10}, H) with respect to the sequence $(1, 10, 4)$, one virtual scheme obtained applying Remark 1 at (P_{11}, H) with respect to the sequence $(4, 10, 1)$, 33 triple points with support on H and 94 triple points. The case $(d, x, y) = (13, 36, 52)$ of Remark 2, gives $h^i(H, \mathcal{I}_{H\cap E}(13)) = 0$ for $i = 0, 1$. Let F be the union of the unreduced components of $\mathrm{Res}_H(E)$. Since $\mathrm{Res}_H(E)\backslash F$ is a general union of 50 points of H (one of them being P_6), it is sufficient to prove $h^1(\mathbb{P}^4, \mathcal{I}_F(12)) = 0$ and $h^0(\mathbb{P}^4, \mathcal{I}_{\mathrm{Res}_H(F)}(11)) \leq 136$. We will only write the proof of the h^1-vanishing, because the inequality for h^0 is true by the case $d = 11$ done before (see below for another h^0-vanishing in degree 13). We degenerate F to a general union of the length 4 schemes $\{2P_7, H\}$ and $\{2P_8, H\}$, two length 14 schemes supported by P_9 and P_{10} and with type $(10, 4)$ with respect to H, a length 11 scheme supported by P_{11} and of type $(10, 1)$ with respect to H, 33 double points with support on H, a virtual scheme obtained applying Remark 1 at (P_{12}, H) with respect to the sequence $(1, 10, 4)$, a virtual scheme obtained applying Remark 1 at (P_{13}, H) with respect to the sequence $(4, 10, 1)$, 38 triple points with support on H and 54 triple points. The case $(d, x, y) = (12, 41, 35)$ of Remark 2, gives $h^i(H, \mathcal{I}_{H\cap G}(12)) = 0$ for $i = 0, 1$. Let A' be the union of the unreduced components of $\mathrm{Res}_H(G)$. Since $\mathrm{Res}_H(G)\backslash A'$ is a general union of 34 general points of H (one of them being P_{11}), it is sufficient to prove $h^1(\mathbb{P}^4, \mathcal{I}_{A'}(11)) = 0$ and $h^0(\mathbb{P}^4, \mathcal{I}_{\mathrm{Res}_H(A')}(10)) \le 186$. We will only check the vanishing of the h^1, because the h^0-inequality is even easier that the proof that $h^0(\mathbb{P}^4, \mathcal{I}_{\mathrm{Res}_H(B)}(13)) = 0$ given below. We degenerate A' to a general union B' of the length 4 schemes $\{2P_9, H\}$ and $\{2P_{10}, H\}$, a length 14 scheme supported by P_{12} and of type $(10, 4)$ with respect to H, a length 11 scheme supported by P_{13} and of type $(10, 1)$ with respect to H, 38 double points with support on H, a virtual scheme obtained applying Remark 1 at (P_{14}, H) with respect to the sequence $(4, 10, 1)$, 18 triple points with support on H and 35 triple points. The case $(d, x, y) = (11, 38, 41)$ of Remark 2, gives $h^i(H, \mathcal{I}_{H\cap B'}(11)) = 0$ for $i = 0, 1$. Let C' be the union of the unreduced components of $\mathrm{Res}_H(B')$. Since $\mathrm{Res}_H(B')\backslash C'$ is a general union of 39 general points of H (one of them being P_{13}), it is sufficient to prove $h^1(\mathbb{P}^4, \mathcal{I}_{C'}(10)) = 0$ and $h^0(\mathbb{P}^4, \mathcal{I}_{\mathrm{Res}_H(A')}(9)) \le 220$. Again, we will only write the h^1-part, because the h^0-part follows from the case $d = 7$. We degenerate C' to a general union D' of the length 4 scheme $\{2P_{12}, H\}$, a length 11 scheme supported by P_{14} and of type $(10, 1)$ with respect to H, 18 double points with support on H, two virtual schemes obtained applying Remark 1 at (P_{15}, H) and (P_{16}, H) with respect to the sequence $(1, 10, 4)$, a virtual scheme obtained applying Remark 1 at (P_{17}, H) with respect to the sequence $(4, 10, 1)$, 10 triple points with support on H and 22 triple points. The case $(d, x, y) = (10, 11, 40)$ of Remark 2, gives $h^i(H, \mathcal{I}_{H\cap D'}(11)) = 0$ for $i = 0, 1$. Let E' be the union of the unreduced components of $\mathrm{Res}_H(D')$. Since $\mathrm{Res}_H(D')\backslash E'$ is a general union of 19 general points of H (one of them being P_{14}), it is sufficient to prove $h^1(\mathbb{P}^4, \mathcal{I}_{E'}(9)) = 0$ and an omitted inequality for h^0. We could continue, but we may stop here for the following reason. A zero-dimensional scheme containing E' is contained in a generalization of the specialization Z' used the the proof of the case $d = 9$ made in part (e). Now we will check that $h^0(\mathbb{P}^4, \mathcal{I}_{\mathrm{Res}_H(B)}(13)) = 0$. $\mathrm{Res}_H(B)$ is a general union of the length 4 schemes $\{2P_4, H\}$ and $\{2P_5, H\}$, 40 general points of H and 182 triple points. It is sufficient to apply the case $(d, x, y) = (13, 182, 0)$ previously done.

(i2) Now assume $0 \le x \le 322$ and $y = 969 - 3x$. The value of x determines at which degree we need to modify the proof of the case $(x, y) = (323, 0)$ just done. For instance, if $82 \le x \le 124$ we make the first step as in part (s1), in the second step we exaust all triple points taking them with support on H (and using them also for the virtual schemes if $x \ge 122$), but we complete the second step adding double points with support on H and making up to 3 virtual constructions with them if necessary. From the next step on we may apply Lemma 2 at each step and quickly get the empty set as residual scheme.

(j) Here we assume $d = 17$. Since $\binom{21}{4} = 5985 \equiv 0 \pmod{15}$, it is sufficient to check all cases with $\epsilon(17, x, y) = 0$. $\epsilon(17, x, y) = 0$ if and only if $0 \le x \le 399$ and $y = 2 + 3(399 - x)$.

(j1) Here we assume $(x, y) = (399, 0)$. We specialize Z to a general union Z' of 114 triple points with support on H and 285 triple points. The case $(d, x, y) = (17, 114, 0)$ of Remark 2, gives $h^i(H, \mathcal{I}_{H\cap Z'}(17)) = 0$ for $i = 0, 1$. We degenerate $\mathrm{Res}_H(Z')$ to a general union A of 114 double points with support on H, 3 virtual schemes obtained applying Remark 1 at (P_1, H), (P_2, H) and (P_3, H) with respect to the sequence $(1, 10, 4)$, 51 triple points with support on H and 231 triple points. The case $(d, x, y) = (16, 51, 114)$ of Remark 2, gives $h^i(H, \mathcal{I}_{H\cap A}(16)) = 0$ for $i = 0, 1$. Let B denote the union of the unreduced components of $\mathrm{Res}_H(A)$. Since $\mathrm{Res}_H(A)\backslash B$ is a general union of 114 points of H, it is sufficient to prove $h^1(\mathbb{P}^4, \mathcal{I}_B(15)) = 0$ and $h^0(\mathbb{P}^4, \mathcal{I}_{\mathrm{Res}_H(B)}(14)) = 0$. We will first do the h^1-vanishing. We degenerate B to a general union C of two virtual schemes obtained applying Remark 1 at (P_4, H) and (P_5, H) with respect to the sequence $(1, 10, 4)$, 3 length 14 schemes supported by P_1, P_2, P_3 and of type $(10, 4)$ with respect to H, 51 double points with support on H, 58 triple points with support on H and 171 triple points. The case $(d, x, y) = (15, 61, 51)$ of Remark 2, gives $h^i(H, \mathcal{I}_{H\cap C}(15)) = 0$ for $i = 0, 1$. Let D denote the union of the unreduced components of $\mathrm{Res}_H(C)$. Since $\mathrm{Res}_H(C)\backslash D$ is a general union of 51 points of H, it is sufficient to prove $h^1(\mathbb{P}^4, \mathcal{I}_D(14)) = 0$ and $h^0(\mathbb{P}^4, \mathcal{I}_{\mathrm{Res}_H(D)}(13)) \le 114$. We will only do the h^1-vanishing, because the inequality for h^0 is easier than the proof that $h^0(\mathbb{P}^4, \mathcal{I}_{\mathrm{Res}_H(B)}(14)) = 0$ just given. Look at part (j1). Let $\mathbb{B}$ denot the scheme called B in part (s1). We checked there that $h^1(\mathbb{P}^4, \mathcal{I}_{\mathbb{D}}(14)) = 0$. $\mathbb{B}$ is a general union of the length 4 schemes $\{2P_4, H\}$ and $\{2P_5, H\}$, 40 double points with support on H and 182 triple points. We cannot quite see D as a subscheme of $\mathbb{D}$, because D contains two length 14 connected subschemes with support on H. Let C' be a general union of 3 length 14 schemes supported by P_1, P_2, P_3 and of type $(10, 4)$ with respect to H, 51 double points with support on H, 58 triple points with support on H and 173 triple points. Since $C' \cap H \subset C \cap H$, $h^1(H, \mathcal{I}_{H\cap C}(15)) = 0$. Hence it is suffcient to prove $h^1(\mathbb{P}^4, \mathcal{I}_{D'}(14)) = 0$ and $h^0(\mathbb{P}^4, \mathcal{I}_{\mathrm{Res}_H(D')}(13)) \le 112$, where D' is obtained from D deleting the two length 14 schemes with support on H and adding 2 general triple points. Since $D' \subset \mathbb{B}$ (in the sense that for any fixed, but general D' there is a sufficiently general $\mathbb{B}$ containing it), $h^1(\mathbb{P}^4, \mathcal{I}_{D'}(14)) = 0$. Now we will check that $h^0(\mathbb{P}^4, \mathcal{I}_{\mathrm{Res}_H(B)}(14)) = 0$. $\mathrm{Res}_H(B)$ is a general union of 2 double points, 231 triple points, the 3 length 4 schemes $\{2P_1, H\}$, $\{2P_2, H\}$, $\{2P_2, H\}$ and 51 general points of H. Just apply the case $d = 14$ to the union of the triple points.

(j2) Here we assume $0 \le x \le 398$ and $y = 2 + 3(398 - x)$. If $x \ge 222$, then we make the same construction and then come in a case coverered by parts (j1), (j2). If $x \le 221$, we use more double points and we arrive earlier in a case covered in parts (j1) and (j2).

(k) Here we check the case $d = 18$. Since $\binom{22}{4} = 7315 \equiv 10 \pmod{15}$, it is sufficient to do all cases with $\epsilon(18, x, y) = 0$ and the case $(x, y) = (488, 0)$. $\epsilon(18, x, y) = 0$ if and only if $0 \le x \le 487$ and $y = 2 + 3(487 - x)$.

(k1) Here we assume $(x, y) = (487, 2)$. We specialize Z to a general union Z' of 2 double points, 133 triple points with support on H and 344 triple points. The case $(d, x, y) = (18, 133, 0)$ of Remark 2, gives $h^i(H, \mathcal{I}_{H\cap Z'}(18)) = 0$ for $i = 0, 1$. We degenerate $\mathrm{Res}_H(Z')$ to a general union A of 2 double points, 133 double points with support on H, two virtual schemes obtained applying Remark 1 at (P_1, H) and (P_2, H) with respect to the sequence $(4, 10, 1)$, 60 triple points with support on H and 282 triple points. The case $(d, x, y) = (17, 60, 135)$ of Remark 2, gives $h^i(H, \mathcal{I}_{H\cap A}(17)) = 0$ for $i = 0, 1$. Let B denote the union of the unreduced components of $\mathrm{Res}_H(A)$. Since $\mathrm{Res}_H(A)\backslash B$ is a general union of 133 points of H, it is sufficient to prove $h^1(\mathbb{P}^4, \mathcal{I}_B(16)) = 0$ and $h^0(\mathbb{P}^4, \mathcal{I}_{\mathrm{Res}_H(B)}(15)) = 0$. We will first do the h^1-part. We degenerate B to a general union C of 2 double points, 60 double points with support on H, 2 length 11 schemes supported by P_1 and P_2 and with type $(10, 1)$ with respect to H, two virtual schemes obtained applying Remark 1 at (P_3, H) and (P_4, H) with respect to the sequence $(1, 10, 4)$, a virtual scheme obtained applying Remark 1 at (P_5, H) with respect to the sequence $(4, 10, 1)$, 55 triple points with support on H and 224 triple points. The case $(d, x, y) = (16, 57, 61)$ of Remark 2, gives $h^i(H, \mathcal{I}_{H\cap B}(16)) = 0$ for $i = 0, 1$. Let C denote the union of the unreduced components of $\mathrm{Res}_H(B)$. Since $\mathrm{Res}_H(B)\backslash C$ is a general union of 62 points of H (two of them being P_1 and P_2), it is sufficient to prove $h^1(\mathbb{P}^4, \mathcal{I}_C(15)) = 0$ and $h^0(\mathbb{P}^4, \mathcal{I}_{\mathrm{Res}_H(C)}(14)) \le 133$. We will only do the h^1-part, because the inequality for h^0 follows from the case $d = 14$ of Theorem 1 proved before. Call $\widetilde{C}$ a general union of two virtual schemes obtained applying Remark 1 at (P_4, H) and (P_5, H) with respect to the sequence $(1, 10, 4)$, 3 length 14 schemes supported by P_1, P_2, P_3 and of type $(10, 4)$ with respect to H, 51 double points with support on H, 58 triple points with support on H and 171 triple points. $\widetilde{C}$ is the scheme called C in step (t1) and hence we know that $h^1(\mathbb{P}^4, \mathcal{I}_{\widetilde{C}}(15)) = 0$. Since $\widetilde{C}$ is a specialization of a scheme containing C, $h^1(\mathbb{P}^4, \mathcal{I}_C(15)) = 0$. Now we will check that $h^0(\mathbb{P}^4, \mathcal{I}_{\mathrm{Res}_H(B)}(15)) = 0$. $\mathrm{Res}_H(B)$ is a general union of

2 double points, 62 general points of H (two of them being P_1 and P_2), and 282 triple points. Hence the case $d = 15$ of Theorem 1 proved before gives $h^0(\mathbb{P}^4, \mathcal{I}_{\mathrm{Res}_H(B)}(15)) = 0$.

(k2) Here we assume $0 \le x \le 486$ and $y = 2+3(487-x)$. If x is sufficiently large, we make one or two steps as in (u1) and then use only double points. After at most one step in which we use only double points,

(k3) Here we assume $(x, y) = (488, 0)$. We copy the proof of part (u1). Here we need to check $h^1(\mathbb{P}^4, \mathcal{I}_B(16)) = 0$ in which B has one more triple point, but no double point with support outside H. We may still use the scheme $\widetilde{C}$.

(l) Assume $d \ge 11$ and fix an integer t such that $8 \le t \le d-1$. We started with a general union $Z \subset \mathbb{P}^4$ of x triple points and y double points for which we wanted to prove that either $h^1(\mathbb{P}^4, \mathcal{I}_Z(d)) = 0$ (case $\epsilon(d,x,y) \ge 0$) or $h^0(\mathbb{P}^3, \mathcal{I}_Z(d)) = 0$ (case $\epsilon(d,x,y) \le 0$). Increasing or decreasing Z we may also assume $\epsilon(d,x,y) \ge -4$, unless $y = 0$ and $\epsilon(d,x,0) < 0$; for $y = 0$ we may assyme $\epsilon(d,x,0) \ge -14$. After $d-t$ steps we arrived at a scheme W and we would have won if either $h^1(\mathbb{P}^4, \mathcal{I}_W(t-1)) = 0$ or $h^0(\mathbb{P}^3, \mathcal{I}_W(t-1)) = 0$ (or $h^0(\mathbb{P}^4, \mathcal{I}_W(t-1))$ small, see below). W has at most $x+y$ connected components. Let m be the number of the simple points that we delete (quoting each time Lemma 2) in the $d-t$ steps which built W from Z. We have $\binom{d+4}{4} - \binom{t+3}{4} = \mathrm{length}(Z) - \mathrm{length}(W) - z$. If $\epsilon(x,y) \ge 0$ we need to prove $h^1(\mathbb{P}^4, \mathcal{I}_W(t-1)) = 0$ and $\binom{t+3}{4} - \mathrm{length}(W) = \epsilon(x,y) + m$. If $\epsilon(x,y) \le 0$, then it is sufficient to prove $h^0(\mathbb{P}^3, \mathcal{I}_W(t-1)) \le m - \epsilon(x,y)$. Hence it is essential to give a good lower bound for z. In the first step $d \to d-1$ there is no scheme sitting in H. Since the restriction to H of the union of all virtual schemes arising using Remark 1 is at most 5, we will apply Remark 1 at most 4 times at each step we apply a virtual construction; the only case in which we need 4 is to get 7 as congruence class modulo 10: a virtual construction at (P_1, H) with respect to the sequence $(4, 10, 1)$ and 3 virtual constructions at (P_2, H), (P_3, H) and (P_4, H) with respect to the sequence $(1, 10, 4)$. Hence $m \ge \lfloor \binom{d+4}{4} - \binom{t+4}{4}/15 \rfloor - 4(d-t)$. Taking $t = 8$ we see that it is sufficient to have $\binom{d+4}{4} \ge 60d - 480 + 15 \cdot \binom{12}{4}$, i.e. it is sufficient to take $d \ge 19$. □

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References

[1] J. Alexander, A. Hirschowitz, La méthode d'Horace éclaté: application à l'interpolation en degré quatre, *Invent. Math.*, **107** (1992), 585-602.

[2] J. Alexander, A. Hirschowitz, Polynomial interpolation in several variables, *J. Algebraic Geom.*, **4** (1995), 201-222.

[3] J. Alexander, A. Hirschowitz, An asymptotic vanishing theorem for generic unions of multiple points, *Invent. Math.*, **140** (2000), 303-325.

[4] E. Ballico, On the postulation of a general union of double points and triple points in $\mathbb{P}^3$, *Int. J. Pure Appl. Math.*, **55**, No. 1 (2009), 121-123.

[5] E. Ballico, Postulation of disjoint unions of lines and a few planes, *Preprint.*

[6] E. Ballico, M. C. Brambilla, Postulation of general quartuple fat point schemes in $\mathbb{P}^3$, *J. Pure Appl. Algebra*, **213**, No. 5 (2009), 1002-1012.

[7] M.C. Brambilla, G. Ottaviani, On the Alexander-Hirschowitz Theorem, *J. Pure Appl. Algebra*, **212**, No. 5 (2008), 1229-1251.

[8] C. De Volder, A. Laface, Base locus of linear systems on the blowing-up of $\mathbb{P}^3$ along at most 8 general points, *Pacific J. Math.*, **223**, No. 1 (2006), 17-34.

[9] C. De Volder, A. Laface, On linear systems of $\mathbb{P}^3$ through multiple points, *J. Algebra*, **310**, No. 1 (2007), 207-217.

[10] K. Chandler, A brief proof of a maximal rank theorem for generic double points in projective space, *Trans. Amer. Math. Soc.*, **353**, No. 5 (2000), 1907-1920.

[11] Daniel Grayson, Michael Stillman, *Macaulay 2*, a software system for research in algebraic geometry, available at http://www.math.uiuc.edu/Macaulay2/

[12] A. Laface, L. Ugaglia, On a class of special linear systems of $\mathbb{P}^3$, *Trans. Amer. Math. Soc.*, **358**, No. 12 (2006), 5485-5500.