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APPROXIMATION AS ATTEMPT TO INFER NONEXISTENCE OF WEAK SOLUTIONS OF RIEMANN PROBLEMS FOR THE "TWO-DIMENSIONAL P-SYSTEM"

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Abstract: A class of Riemann problems for the two-dimensional p-system is considered, for which the existence of a traditional weak solution is at best uncertain. Regarding the existence of such a solution as a postulated hypothesis, we attempt to prove the hypothesis wrong by experiment.

In this case, experiment means the construction and analysis of one-parameter sequences of ostensibly approximate solutions, obtained by both vanishing viscosity and discretization methods. In each case, failure of the method to produce a sequence provably converging to the desired solution is shown to be readily observable in the sense of numerical computations. The hypothesis of existence of a solution thus survives to the extent that no such failure is observed.

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1. Introduction

Decades of effort notwithstanding, the solution of two-dimensional Riemann problems for nonlinear systems of conservation laws remains almost entirely an open question. While apparently successful computations of weak solutions

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thereof are routinely performed, ambiguous, unconvincing such results remain commonplace. This motivates posing two questions. If an existence theorem for such problems is at present elusive, what should we be trying to prove? Alternatively, if the existence of a traditional weak solution of a given problem is uncertain, what are appropriate priorities in constructing an approximation method?

An observation from physics suggests that these two questions are related. Specifically, physics experiments do not prove theories right, they prove theories wrong. At any given time, candidates for acceptance are those theories which have survived all such experiments to date.

In this case our theory is that an admissible weak solution exists for some specific, given problem. Attempt to approximate such a solution is an experimental test of the theory. According to the above principle, the first priority of such an experiment is an unambiguous identification of failure thereof. Thus we seek approximation schemes which either provably converge to a solution of some postulated form or else provide convincing evidence of what went wrong.

Below we present such results for an example, the two-dimensional "p-system". Boundary and initial conditions are taken corresponding to a familiar problem, the reflection of an incident shock at a wedge, Chen et al [2], Henderson [5], Morawetz [9], von Neumann [10], Zheng [13].

Basic properties of this system are reviewed in Section 2; some needed results on the strong convergence of approximations are presented in Section 3. The specific initial-boundary problem of interest here is described in Section 4, with emphasis on the possible nonexistence of a solution.

Approximation by viscous regularization is considered in Section 5; approximation by discretization in Section 6. We conclude with comments on identification and implications of failure of each method to find a solution.

2. The Two-Dimensional p-System

Extended to two space dimensions with rotational symmetry, the familiar "p-system" assumes the form

$$v_t + \bigtriangledown_x \cdot u = 0_1, \tag{2.1}$$

$$u_t + \bigtriangledown_x p = 0_1, \tag{2.2}$$

with $u = (u_1, u_2), v, p$ functions of $x = (x_1, x_2), t$. When the solutions satisfy

$$v_{tt} = \Delta_x p \tag{2.3}$$

this is also called the nonlinear wave system. Riemann problems for this system have been of considerable recent interest, Canic et al [1], Jegdic [6], Jegdic et al [7], Sever [11], Tesdall et al [12].

Nonlinearity results from a given relation between v, p, of the form

$$p = P(v) \tag{2.4}$$

with P smooth and satisfying

$$P(0) = P_v(0) = 0. \tag{2.5}$$

For all $v > 0$, some $\gamma > 1$ and c a generic constant here and throughout, we assume that the function P satisfies

$$P_v(v) > 0, \tag{2.6}$$

$$P_{vv}(v) > 0, \tag{2.7}$$

$$\left(\frac{P P_{vv}}{P_v^2} \right)_v \geq 0, \tag{2.8}$$

$$P(v) \leq c + c v^{\gamma}, \tag{2.9}$$

$$P^{-1} \in C^{1/\gamma}(0, \infty). \tag{2.10}$$

Throughout we use

$$z = \begin{pmatrix} p \\ u \end{pmatrix} \tag{2.11}$$

as coordinates for phase space, isomorphic to $\mathbb{R}^3$, understanding P, P^{-1} extended to negative argument as odd functions and v obtained from

$$v = P^{-1}(p). \tag{2.12}$$

Self-similar solutions of (2.1), (2.2), depending only on $r = |x|/t, \theta : \tan\theta = x_2/x_1$, are anticipated when the initial data and any boundary conditions are independent of $|x|$. Such solutions satisfy

$$-r v_r + \nabla \cdot u = 0, \tag{2.13}$$

$$-r u_r + \nabla p = 0, \tag{2.14}$$

where, here and throughout, $\nabla, \nabla \cdot, \triangle$ are with respect to the self-similar polar coordinates r, θ.

The system (2.13), (2.14), nominally of dimension three, can be reduced to a scalar second order equation by introduction of a potential $\psi = \psi(r, \theta)$. We simplify the discussion of Sever [11] by assuming

$$\nabla_x \times u \Big|_{t=0} = 0. \tag{2.15}$$

Then (2.14) is satisfied identically by taking

$$u = \nabla\psi, \tag{2.16}$$

$$p = r\psi_r - \psi, \tag{2.17}$$

and (2.13) provides an equation for ψ, using (2.16),

$$\triangle\psi - rv_r = 0. \tag{2.18}$$

Weak solutions of (2.13), (2.14), (2.18) are anticipated, with $\psi \in W^{1,\kappa}, u, p \in L_{\kappa,loc}, v \in L_{\gamma\kappa,loc}$ for some $\kappa \geq 2$. The condition (2.6) implies that the system (2.1), (2.2) is hyperbolic, and isolated singularities appearing in self-similar solutions will be among the three familiar forms.

For this system, contact discontinuities correspond to a Dirac measure in $\bigtriangledown_x \times u$. Given the assumption (2.15), and

$$\frac{\partial}{\partial t} \bigtriangledown_x \times u = 0$$

immediately obtained from (2.2), such are precluded throughout.

Shocks may appear on curves of the form

$$r = \Lambda(\theta) \tag{2.19}$$

and move with nonzero local speed σ, given by

$$\sigma^2 = \Lambda^2/(1 + \Lambda_\theta^2/\Lambda^2). \tag{2.20}$$

The corresponding Rankine-Hugoniot conditions are Canic et al [1], Sever [11]

$$(1 + \Lambda_\theta^2/\Lambda^2)[p] = \Lambda^2[v], \tag{2.21}$$

and

$$[\psi] = 0, \tag{2.22}$$

where [ ] denotes jump across the discontinuity.

Within a rarefaction fan, parameterized by $\beta \in [\underline{\beta}, \overline{\beta}]$ and centered at a point $r = R, \theta = \Theta$, the value $v(\beta)$, which appears on whatever segment of the line

$$r\cos(\theta + \beta) = R\cos(\Theta + \beta) \tag{2.23}$$

satisfies

$$P_v(v(\beta)) = R\cos^2\beta. \tag{2.24}$$

An entropy condition, of the form Lax [8]

$$-rU_r + \nabla \cdot Q \leq 0 \tag{2.25}$$

in the sense of distributions, is used to distinguish admissible weak solutions.

For the system (2.1), (2.2),

$$U = H + \frac{1}{2}|u|^2, \tag{2.26}$$

$$Q = up, \tag{2.27}$$

$$H(v) = \int_0^v P(v')dv'. \tag{2.28}$$

From (2.28), using (2.9), we have

$$|H(v)| \le c + c|v|^{\gamma+1}. \tag{2.29}$$

Given the condition (2.6), the entropy density U is strictly convex in u, v. Nevertheless, the self-similar system (2.13), (2.14) and (2.18) exhibit a change of type. Each is hyperbolic, with $-r$ time-like, for

$$P_v(v(r,\cdot)) < r^2 \tag{2.30}$$

and elliptic for

$$P_v(v(r,\cdot)) > r^2. \tag{2.31}$$

3. Convergence of Approximations

The potential form (2.18) is useful in obtaining results on strong convergence of approximate solutions. Throughout this section, we assume a bounded open region $\omega \subset \mathbb{R}^2$ and a sequence $\{\psi_j, p_j, v_j\}_{j=1}^{\infty}$ satisfying (2.17), (2.12) and

$$||\triangle\psi_j - rv_{j,r}||_{H^{-1}(\omega)} \overset{j\to\infty}{\longrightarrow} 0. \tag{3.1}$$

Following are two lemmas, each giving sufficient conditions for strong convergence, determining a weak local solution of (2.18) or (2.13), (2.14) in the limit. In neither case is an entropy condition used.

Lemma 3.1. *Assume that*

$$||\psi_j||_{W^{1,\kappa}(\omega)} \le c \tag{3.2}$$

for some $\kappa > 2$, and

$$\underset{\omega}{\text{measure}} \quad v_{j,r} \le c. \tag{3.3}$$

Then taking a subsequence as necessary,

$$\psi_j \overset{j\to\infty}{\longrightarrow} \psi \tag{3.4}$$

strongly in $H^1(\omega)$.

Proof. From (3.3), (3.1), the sequence $\triangle\psi_j$ is the sum of two sequences, one of bounded measure in ω and one converging to zero strongly in $H^{-1}(\omega)$. By application of Lemma 15.2.1 of Dafermos [3], using (3.2), it follows that $\triangle\psi_j$ lies in a compact subset of $H^{-1}(\omega)$, and (3.4) follows. □

Lemma 3.2. *Assume*

$$\|\psi_j\|_{H^1(\omega)} \leq c \tag{3.5}$$

and for each j, almost everywhere in ω, that

$$v_j \geq 0 \tag{3.6}$$

and

$$P_v(v_j(r,\cdot)) + \frac{r^2}{2} \quad \frac{P(v_j(r,\cdot))P_{vv}(v_j(r,\cdot))}{P_v(v_j(r,\cdot))^2} \geq r^2. \tag{3.7}$$

Then the conclusion (3.4) holds.

Remark. Because of the second left-hand term in (3.7) and the assumption (2.7), the assumption (3.7) is weaker than the ellipticity condition (2.31). For example, for (2.4) of the form

$$P(v) = \frac{1}{\gamma} v^\gamma \tag{3.8}$$

the condition (2.31) is $v(r,\cdot)^{\gamma-1} > r^2$, while (3.7) is

$$v(r,\cdot)^{\gamma-1} \geq \frac{\gamma+1}{2\gamma} r^2. \tag{3.9}$$

Proof. Extracting a subsequence as necessary, we have

$$\psi_j \overset{j\to\infty}{\longrightarrow} \psi \tag{3.10}$$

strongly in $L_2(\omega)$, and

$$\psi_j \overset{j\to\infty}{\rightharpoonup} \psi \tag{3.11}$$

weakly in $H^1(\omega)$.

Using (2.17), (3.10), such a bounded, weakly convergent sequence ψ_j, p_j, v_j determines a Young measure $\nu = \nu(\eta, \tau; r, \theta)$ such that for any function f satisfying

$$|f(p, \psi_r, \frac{1}{r}\psi_\theta)| \leq c(p^2 + |\nabla\psi|^2), \tag{3.12}$$

the weak limit in $L_1(\omega)$, denoted by

$$f(p_v, \psi_{j,r}, \frac{1}{r}\psi_{j,\theta}) \overset{j\to\infty}{\rightharpoonup} \langle f \rangle \tag{3.13}$$

is given in terms of ν by

$$\langle f\rangle(r,\theta)=\int\limits_{\mathbb{R}^2} f(r\eta-\psi(r,\theta),\eta,\tau)\nu(\eta,\tau;r,\theta)d\eta d\tau. \tag{3.14}$$

From the assumptions (2.5), (2.6), (2.7), (2.8), there exists a unique $\mu(r)>0$ such that (3.7) holds with equality at $v_j(r,\cdot)=\mu(r)$,

$$P_v(\mu(r))+\frac{r^2}{2}\quad\frac{P(\mu(r))P_{vv}(\mu(r))}{P_v(\mu(r))^2}=r^2. \tag{3.15}$$

From the assumptions (3.6), (3.7), (2.8), using (3.10), (2.17), we have a restriction on the support of ν in phase space, pointwise within ω, of the form

$$\operatorname{supp}\nu(\cdot,\tau;r,\theta)\subset\{r\eta-\psi(r,\theta)\geq P(\mu(r))\}. \tag{3.16}$$

It will suffice to show that pointwise within ω,ν is supported at only one point in phase space, thus necessarily of the form

$$\nu(\eta,\tau;r,\theta)=\delta(\eta-\psi_r(r,\theta))\delta(\tau-\frac{1}{r}\psi_\theta(r,\theta)) \tag{3.17}$$

with δ here the Dirac measure. This is an application of compensated compactness.

Denote a two-dimensional vector function

$$G(p,\psi_r,\frac{1}{r}\psi_\theta)=\begin{pmatrix}\psi_r-rv\\ \frac{1}{r}\psi_\theta\end{pmatrix}. \tag{3.18}$$

Then

$$\nabla\cdot G(p_j,\psi_{j,r},\frac{1}{r}\psi_{j,\theta})=\triangle\psi_j-rv_{j,r}-2v_j. \tag{3.19}$$

Using (3.1), (3.5), (2.10), the right side of (3.19) is the sum of two terms, one vanishing in $H^{-1}(\omega)$ as $j\to\infty$ and one bounded in $L_{2\gamma}(\omega)$, uniformly with respect to j. Thus G qualifies for application of the div-curl lemma. From (3.18), each component of G satisfies (3.12), so in the space of measures on ω,

$$\langle G\cdot\nabla\psi\rangle=\langle G\rangle\cdot\langle\nabla\psi\rangle. \tag{3.20}$$

Using (3.18), we rewrite (3.20)

$$\langle\psi_r^2-rv\psi_r\rangle+\langle\frac{1}{r^2}\psi_\theta^2\rangle=\langle\psi_r-rv\rangle\langle\psi_r\rangle+\langle\frac{1}{r}\psi_\theta\rangle^2. \tag{3.21}$$

Using (3.10), (2.17), the condition (3.21) is equivalent to

$$\langle p^2-r^2vp\rangle+\langle\psi_\theta^2\rangle=\langle p-r^2v\rangle\langle p\rangle+\langle\psi_\theta\rangle^2. \tag{3.22}$$

The function p^2-r^2vp is strictly convex in p for $p>P(\mu(r))$ as determined from (3.15). Thus the only way that (3.22), (3.14), (3.16) are compatible is that ν is of the form (3.17). □

4. The Given Problem

We choose initial data depending on two parameters $p_+ > p_- > 0$,

$$z\Big|_{t=0} = \begin{cases} z_+, & x, < 0, \\ z_-, & x, > 0, \end{cases} \tag{4.1}$$

with $z_\pm$ of the form (2.11) with

$$u_{1-} = u_{2-} = u_{2+} = 0, \tag{4.2}$$

$$u_{1+} = ((p_+ - p_-)(v_+ - v_-))^{\frac{1}{2}}. \tag{4.3}$$

In the absence of physical boundaries, the corresponding entropy solution of (2.1), (2.2) or (2.13), (2.14), (2.18), denoted by $\hat{z}$, is a single shock of speed $s > 0$,

$$s = \left(\frac{p_+ - p_-}{v_+ - v_-}\right)^{\frac{1}{2}} \tag{4.4}$$

corresponding to

$$\Lambda(\theta) = s/\cos\theta \tag{4.5}$$

in (2.19). This "far-field" solution

$$\hat{z}(r,\theta) = \begin{cases} z_+, & r\cos\theta < s, \\ z_-, & r\cos\theta > s, \end{cases} \tag{4.6}$$

satisfies (2.18), (2.17),

$$\triangle\hat{\psi} - r\hat{v}_r = 0, \tag{4.7}$$

$$\hat{v} = P^{-1}(r\hat{\psi}_r - \hat{\psi}), \tag{4.8}$$

with the "far-field potential" obtained from (2.17), (2.16), (4.3), (4.6)

$$\hat{\psi}(r,\theta) = \begin{cases} -p_+ + u_{1+} r\cos\theta, & r\cos\theta < s, \\ -p_-, & r\cos\theta \geq s. \end{cases} \tag{4.9}$$

However, we restrict the domain to a wedge

$$\Omega_\infty \stackrel{def}{=} \{r > 0, \alpha < \theta < \pi\} \tag{4.10}$$

with the wedge angle $\alpha > 0$ an additional input parameter. On the physical boundaries, we impose a condition of no normal component of u, using (2.16)

$$\psi_\theta(\cdot,\alpha) = 0, \tag{4.11}$$

$$\psi_\theta(\cdot,\pi) = 0. \tag{4.12}$$

We observe that the far-field potential does not satisfy (4.11) for $0 < r < s/\cos\alpha$.

This problem, interpreted as the reflection of an incident shock by a wedge for the nonlinear wave system, has received considerable recent study Canic et al [1], Jegdic [6], Jegdic et al [7], Sever [11], Tesdall et al [12]. A "free-boundary" approach as previously been adopted, anticipating the form of a solution and using the Rankine-Hugoniot conditions (2.21), (2.22) to locate the free boundary.

Here we shall adopt an alternative, "fixed-boundary" approach, choosing an additional input parameter L,

$$L > s/\cos\alpha. \tag{4.13}$$

Given a value for L, we seek a solution of (2.18) within

$$\Omega \stackrel{def}{=} \{r < L, \alpha < \theta < \pi\} \tag{4.14}$$

such that the extension

$$\psi(r,\theta) = \hat{\psi}(r,\theta), \quad r > L, \tag{4.15}$$

determines a solution of (2.18), (4.11), (4.12) throughout Ω_∞ given in (4.10). We are thus seeking an entropy solution of (2.18), (4.11), (4.12) corresponding to initial data (4.1) and satisfying the additional condition (4.15).

The condition (4.13) assures compatibility of (4.11), (4.15), (4.9). Using (4.7), the extension (4.15) determines a global solution of (2.18) if two conditions are satisfied on the artificial boundary $r = L$:

$$\psi(L,\cdot) = \hat{\psi}(L,\cdot), \tag{4.16}$$

and

$$\psi_r(L-0,\cdot) - Lv(L-0,\cdot) = \hat{\psi}_r(L,\cdot) - L\hat{v}(L,\cdot). \tag{4.17}$$

Using (2.17), we observe that the conditions (4.16), (4.17) correspond to the Rankine-Hugoniot conditions (2.21), (2.22) for a shock on the artificial boundary, corresponding to $\Lambda(\theta) = L$ in (2.19).

We consider problems with s fixed, $|z_+ - z_-|$ and α small, and with the strength of the incident shock bounded,

$$P_v(v+) \le s^2/\cos^2\alpha. \tag{4.18}$$

Using (2.30), (4.6), the conditions (4.13), (4.18) assure that the systems (2.13), (2.14) and (2.18) are hyperbolic, with $-r$ timelike, outside of Ω. This motivates, if not justifies, the specification of two boundary conditions (4.16), (4.17) on the artificial boundary $r = L$.

In this generality, however, solutions do not exist, as least within the class of traditional weak solutions.

Theorem 4.1. *For fixed* $s > 0$, *there exist arbitrarily small* $|z_+ - z_-|, \alpha$ *satisfying* (4.2), (4.3), (4.4), (4.18), L *satisfying* (4.13), P *satisfying* (2.5), (2.6), (2.7), (2.8), (2.9), (2.10) *such that there is no traditional weak solution of* (2.18), (4.11), (4.12), (4.15).

Remark. The entropy condition (2.25) is not applied here.

Proof. Fix $s > 0, \gamma > 1$, and then v_+, p_+ satisfying (2.4) and such that

$$\frac{p_+}{v_+} < s^2 < P_v(v_+) \tag{4.19}$$

with P of the form (3.8) in the interval $[0, v_+]$.

Next we choose α such that (4.18) holds with equality. By making $P_v(v_+) - s^2$ arbitrarily small, we make $|z_+ - z_-|$ and α arbitrarily small.

For any L satisfying (4.13), denote by

$$a_+(L) = \text{area of } (\Omega \cap \{r \cos\theta < L\}), \tag{4.20}$$

$$a_-(L) = \text{area of } (\Omega \cap \{r \cos\theta > L\}). \tag{4.21}$$

Clearly

$$a_+(L) + a_-(L) = \frac{L^2}{2}(\pi - \alpha). \tag{4.22}$$

the area of Ω. As a function of L, $a_-(\cdot)$ is increasing, unbounded, and vanishing in the limit $L \downarrow s/\cos\alpha$. Thus we determine L uniquely from (4.13) and the condition

$$a_-(L) = \frac{s^2}{4}\tan\alpha. \tag{4.23}$$

With these parameters, assume that for any extension of P to $v > v_+$, there exists a solution ψ, v, p of (2.18), (4.11), (4.12), (4.15).

The condition (4.23) implies that the average of v within Ω exceeds v_+. Using (4.15), then (2.18), (4.7), then (4.11), (4.12), then (4.9),(4.3), (4.4), partial integrations give

$$\begin{aligned}\iint\limits_{\Omega}(v - \hat{v})r\,dr\,d\theta &= \iint\limits_{\Omega_\infty}(v - \hat{v})r\,dr\,d\theta \\ &= -\frac{1}{2}\iint\limits_{\Omega_\infty}(v - \hat{v})_r r^2\,dr\,dr\theta\end{aligned}$$

$$= -\frac{1}{2} \iint_{\Omega_\infty} \triangle(\psi - \hat{\psi}) r^2 drdr\theta$$

$$= -\frac{1}{2} \int_0^{s/\cos\alpha} \frac{1}{r} \hat{\psi}_\theta(r,\alpha) dr$$

$$= \frac{1}{2}(p_+ - p_-) \tan\alpha. \tag{4.24}$$

Then using (4.22), (4.23), (4.24)

$$\iint_\Omega vrdrd\theta = \frac{1}{2}(p_+ - p_-) \tan\alpha + \iint_\Omega \hat{v} rdrd\theta$$

$$= \frac{1}{2}(p_+ - p_-) \tan\alpha + v_+ a_+(L) + v_- a_-(L)$$

$$= v_+(a_+(L) + a_-(L)) + \frac{1}{4}(p_+ - p_-) \tan\alpha. \tag{4.25}$$

Thus

$$v'_+ \overset{def}{=} \iint_\Omega vrdrd\theta / (a_+(L) + a_-(L)) > v_+. \tag{4.26}$$

Next extend P to $\{v > v_+\}$ satisfying (2.6), (2.7), (2.8), (2.9), (2.10) such that

$$\inf_{\underline{v} < v'_+} \frac{P(v'_+) - P(\underline{v})}{v'_+ - \underline{v}} > L^2 \tag{4.27}$$

otherwise arbitrary. The infimum in (4.27) will occur at some $\underline{v} < 0$, which is permitted.

From (2.7), (4.27), necessarily

$$P_v(v'_+) > L^2, \tag{4.28}$$

so from (2.31), the equation (2.18) is elliptic in a nonempty region

$$\Omega_E \overset{def}{=} \{(r,\theta) \in \Omega | v(r,\theta) \geq v'_+\}. \tag{4.29}$$

The self-similar form of (2.3)

$$\triangle p = (r^2 v_r)_r \tag{4.30}$$

is also necessarily satisfied within Ω and elliptic within Ω_E. Here (4.30) is elliptic, a maximum principle applies. Thus the value

$$v''_+ \overset{def}{=} \underset{\Omega_E}{lub} \quad v \geq v'_+ \tag{4.31}$$

is necessarily approached at a boundary point of Ω_E, not a point on the physical boundary. In particular, v''_+ must be a limiting value on one side of a shock curve of the form (2.19).

Let Λ be the value of r at which a shock of local speed σ has limiting values v''_+ and some v_0. As such occurs within Ω, necessarily

$$\Lambda \leq L. \tag{4.32}$$

From (2.20), (2.21), using (4.31), (2.7)

$$\begin{aligned}\sigma^2 = & \frac{P(v''_+) - P(v_0)}{v''_+ - v_0} \\ \geq & \frac{P(v'_+) - P(v_0)}{v'_+ - v_0}.\end{aligned} \tag{4.33}$$

From (4.32), (2.20)

$$\sigma^2 < L^2. \tag{4.34}$$

Now (4.28), (4.33), (4.34) are incompatible unless

$$v_0 < v'_+. \tag{4.35}$$

But for v_0 satisfying (4.35), from (4.27),

$$\frac{P(v'_+) - P(v_0)}{v'_+ - v_0} > L^2 \tag{4.36}$$

and (4.36), (4.33), (4.34) are incompatible. □

4.1. A Second Opinion

Two weaknesses in the above argument are apparent. First the condition (4.27), although legal, is pathological. Second, the result may follow simply by the insufficiently small value of L chosen in (4.23).

There is reason to question the existence of an entropy weak solution of (2.18), (4.11), (4.12), (4.15), with small data satisfying (4.18), irrespective of the value of L and the specific form of the function P. This follows from consideration of the behavior of the anticipated solutions as the strength of the incident shock (4.1), (4.2), (4.3) is reduced.

We fix a function P – the form (3.8) for definiteness – and the parameters $\alpha > 0, s > 0, L$ satisfying (4.13) but otherwise arbitrary. The values of $z_\pm, \hat{z}, \hat{\psi}$, are obtained from (4.2), (4.3), (4.4), (4.6), (4.9), depending smoothly on $\lambda \in [0, \bar{\lambda}]$ with

$$\lambda \stackrel{def}{=} v_+ - v_- \tag{4.37}$$

and $\bar{\lambda}$ such that (4.18), (4.19) hold for all $\lambda < \bar{\lambda}$. By taking α small, $\bar{\lambda}$ is necessarily small, and we obtain small data. This is the case of primary interest.

Let $\Sigma \subseteq [0, \bar{\lambda}]$ be such that for $\lambda \in \Sigma$, there exists a weak solution of (2.18) within Ω, satisfying boundary conditions (4.11), (4.12), (4.16), (4.17) and with the following properties.

The incident shock satisfying (4.5), and the boundary of the open subset of Ω within which $\psi \neq \hat{\psi}$ are a finite set of shock curves of the form (2.19), with

$$r_i = \Lambda_i(\theta), \quad \underline{\theta}_i \leq \theta \leq \overline{\theta}_i, \tag{4.38}$$

$$\Lambda_i \in \quad W^{1,\infty}(\underline{\theta}_i, \overline{\theta}_i), \tag{4.39}$$

$i = 1, \cdots,$ with $\{\underline{\theta}_i, \overline{\theta}_i, \Lambda_i\}$ all continuously dependent on λ. The solution, however, may contain any number of additional shocks and rarefaction centers.

Uniformly with respect to $\lambda \in \Sigma$, the solutions satisfy

$$\|\psi\|_{W^{1,\kappa}(\Omega)} \leq c \tag{4.40}$$

for some $\kappa > 2$, and

$$v \geq c > 0 \tag{4.41}$$

almost everywhere in Ω.

There exists a finite number of open sets $\Sigma_i \subset \mathbb{R}, \omega_i', \omega_i'' \subset \mathbb{R}^2, i = 1, \cdots, \bar{i}$, such that

$$\Sigma \subset \bigcup_i \Sigma_i \tag{4.42}$$

and for each value of i,

$$\Omega \subset \omega_i' \cup \omega_i''. \tag{4.43}$$

For each i, uniformly with respect to $\lambda \in \Sigma_i$,

$$\underset{\omega_i'}{\text{measure}} \quad v_r \leq c \tag{4.44}$$

and (3.7) holds almost everywhere in ω_i''.

We observe that the sets ω_i', ω_i'' necessarily have nonempty intersection. For small data, we expect all points with r sufficiently small in ω_i'', and ψ very close to $\hat{\psi}$ within ω_i', thus justifying (4.44).

Lemma 4.2. *The set Σ is closed.*

Proof. Given $\underline{\lambda} \in \partial\Sigma$, using (4.42) fix i so that $\underline{\lambda} \in \Sigma_i$. We consider a sequence of solutions ψ_λ of (2.18) as $\lambda \to \underline{\lambda}$. As Σ_i is open, without loss of generality we take $\lambda \in \Sigma_i$. As $\hat{\psi}_\lambda \longrightarrow \hat{\psi}_{\underline{\lambda}}$ strongly in H^1_{loc}, the condition (3.1) holds with $\omega = \Omega$ and ψ_j replaced by ψ_λ.

Using (4.40), (4.44), we apply Lemma 3.1 with $\omega = \Omega \cap \omega_i'$. Using (4.40), (4.41), we apply Lemma 3.2 with $\omega = \Omega \cap \omega_i''$. From (4.43), it follows, taking a subsequence as necessary, that $\psi_\lambda \to \psi_{\underline{\lambda}}$ strongly in $H^1(\Omega)$ and that the limit $\psi_{\underline{\lambda}}$ satisfies (2.18).

It will thus suffice to show, again taking a subsequence as necessary, that the specified shock curves $\{\Lambda_i\}$ converge as $\lambda \to \underline{\lambda}$, and that the limit shock curves satisfy (4.39). This will follow from a condition

$$|\Lambda_{i,\theta}| \le c \tag{4.45}$$

with a constant c independent of i, θ, λ.

In the Rankine-Hugoniot condition (2.21), the condition (4.32) holds and $[p]/[v]$ is bounded below, using (4.41). Thus (4.45) is established. □

For the nonlinear wave system, solutions corresponding to reflection of an incident shock by a wedge are anticipated in only two such forms Canic et al [1], Jegdic [6], Jegdic et al [7].

Solutions corresponding to regular reflection (weak or strong) contain exactly two shocks: the incident shock

$$\Lambda_1(\theta) = s/\cos\theta, \quad \underline{\theta}_1 = \alpha, \;\; \overline{\theta}_1 = \cos^{-1}(s/L), \tag{4.46}$$

and a reflected shock, for which

$$\Lambda_2(\alpha) = s/\cos\alpha, \quad \underline{\theta}_2 = \alpha, \;\; \overline{\theta}_2 = \pi. \tag{4.47}$$

For the reflected shock, uniformly with respect to λ,

$$\Lambda_{2,\theta}(\alpha + 0) \le c < 0. \tag{4.48}$$

In addition to the analytical results cited, such solutions are routinely computed numerically; arguably, their existence is established beyond reasonable doubt. However, regular reflection is limited to $\lambda \le \lambda_R$, with $\lambda_R < \overline{\lambda}$ as determined from (4.18), (4.19), (4.37).

Irregular or Mach reflection solutions are characterized by a triple point $\theta = \theta_T > \alpha, r = s/\cos\theta_T$, which is the intersection of three shocks: the incident shock

$$\Lambda_1(\theta) = s/\cos\theta, \quad \underline{\theta}_1 = \theta_T, \quad \overline{\theta}_1 = \cos^{-1}(s/L), \tag{4.49}$$

a reflected shock, with

$$\Lambda_2(\theta_T) = s/\cos\theta_T, \underline{\theta}_2 = \theta_T, \overline{\theta}_2 = \pi, v(\Lambda_2(\theta) + 0, \theta) = v_+, \tag{4.50}$$

and a Mach stem,with

$$\Lambda_3(\theta_T) = s/\cos\theta_T, \underline{\theta}_3 = \alpha, \overline{\theta}_3 = \theta_T, v(\Lambda(\theta) + 0, \theta) = v_-. \tag{4.51}$$

The triple point is also a rarefaction center, of the form (2.23), (2.24), with

$$R = s/\cos\theta_T, \Theta = \theta_T, \underline{\beta} \ge 0. \tag{4.52}$$

Additional shocks and rarefaction centers are expected, but not at the triple point.

In contrast to regular reflection (4.48), the following holds.

Lemma 4.3.

$$\Lambda_{2,\theta}(\theta_T + 0) > 0. \tag{4.53}$$

Proof. Denote by

$$\underline{v} = v(\Lambda_2(\theta_T + 0) - 0, \theta_T + 0), \tag{4.54}$$

the limiting value of v as the triple point is approached from the region between the reflected shock and the rarefaction fan. From (2.24)

$$P_v(\underline{v}) = \frac{s^2}{\cos^2\theta_T} \quad \cos^2\underline{\beta}. \tag{4.55}$$

From (2.21), applied to Λ_2 in the limit $\theta \downarrow \theta_T$, using (4.54), (4.50),

$$\frac{P(\underline{v}) - P(v_+)}{\underline{v} - v_+} = \frac{s^2}{\cos^2\theta_T} \quad \left(1 + \frac{\Lambda_{2,\theta}(\theta_T+0)^2}{\Lambda_2(\theta_T)^2}\right)^{-1}. \tag{4.56}$$

As $\underline{v} > v_+$ from the entropy condition, compatibility of (4.55), (4.56) requires

$$\frac{\Lambda_{2,\theta}(\theta_T+0)^2}{\Lambda_2(\theta_T)^2} > \tan^2\underline{\beta}. \tag{4.57}$$

However, as the reflected shock does not pass through the rarefaction fan, necessarily

$$\tan^{-1}\left(-\frac{\Lambda_{2,\theta}(\theta_T+0)}{\Lambda_2(\theta_T)}\right) \le \underline{\beta}. \tag{4.58}$$

Compatibility of (4.57), (4.58) implies (4.53). □

The best computational evidence for the existence of such solutions is Tesdall et al [12], which indeed admits interpretation as Guderley Mach reflection [*G*]. A rarefaction center (4.52) and the condition (4.53) are observable in the results. However, this computation was done with a very strong incident shock, with parameters far from satisfying (4.18), (4.19), and notwithstanding the exceedingly fine mesh employed, the solution is less than well resolved.

The conventional wisdom for this problem is that for fixed $s > 0, \alpha$ small and L sufficiently large, there exist solutions of (2.18), (4.11), (4.12), (4.15) continuously dependent on λ up to $\overline{\lambda}$ determined from (4.18), (4.19). Something is lacking.

For $\lambda_R < \lambda \le \overline{\lambda}$, such solutions cannot correspond to regular reflection. In the absence of some additional form of solution, such solutions must correspond to Mach reflection (4.49), (4.50), (4.51), (4.53).

For small data, the conditions (4.40), (4.41), (4.43) are expected. Then from Lemma 4.2, the subset of Σ corresponding to Mach reflection contains an interval $[\underline{\lambda}, \overline{\lambda}]$ with $0 \leq \underline{\lambda} \leq \lambda_R$.

If $\underline{\lambda} = 0$, then Mach reflection is possible for arbitrarily weak incident shocks, which seems very unlikely. Indeed, using Lemma 4.3, such is provably impossible in the class of solutions for which

$$\|v - v_{\pm}\|_{L_\infty} = o(1) \tag{4.59}$$

as $\lambda \downarrow 0$.

But if $\underline{\lambda} > 0$, continuous dependence on λ requires that

$$\theta_T \downarrow \alpha \quad \text{as} \quad \lambda \downarrow \underline{\lambda} \tag{4.60}$$

and that the solution for $\lambda = \underline{\lambda}$ corresponds to regular reflection. However, the conditions (4.48), (4.53) are incompatible in this context; we omit further discussion in the interest of brevity.

We simply observe that for problem parameters such that regular reflection is impossible, the existence of a traditional weak solution of (2.18), (4.11), (4.12), (4.15) is less than certain.

5. Viscous Approximation

Notwithstanding the possible nonexistence of a traditional weak solution of (2.18), (4.11), (4.12), (4.15), approximate solutions with attractive properties are readily constructed by (at least) two methods. Here we employ viscous regularization, replacing (2.13), (2.14) by

$$-rv_{\epsilon,r} + \nabla \cdot u_\epsilon = 0, \tag{5.1}$$

$$-ru_{\epsilon,r} + \nabla p_\epsilon = \epsilon \triangle u_\epsilon, \tag{5.2}$$

with $\epsilon > 0$. The potential representation survives such regularization, with (2.16), (2.18) unchanged in form,

$$u_\epsilon = \nabla \psi_\epsilon, \tag{5.3}$$

$$\triangle \psi_\epsilon - rv_{\epsilon,r} = 0, \tag{5.4}$$

and (2.17) replaced by

$$p_\epsilon = r\psi_{\epsilon,r} - \psi_\epsilon + \epsilon \triangle \psi_\epsilon. \tag{5.5}$$

Regularization is also applied to the "far-field" solution, choosing $\hat{\psi}_\epsilon, \hat{v}_\epsilon$ satisfying the following conditions:

$$\|\hat{\psi}_\epsilon - \hat{\psi}\|_{H^1(\Omega^c)} \le c\epsilon^{\frac{1}{2}}, \tag{5.6}$$

with

$$\Omega^c \stackrel{def}{=} \{r > L, \quad \alpha < \theta < \pi\}, \tag{5.7}$$

and understanding c independent of ϵ throughout;

$$\|\hat{\psi}_\epsilon(L,\cdot) - \hat{\psi}(L,\cdot)\|_{H^1(\alpha,\pi)} \le c\epsilon^{\frac{1}{2}}; \tag{5.8}$$

$$\|\hat{\psi}_{\epsilon,r}(L,\cdot) - \hat{\psi}_r(L,\cdot)\|_{L_2(\alpha,\pi)} \le c\epsilon^{\frac{1}{2}}; \tag{5.9}$$

$$\|\hat{\psi}_\epsilon(L,\cdot)\|_{H^2(\alpha,\pi)} \le c\epsilon^{-\frac{1}{2}}; \tag{5.10}$$

$$\hat{\psi}_{\epsilon,\theta}(r,\alpha) = \hat{\psi}_{\epsilon,\theta}(r,\pi) = 0 \tag{5.11}$$

for all $r \ge L$;

$$\|\hat{v}_\epsilon(L,\cdot) - \hat{v}(L,\cdot)\|_{L_2(\alpha,\pi)} \le c\epsilon^{\frac{1}{2}}; \tag{5.12}$$

$$\|\hat{v}_\epsilon(L,\cdot)\|_{H^1(\alpha,\pi)} \le c\epsilon^{-\frac{1}{2}}. \tag{5.13}$$

The regularized far-field solution is used in a regularization of the boundary conditions (4.16), (4.17),

$$\psi_\epsilon(L,\cdot) = \hat{\psi}_\epsilon(L,\cdot) \tag{5.14}$$

and

$$\begin{aligned} &\frac{1}{\delta}(\psi_\epsilon(L,\cdot) - \psi_\epsilon(L-\delta,\cdot)) - Lv_\epsilon(L,\cdot) \\ &\qquad = \hat{\psi}_{\epsilon,r}(L,\cdot) - L\hat{v}_\epsilon(L,\cdot) \end{aligned} \tag{5.15}$$

with here and throughout, for some fixed $c > 0$,

$$\delta \stackrel{def}{=} c\epsilon^{(3\gamma+1)/(2\gamma+2)}. \tag{5.16}$$

Theorem 5.1. *For any $\epsilon > 0$, there exist $\psi_\epsilon, p_\epsilon, v_\epsilon$ satisfying* (5.4), (5.5), (2.12) *within Ω, and boundary conditions* (4.11), (4.12), (5.14), (5.15).

Furthermore

$$\|p_\epsilon\|_{L_2(\Omega)} \le c \tag{5.17}$$

and

$$\|\psi_\epsilon\|_{W^{1,2\gamma}(\Omega)} \le c. \tag{5.18}$$

Proof. The functions $\psi_\epsilon, p_\epsilon, v_\epsilon$ are obtained from a fixed point of a mapping T_1, where for any $0 \le \lambda \le 1$, the mapping $T_\lambda : L_2(\Omega) \to L_2(\Omega)$ is as follows.

Given $p \in L_2(\Omega)$, we obtain $v \in L_{2\gamma}(\Omega)$ from (2.12), (2.10), then $\psi \in W^{1,2\gamma}(\Omega)$ from

$$\triangle\psi - \lambda r v_r = 0 \tag{5.19}$$

weakly in Ω, with boundary conditions (4.11), (4.12), (5.14). From (5.19), (5.8), (2.10), uniformly with respect to $\lambda \in [0,1]$ and $\epsilon > 0$,

$$\begin{aligned} \|\psi\|_{W^{1,2\gamma}(\Omega)} &\le c + c\|v\|_{L_{2\gamma}(\Omega)} \\ &\le c + c\|p\|^{1/\gamma}_{L_2(\Omega)}. \end{aligned} \tag{5.20}$$

Next we determine $\zeta \in H^1(\Omega)$ from an ordinary differential equation in r for each value of θ,

$$r\zeta_r = P(\frac{\zeta - \psi}{\epsilon}) + \psi, \quad 0 < r < L, \tag{5.21}$$

with an initial condition at $r = L$,

$$\begin{aligned} \zeta(L,\theta) = \hat{\psi}_\epsilon(L,\theta) + \epsilon(\hat{v}_\epsilon(L,\theta) + \frac{1}{L}(\frac{1}{\delta}(\hat{\psi}_\epsilon(L,\theta) \\ - \psi(L-\delta,\theta)) - \hat{\psi}_{\epsilon,r}(L,\theta))). \end{aligned} \tag{5.22}$$

Then the image $T_\lambda p \in H^1(\Omega)$ is obtained from

$$T_\lambda p = P(\frac{\zeta - \psi}{\epsilon}). \tag{5.23}$$

Using (5.13), (5.9), for any fixed $\epsilon > 0$, T_λ maps bounded sets of $L_2(\Omega)$ into bounded sets of $H^1(\Omega)$, uniformly with respect to λ. Thus the $\{T_\lambda\}$ are collectively compact. From (5.19), the range of T_0 is a single point. Thus by a standard application of topological degree, existence of a fixed point of T_1 will follow from an a priori estimate for a fixed point of T_λ, uniform with respect to λ.

For such a fixed point satisfying $p = T_\lambda p$, from (5.23), (2.4), we may identify

$$\zeta = \psi + \epsilon v. \tag{5.24}$$

Then from (5.21), (5.24), within Ω the functions p, v satisfy (2.4) and

$$p = r\psi_r - \psi + \epsilon r v_r. \tag{5.25}$$

From (5.22), (5.14), (5.24),

$$\begin{aligned} v(L,\theta) = \hat{v}_\epsilon(L,\theta) + \frac{1}{L}(\frac{1}{\delta}(\psi(L,\theta) - \psi(L-\delta,\theta)) \\ - \hat{\psi}_{\epsilon,r}(L,\theta)). \end{aligned} \tag{5.26}$$

We multiply (5.25) by p and integrate over Ω, using (2.29), (5.20) and a partial integration to obtain

$$\begin{aligned}\|p\|^2_{L_2(\Omega)} &= \iint_\Omega (r\psi_r - \psi)prdrd\theta + \epsilon \iint_\Omega pv_r r^2 drd\theta \\ &\le c\|p\|_{L_2(\Omega)}\|\psi\|_{H^1(\Omega)} + cL^2 \int_\alpha^\pi H(v(L,\theta))d\theta \\ &- 2\epsilon \iint_\Omega H(v)rdrd\theta \\ &\le c\|p\|^{1+1/\gamma}_{L_2(\Omega)} + \epsilon\|v(L,\cdot)\|^{\gamma+1}_{L_{\gamma+1}(\alpha,\pi)}. \qquad (5.27)\end{aligned}$$

In (5.26), we use (5.12), (5.9), (5.20), (5.16) to obtain

$$\begin{aligned}\epsilon\|v(L,\cdot)\|^{\gamma+1}_{L_{\gamma+1}(\alpha,\pi)} &\le c\epsilon + c\epsilon\delta^{-(\gamma+1)/(2\gamma)}\|\psi\|^{\gamma+1}_{W^{1,2\gamma}} \\ &\le c\epsilon + c\epsilon^{(\gamma-1)/(4\gamma)}\|p\|^{1+1/\gamma}_{L_2(\Omega)}. \qquad (5.28)\end{aligned}$$

From (5.27), (5.28)

$$\|p\|^2_{L_2(\Omega)} \le c + c\|p\|^{1+1/\gamma}_{L_2(\Omega)} \qquad (5.29)$$

uniformly with respect to λ and ϵ. This establishes the existence of a fixed point of T_1, and also the claim (5.17).

For $\lambda = 1$, the equations (5.19), (5.4) are equivalent, and thus (5.25) is equivalent to (5.5). The claim (5.18) thus follows from (5.17), (5.20). □

5.1. Approximation of a Global Solution

We obtain some additional estimates for the viscous approximations. Motivated by (2.26), (2.27), (2.17), we denote by

$$U_\epsilon \stackrel{def}{=} H(v_\epsilon) + \frac{1}{2}|\nabla\psi_\epsilon|^2, \qquad (5.30)$$

$$Q_\epsilon \stackrel{def}{=} (r\psi_{\epsilon,r} - \psi_\epsilon)\nabla\psi_\epsilon. \qquad (5.31)$$

An elementary computation using (5.4), (5.30), (5.31) gives

$$\begin{aligned}0 &= p_\epsilon(\triangle\psi_\epsilon - rv_{\epsilon,r}) \\ &= -rU_{\epsilon,r} + \nabla\cdot Q_\epsilon + \epsilon(\triangle\psi_\epsilon)^2 \qquad (5.32)\end{aligned}$$

mirroring the entropy inequality (2.25).

Integrating (5.32) over Ω, using (5.30), (5.31) and partial integrations, we obtain

$$\begin{aligned}
&\epsilon\|\triangle\psi_\epsilon\|^2_{L_2(\Omega)} + 2\iint\limits_{\Omega} U_\epsilon r dr d\theta \\
&= \int\limits_{\alpha}^{\pi} (L^2 H(v_\epsilon(L,\theta)) + L\psi_\epsilon(L,\theta)\psi_{\epsilon,r}(L-0,\theta) \\
&\quad - \frac{L^2}{2}\psi_{\epsilon,r}(L-0,\theta)^2 + \frac{1}{2}\psi_{\epsilon,\theta}(L,\theta)^2) d\theta \\
&\le c + c\|v_\epsilon(L,\cdot)\|^{\gamma+1}_{L_{\gamma+1}(\alpha,\pi)} \\
&\le c + c\|p_\epsilon(L,\cdot)\|^{1+1/\gamma}_{L_{1+1/\gamma}(\alpha,\pi)}
\end{aligned} \tag{5.33}$$

using (2.29), (2.10), (5.8), (5.14) in the last steps.

Combining (5.33), (5.28), (5.17)

$$\epsilon\|\triangle\psi_\epsilon\|_{L_2(\Omega)} \le c\epsilon^{(\gamma-1)/(8\gamma)}. \tag{5.34}$$

Then from (2.10), (5.5), (5.34)

$$\begin{aligned}
\|v_\epsilon - P^{-1}(r\psi_{\epsilon,r} - \psi_\epsilon)\|_{L_2(\Omega)} &\le c\|p_\epsilon - r\psi_{\epsilon,r} + \psi_\epsilon\|^{1/\gamma}_{L_2(\Omega)} \\
&= c(\epsilon\|\triangle\psi_\epsilon\|_{L_2(\Omega)})^{1/\gamma} \\
&\le c\epsilon^{(\gamma-1)/(8\gamma^2)}.
\end{aligned} \tag{5.35}$$

We denote by

$$H^1_0(\Omega) \stackrel{def}{=} \{\phi \in H^1(\Omega) | \phi(L,\cdot) = 0\} \tag{5.36}$$

with dual $H_0^{-1'}(\Omega)$.

Then from (5.4), (5.35)

$$\|\triangle\psi_\epsilon - r(P^{-1}(r\psi_{\epsilon,r} - \psi_\epsilon))_r\|_{H_0^{-1}(\Omega)} \le c\epsilon^{(\gamma-1)/(8\gamma^2)}. \tag{5.37}$$

We extend ψ_ϵ to Ω^c given in (5.7) by

$$\psi_\epsilon(r,\theta) \stackrel{def}{=} \hat{\psi}_\epsilon(r,\theta), \quad r > L, \quad \alpha < \theta < \pi. \tag{5.38}$$

The following is an expression of the accuracy to which ψ_ϵ satisfies (2.18) in all of the wedge Ω_∞.

Theorem 5.2.

$$\begin{aligned}
&\|\triangle\psi_\epsilon - r(P^{-1}(r\psi_{\epsilon,r} - \psi_\epsilon))_r\|_{H^{-1}(\Omega_\infty)} \\
&\quad \le c\epsilon^{(\gamma-1)/(8\gamma^2)} + c\epsilon^{(\gamma-1)/(4\gamma+4)}\|p_\epsilon(L,\cdot)\|^{1+1/\gamma}_{L_{1+1/\gamma}(\alpha,\pi)}.
\end{aligned} \tag{5.39}$$

Proof. For any $\phi \in H^1(\Omega_\infty)$, using (4.11), (4.12), (5.38), (5.11) and partial integrations

$$-\iint\limits_{\Omega_\infty} \phi(\triangle\psi_\epsilon - r(P^{-1}(r\psi_{\epsilon,r} - \psi_\epsilon))_r) r dr d\theta$$
$$= \iint\limits_{\Omega} (\nabla\phi \cdot \nabla\psi_\epsilon - (r\phi_r + 2\phi)P^{-1}(r\psi_{\epsilon,r} - \psi_\epsilon)) r dr d\theta$$
$$+ \iint\limits_{\Omega^c} (\nabla\phi \cdot \nabla\hat{\psi}_\epsilon - (r\phi_r + 2\phi)P^{-1}(r\hat{\psi}_{\epsilon,r} - \hat{\psi}_\epsilon)) r dr d\theta. \tag{5.40}$$

Using (5.6), (4.8) within Ω^c,

$$|\iint\limits_{\Omega^c} (\nabla\psi \cdot \nabla(\hat{\psi}_\epsilon - \hat{\psi}) - (r\phi_r + 2\phi)(P^{-1}(r\hat{\psi}_{\epsilon,r} - \hat{\psi}_\epsilon) - \hat{v}) r dr d\theta|$$
$$\leq c\epsilon^{1/2} \|\phi\|_{H^1(\Omega^c)}. \tag{5.41}$$

From (4.7), (4.9)

$$\iint\limits_{\Omega^c} (\nabla\psi \cdot \nabla\hat{\psi} - (r\phi_r + 2\phi)\hat{v}) r dr d\theta = -L \int\limits_\alpha^\pi \phi(L,\theta)(\hat{\psi}_r(L,\theta) - L\hat{v}(L,\theta)) d\theta. \tag{5.42}$$

From (5.9), (5.12)

$$|\int\limits_\alpha^\pi \phi(L,\theta)(\hat{\psi}_{\epsilon,r}(L,\theta) - \hat{\psi}_r(L,\theta) - L\hat{v}_\epsilon(L,\theta) + L\hat{v}(L,\theta)) d\theta|$$
$$\leq c\epsilon^{1/2} \|\phi(L,\cdot)\|_{L_2(\alpha,\pi)}. \tag{5.43}$$

As $\psi_\epsilon, v_\epsilon$ satisfy (5.4) within Ω and (5.15) on $r = L$,

$$\iint\limits_{\Omega} (\nabla\phi \cdot \nabla\psi_\epsilon - (r\phi_r + 2\phi)v_\epsilon) r dr d\theta$$
$$= L \int\limits_\alpha^\pi \phi(L,\theta)(\psi_{\epsilon,r}(L-0,\theta) - Lv_\epsilon(L,\theta)) d\theta$$
$$= L \int\limits_\alpha^\pi \phi(L,\theta)(\hat{\psi}_{\epsilon,r}(L,\theta) - L\hat{v}_\epsilon(L,\theta)) d\theta$$
$$+ L \int\limits_\alpha^\pi \phi(L,\theta)(\psi_{\epsilon,r}(L-0,\theta) - \frac{1}{\delta}(\psi_\epsilon(L,\theta) - \psi_\epsilon(L-\delta,\theta))) d\theta. \tag{5.44}$$

We use (5.44) in the first right-hand term of (5.40), (5.41) in the second term, and then (5.42), (5.43) to simplify the terms from the artificial boundary $r = L$, obtaining

$$\begin{aligned}&\|\triangle\psi_\epsilon - r(P^{-1}(r\psi_{\epsilon,r} - \psi_\epsilon))_r\|_{H^{-1}(\Omega_\infty)}\\&\quad\le c\epsilon^{1/2} + c\|v_\epsilon - P^{-1}(r\psi_{\epsilon,r} - \psi_\epsilon)\|_{L_2(\Omega)}\\&\quad+ c\|\psi_{\epsilon,r}(L-0,\cdot) - \frac{1}{\delta}(\psi_\epsilon(L,\cdot) - \psi_\epsilon(L-\delta,\cdot))\|_{L_2(\alpha,\pi)}.\end{aligned} \tag{5.45}$$

The last right-hand term of (5.45) is estimated

$$\begin{aligned}&\|\psi_{\epsilon,r}(L-0,\cdot) - \frac{1}{\delta}(\psi_\epsilon(L,\cdot) - \psi_\epsilon(L-\delta,\cdot))\|_{L_2(\alpha,\pi)}\\&= \Big(\int_\alpha^\pi (\psi_{\epsilon,r}(L-0,\theta) - \frac{1}{\delta}\int_{L-\delta}^L \psi_{\epsilon,r}(r,\theta)dr)^2 d\theta\Big)^{1/2}\\&= \Big(\int_\alpha^\pi (\frac{1}{\delta}\int_{L-\delta}^L (L-r)\psi_{\epsilon,rr}(r,\theta)dr)^2 d\theta\Big)^{1/2}\\&\le c\delta^{1/2}\|\psi_{\epsilon,rr}\|_{L_2(\Omega)}.\end{aligned} \tag{5.46}$$

Using the boundary conditions (5.10), (5.11) and then (5.33), (5.16), the right-hand side of (5.46) is majorized by

$$\begin{aligned}&\delta^{1/2}(\|\triangle\psi_\epsilon\|_{L_2(\Omega)} + \epsilon^{-1/2})\\&\quad\le c\epsilon^{(\gamma-1)/(4\gamma+4)}(1 + \|p_\epsilon(L,\cdot)\|^{1+1/\gamma}_{L_{1+1/\gamma}(\alpha,\pi)}).\end{aligned} \tag{5.47}$$

We finally obtain (5.39) by using (5.35) in the second right-hand term of (5.45) and (5.46), (5.47) in the third. □

5.2. Convergence in the Limit of Vanishing Viscosity

Following is a sufficient condition for strong convergence of the viscous approximations. The limit necessarily satisfies (2.18), (2.17), (4.11), (4.12) weakly in $H^1(\Omega)$.

Theorem 5.3. *Assume a sequence $\epsilon_j \downarrow 0$ as $j \to \infty$, and the existence of ω', ω'' such that*

$$\Omega \subset \omega' \cup \omega'', \tag{5.48}$$

$$\underset{\omega'}{\text{measure}} \quad v_{\epsilon_j,r} \quad \le c, \tag{5.49}$$

and such that for each j, the conditions (3.6), (3.7) *(with $v_j = v_{\epsilon_j}$) hold almost everywhere in ω''.*

Then extracting a subsequence as necessary

$$\psi_{\epsilon_j} \stackrel{j\to\infty}{\longrightarrow} \psi \tag{5.50}$$

strongly in $H^1(\Omega)$.

Proof. Given the bound (5.18) with $\gamma > 1$, it will suffice to prove that (5.50) holds strongly in $H^1(\Omega')$ for $\Omega' \subset \Omega$ with

$$\operatorname{dist}(\Omega', \{r = L\}) > 0 \tag{5.51}$$

otherwise arbitrary.

For any such Ω', using (5.37), (5.51)

$$\begin{aligned} &\|\triangle\psi_{\epsilon_j} - r(P^{-1}(r\psi_{\epsilon_j,r} - \psi_{\epsilon_j}))_r\|_{H^{-1}(\Omega')} \\ &\quad \le c(\Omega')\|\triangle\psi_{\epsilon_j} - r(P^{-1}(r\psi_{\epsilon_j,r} - \psi_{\epsilon_j}))_r\|_{H_0^{-1}(\Omega)} \\ &\quad \le c(\Omega')\epsilon^{(\gamma-1)/(8\gamma^2)}. \end{aligned} \tag{5.52}$$

Using (5.18), (5.49), (5.52), we apply Lemma 3.1 with $\omega = \omega' \cap \Omega'$. Using (5.18), (5.52) and the assumption that (3.6), (3.7) hold almost everywhere in ω'', we apply Lemma 3.2 with $\omega = \omega'' \cap \Omega'$. Then as $\Omega' \subset \omega' \cup \omega''$ follows from (5.48), it follows that (5.50) holds strongly in Ω'. □

6. Approximation by Discretization

An alternative to regularization of a given system is discretization, obtaining approximating functions from finite-dimensional spaces, generally of lower regularity than that of the anticipated solution. An example of such a procedure, combining "finite-element" and "finite-volume" ideas, is here applied to the problem posed in Section 4.

Throughout let h be the mesh parameter, assuming a sequence of values decreasing to zero. For each value of h, we construct a union of triangles

$$\Omega_h \subset \Omega \tag{6.1}$$

with the following properties.

The discrete boundary $\partial\Omega_h$ approximates $\partial\Omega$,

$$\partial\Omega_h \subset \{\theta = \alpha\} \cup \{\theta = \pi\} \cup \{r = \Gamma_h(\theta), \quad \alpha \le \theta \le \pi\}, \tag{6.2}$$

where Γ_h is continuous and composed of straight line segments with

$$\Gamma_h(\alpha) = \Gamma_h(\pi) = L, \tag{6.3}$$

$$L - h \le \Gamma_h(\theta) \le L, \quad \alpha < \theta < \pi. \tag{6.4}$$

The origin and the two corners $(\theta = \alpha, r = L)$ and $(\theta = \pi, r = L)$ are among the vertices of Ω_h.

The line

$$r \cos\theta = s \tag{6.5}$$

on which the far-field functions $\hat{v}, \hat{\psi}$ are not smooth, does not intersect the interior of any triangles $A_{h,i}, i = 1, \cdots,$ of which Ω_h is composed.

Finally, we assume that

$$X_h \quad \text{becomes dense in} \quad H^1(\Omega_h) \quad \text{as} \quad h \downarrow 0, \tag{6.6}$$

where X_h is the space of continuous functions within Ω_h, affine in $r\cos\theta, r\sin\theta$ within each of the triangles $A_{h,i}$.

We denote by X_h^0 the subspace of X_h vanishing on the discrete artificial boundary $\{r = \Gamma_h(\theta)\}$.

The space of piecewise constant functions on the triangles $A_{h,\cdot}$ is denoted by Y_h.

Finally, Z_h is the space of piecewise constant measures on the boundary segments of the triangles $A_{h,\cdot}$, including those in the boundary $\partial\Omega_h$.

From the assumption (6.5) and the explicit forms (4.6), (4.9), within Ω_h

$$\hat{\psi} \in X_h \tag{6.7}$$

and

$$\hat{v} \in Y_h. \tag{6.8}$$

Our approximate potential functions satisfy a discrete form of (4.16),

$$\psi_h - \hat{\psi} \in X_h^0 \tag{6.9}$$

and a discrete form of (4.11), (4.12)

$$\psi_{h,\theta}(\cdot, \alpha - 0) = \psi_{h,\theta}(\cdot, \pi + 0) = 0. \tag{6.10}$$

The extension of ψ_h to Ω^c is obtained directly from (4.15),

$$\psi_h(r,\theta) = \hat{\psi}(r,\theta), \quad r > \Gamma_h(\theta), \quad \alpha < \theta < \pi. \tag{6.11}$$

For any such ψ_h, from (2.17), (2.12)

$$p_h \stackrel{def}{=} r\psi_{h,r} - \psi_h \in Y_h; \tag{6.12}$$

$$v_h \stackrel{def}{=} P^{-1}(p_h) \in Y_h. \tag{6.13}$$

From (6.11), for $r > \Gamma_h(\theta)$,

$$p_h(r,\theta) = \hat{p}(r,\theta), \tag{6.14}$$

$$v_h(r,\theta) = \hat{v}(r,\theta). \tag{6.15}$$

Using (6.9), (6.10), (6.11), (6.15)

$$e(\psi_h) \stackrel{def}{=} \triangle\psi_h - r v_{h,r} \in Z_h. \tag{6.16}$$

As dim $Z_h \approx 3$ dim X_h^0, we cannot expect to find ψ_h for which $e(\psi_h)$ vanishes. However, observing that

$$Z_h \subset H^{-1}(\mathbb{R}^2) \tag{6.17}$$

we have the following.

Lemma 6.1. *Assume a solution $\psi \in H^1_{loc}$ of* (2.18), (4.11), (4.12), (4.15), *with*

$$v \geq c > 0 \tag{6.18}$$

almost everywhere in Ω_∞.

Then as $h \downarrow 0$,

$$\inf_{\phi_h \in X_h^0} \quad \|e(\phi_h + \hat{\psi})\|_{H^{-1}(\mathbb{R}^2)} = o(1). \tag{6.19}$$

Remark. If $\psi \in H^2_{loc}$ except on a set of curves of finite total length within Ω, and the mesh is quasiuniform, then the right side of (6.19) is $O(h^{1/2})$.

Proof. For any $\phi_h \in X_h^0$, we obtain the corresponding $\psi_h = \phi_h + \hat{\psi}, p_h, v_h$ from (6.9), (6.12), (6.13) respectively. From (6.12), (6.13), (6.18)

$$\|v - v_h\|_{L_2(\Omega_h)}\| \leq c \quad \|\psi - \psi_h\|_{H^1(\Omega_h)}. \tag{6.20}$$

Then from (2.18), (6.16), (6.20)

$$\|e(\phi_h + \hat{\psi})\|_{H^1(\mathbb{R}^2)} \quad \leq c \quad \|\psi - \phi_h - \hat{\psi}\|_{H^1(\Omega_h)} \tag{6.21}$$

which becomes small for small h using (6.6). □

Throughout this section, constants c such as appearing in (6.21) are understood independent of h.

Obtaining control of $\|e\|_{H^{-1}}$ is assisted by the following.

Lemma 6.2. *Assume that as $h \downarrow 0$,*

$$\|\psi_h\|_{W^{1,\kappa}(\Omega_h)} \leq c \tag{6.22}$$

for some $\kappa > 2$, that

$$\text{measure} \quad e(\psi_h) \leq c \tag{6.23}$$

that

$$v_h \geq c > 0 \tag{6.24}$$

and that

$$\|e(\psi_h)\|_{X_h^*} \xrightarrow{h\downarrow 0} 0 \tag{6.25}$$

with

$$\|e\|_{X_h^*} \overset{def}{=} \sup_{\phi_h \in X_h} \frac{\iint e\phi_h r dr d\theta}{\|\phi_h\|_{H^1}}. \tag{6.26}$$

Then

$$\|e(\psi_h)\|_{H^{-1}} \xrightarrow{h\downarrow 0} 0. \tag{6.27}$$

Remark. The left side of (6.25) is easily computed. For $\chi_1, \chi_2, \cdots$ a basis for X_h, set

$$E_k = \iint \chi_k e(\psi_h) r dr d\theta, \quad k = 1, \cdots, \dim X_h \tag{6.28}$$

and

$$S_{k,k'} = \iint (\nabla\chi_k \cdot \nabla\chi_{k'} + \chi_k\chi_{k'}) r dr d\theta, \quad 1 \leq k, k' \leq \dim X_h. \tag{6.29}$$

Then we can take

$$\|e(\psi_h)\|_{X_h^*}^2 = E \cdot S^{-1} E. \tag{6.30}$$

However, as $\dim X_h > \dim X_h^0$, we cannot expect to find ψ_h such that $\|e(\psi_h)\|_{X_h^*}$ vanishes.

Proof. Suppose not; then as $h \downarrow 0$ there exists a sequence ψ_h satisfying (6.22), (6.24), (6.23), (6.25) but for which

$$\|e(\psi_h)\|_{H^{-1}(\Omega_h)} \geq c > 0 \tag{6.31}$$

From (6.12), (6.13), (6.22), (6.24)

$$\|v_{h,r}\|_{W_{loc}^{-1,\kappa}} \leq c. \tag{6.32}$$

From (6.16), (6.22), (6.32)

$$\|e(\psi_h)\|_{W_{loc}^{-1,\kappa}} \leq c \tag{6.33}$$

Now from (6.23), (6.33), by application of Lemma 15.2.1 of Dafermos [3], the sequence $\{e(\psi_h)\}$ lies in a compact subset of H_{loc}^{-1}, so extracting a sequence as necessary,

$$e(\psi_h) \xrightarrow{h\downarrow 0} \underline{e} \tag{6.34}$$

strongly in $H^{-1}(\Omega)$.

From (6.31), $\underline{e} \neq 0$. But then using (6.6), we can choose

$$\phi_h \approx (1 - \triangle)^{-1} \underline{e} \tag{6.35}$$

obtaining a contradiction with (6.25). □

From Lemmas 6.1 and 6.2, it follows that finding ψ_h to minimize $\|e(\psi_h)\|_{X_h^*}$ is unnecessary. It is more than sufficient to find ψ_h satisfying (6.9), (6.22), (6.24), (6.23) such that

$$\|e(\psi_h)\|_{X_h^*} \quad \leq \inf_{\phi_h \in X_h^0} \quad \|e(\phi_h + \hat{\psi})\|_{X_h^*}. \tag{6.36}$$

From (6.16), (6.12), (6.13), a partial integration using (6.9), (6.10) gives a standard "energy estimate"

$$\|e(\phi_h + \hat{\psi})\|_{X_h^*}^2 \geq c' \|\phi_h + \hat{\psi}\|_{H^1(\Omega_h)}^2 - c'' \tag{6.37}$$

with $c' > 0, c''$ independent of $\phi_h \in X_h^0$. Then from (6.37), for any ψ_h satisfying (6.36),

$$\|\psi_h\|_{H^1(\Omega)} \leq c. \tag{6.38}$$

A sufficient condition for the convergence of a subsequence of the ψ_h as $h \downarrow 0$ is similar to Theorem 5.3 for the viscous approximations.

Theorem 6.3. *For a sequence of values of $h \downarrow 0$, assume that* (6.22), (6.24), (6.23), (6.25) *hold. Assume in addition that* (5.48), (5.49) *hold (with v_{ξ_j} replaced by v_j) and that* (3.7) *holds almost everywhere in ω'' (with v_j replaced by v_h).*

Then extracting a subsequence as necessary

$$\psi_h \xrightarrow{h \downarrow 0} \psi \tag{6.39}$$

strongly in $H^1(\Omega)$.

Proof. By application of Lemma 6.2, we obtain the condition (3.1) (with ψ_j replaced by ψ_h and v_j by v_h). Then applying Lemma 3.1 with $\omega = \omega' \cap \Omega$ and Lemma 3.2 with $\omega = \omega'' \cap \Omega$ we obtain (6.39). □

7. What Went Wrong?

In the language of Introduction, Section 4 is understood as posing a hypothesis: given a specific function P and parameters $v_\pm, \alpha, L$, there exists a weak solution of (2.18), (2.17) in the entire wedge Ω_∞ with boundary conditions (4.11), (4.12) and the "far-field" condition (4.15). While we are unable to devise an experimental test of this hypothesis as stated, Sections 5 and 6 each admit interpretation as proposed experimental tests of closely related hypotheses. In each section, a proposed method will find a solution if the modified hypothesis is correct. In each case, success of the adopted method will be difficult to observe, but failure would be readily apparent. Thus each method is reasonably viewed as an attempt to prove the corresponding modified hypothesis wrong.

Section 5 is based on the additional presumption that the solution is obtained as the strong limit of viscous approximations $\psi_\epsilon, p_\epsilon, v_\epsilon$ obtained from (5.4), (5.5), (5.14), (5.15). Such is shown to be the case if no blowup occurs on the artificial boundary, meaning that the right side of (5.39) vanishes in the limit of vanishing ϵ, and if the hypotheses of Theorem 5.3 hold.

Either of two symptoms of failure of this method to find a solution thus proves the modified hypothesis wrong. If blowup occurs on the artificial boundary, then from (5.39)

$$\epsilon^{\gamma(\gamma-1)/(4(\gamma+1)^2)}\|p_\epsilon(L,\cdot)\|_{L_{1+1/\gamma}(\alpha,\pi)} \geq c > 0 \tag{7.1}$$

with a constant independent of ϵ. And if the hypotheses of Theorem 5.3 fail, then there exists $\omega \subset \Omega$ such that

$$\underset{\omega}{\text{measure}}\; v_{\epsilon,r} \xrightarrow{\epsilon\downarrow 0} \infty \tag{7.2}$$

and such that (3.6) or (3.7) (with v_j replaced by v_ϵ) fail on a subset of ω becoming dense in the limit $\epsilon \downarrow 0$. To the extent that neither of these can be ascertained, the experiment “fails” and the modified hypothesis survives with increased credibility.

In Section 6, we assume the existence of a solution satisfying

$$\|\psi\|_{W^{1,\kappa}(\Omega)} \leq c \tag{7.3}$$

for some $\kappa > 2$; that

$$v \geq c > 0 \tag{7.4}$$

holds almost everywhere in Ω; and that (5.48) holds, with

$$\underset{\omega'}{\text{measure}} \quad v_r < \infty \tag{7.5}$$

and (3.7) (with v_j replaced by v) almost everywhere in ω''.

The discretization algorithm adopted fails to find a solution, thus proving the modified hypothesis wrong, if for any sequence ψ_h, as $h \downarrow 0$ any of the conditions (6.22), (6.24), (6.23), (6.25) fails, or if there exists $\omega \subset \Omega$ such that

$$\underset{\omega}{\text{measure}} \quad v_{h,r} \xrightarrow{h\downarrow 0} \infty \tag{7.6}$$

and such that (3.7) (with v_j replaced by v_h) fails on a subset of ω becoming dense as $h \downarrow 0$.

By imposing constraints in the minimization of $\|e(\psi_h)\|_{X_h^*}$, it can be assured that the conditions (6.22), (6.24), (6.23) hold, that (5.48), (5.49) (with v_{ξ_j} replaced by v_h in (5.49)), and that (3.7) holds (with v_j replaced by v_h) almost everywhere in ω''. Then the outcome of the experiment is determined by the condition (6.25). We tacitly assume that the minimization is done to satisfy (6.36). Then the experiment "succeeds", and proves the modified hypothesis (7.3), (7.4), (7.5) wrong, to the extent that (6.25) fails. Otherwise the hypothesis survives, again with increased credibility.

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