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THE AK RAMSEY MODEL WITH EXPONENTIAL UTILITY AND LOGISTIC POPULATION CHANGE

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Abstract: This paper investigates a Ramsey growth model with exponential utility and logistic population growth. The model's solution is found analytically and its long-run behavior is determined.

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1. Introduction

The growth model developed by Ramsey [6], and subsequently adapted by Cass [2] and Koopmans [5], is arguably the single most important theoretical construct in modern macroeconomics (see Barro and Sala-i-Martin [1]). Recently, Guerrini [4] have explored the implications of studying the Ramsey model within a framework where the change over time of the labor force is governed by the logistic growth law. In this paper, we want to analyze the Ramsey model with logistic population growth by replacing the CARA utility function with an exponential utility function. This leads to a non-linear dynamical system with no unique or isolated rest point. Consequently, even if we cannot solve the linear approximation, which retains the main dynamic properties of the original non-linear system, we will be able to write a general solution for the model as well as to study its long-run behavior.

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2. The Model

We start by considering the modified Ramsey's [6] model of a representative consumer whose objective is to choose per capita consumption c so as to maximize

$$\int_0^\infty -\frac{\left(1-e^{-\theta c}\right)}{\theta}\, e^{-\rho t} dt,$$

subject to

$$\dot{k} = Ak - (a - bL)k - c, \tag{1}$$

where k is per capita capital stock, $\rho > 0$ is the rate of time preference, $\theta > 0$, and L is population. For simplicity, there is no capital depreciation. The utility function is now assumed to take the constant absolute averse form. As well population dynamic is supposed to evolve according the logistic growth model, i.e., $\dot{L} = L(a-bL)$, $a > b > 0$, where $L_0 = 1$. In order to solve this optimization problem we use the Pontryagin Maximum Principle. We can derive for the Pontryagin paths by first setting up the current-value Hamiltonian, which is

$$H(k, c, L, \lambda) = -\frac{\left(1-e^{-\theta c}\right)}{\theta} + \lambda \left[Ak - (a - bL)k - c\right],$$

where λ is the costate variable associated to the budget constraint (1). Necessary and sufficient conditions for a path to be optimal under the assumptions on the utility, the production function, and the population made here are that

$$H_c = 0 \quad \Rightarrow \quad -e^{-\theta c} = \lambda, \tag{2}$$

$$\dot{\lambda} = \rho\lambda - H_k \quad \Rightarrow \quad \dot{\lambda} = -\lambda \left[A - \rho - (a - bL)\right], \tag{3}$$

together with equation (1), the boundary condition $k_0 > 0$, and the transversality condition $\lim_{t\to\infty} e^{-\rho t}\lambda k = 0$. Totally differentiating (2), and using it in conjunction with (3) yields $\dot{c} = [A - \rho - (a - bL)]/\theta$. Hence, the dynamic system, which describes the economy, comes down to

$$\dot{k} = Ak - (a - bL)k - c, \tag{4}$$

$$\dot{c} = \frac{A - \rho - (a - bL)}{\theta}, \tag{5}$$

$$\dot{L} = L(a - bL), \tag{6}$$

plus the transversality condition

$$\lim_{t\to\infty} e^{-\rho t} e^{-\theta c} k = 0. \tag{7}$$

3. Steady State and Dynamics

We start with the steady state. In steady state, both the per capita capital stock, the level of consumption per capita, and population are constant. We denote the steady state values of these variables by k_*, c_*, and L_*, respectively.

Lemma 1. *The economy has infinite steady state equilibria if $A = \rho$, and no steady state if $A \neq \rho$.*

Proof. Setting $\dot{k} = \dot{c} = \dot{L} = 0$, we get $c = Ak$, $A - \rho = 0$, $L = a/b$. The statement is now immediate. □

Remark 1. For any $k_* > 0$, the point (k_*, c_*, L_*), where $c_* = Ak_*$, and $L_* = a/b$, is a non-trivial steady state equilibrium.

Henceforth, let $A = \rho$. Linearizing the dynamical system (4)-(6) around a steady state (k_*, c_*, L_*) yields the following approximated dynamic system

$$\begin{bmatrix} \dot{k} \\ \dot{c} \\ \dot{L} \end{bmatrix} = J^* \begin{bmatrix} k - k_* \\ c - c_* \\ L - L_* \end{bmatrix}, \text{ with } J^* = \begin{bmatrix} A & -1 & bk_* \\ 0 & 0 & b/\theta \\ 0 & 0 & -a \end{bmatrix}. \tag{8}$$

In order to characterize the local stability of our system, we need to compute the eigenvalues of the Jacobian matrix J^*. We recall that an equilibrium point of a system of differential equations is called hyperbolic if the Jacobian matrix calculated at that point has no zero or purely imaginary eigenvalues (no eigenvalue has real part equal to zero). There exists a general result in the theory of differential equations, known as the Hartman-Grobman Theorem (see, e.g., Guckenheimer and Holmes [3]), which guarantees that, if the equilibrium point is hyperbolic, in a neighbourhood of the equilibrium point the qualitative properties of the non-linear system (4)-(6) are preserved by the linearization (8). In our case, it is immediately seen that the eigenvalues of J^* are given by $\lambda_1 = A > 0$, $\lambda_2 = 0$, $\lambda_3 = -a < 0$. Hence, the equilibrium point (k_*, c_*, L_*) is non-hyperbolic, and, consequently, the linear approximation cannot be used to study the dynamic properties of the original non-linear system. The next result will help investigating the dynamics.

Proposition 1.

$$k = e^{At} L^{-1} \left[k_0 - \int_0^t e^{-At} L \left(c_0 - \frac{\ln L}{\theta} \right) dt \right], \tag{9}$$

$$c = c_0 - \frac{\ln L}{\theta}, \qquad L = \frac{ae^{at}}{a - b + be^{at}}. \tag{10}$$

Proof. As a function of time, equation (6) is a Bernoulli's differential equation. It is known that the change of variables $w = L^{-1}$ transforms it into a linear first-order differential equation in w, whose solution is easily found to be $L = ae^{at}/(a - b + be^{at})$. Next, substituting for L in equation (5), and integrating, starting from some constant initial level of consumption c_0, still to be determined, we obtain $c = c_0 - (\ln L)/\theta$. Plugging this expression into equation (4) yields a first order linear differential equation in k, which can be solved explicitly as

$$k = e^{\int_0^t (A - \dot{L}/L)dt} \left[k_0 - \int_0^t e^{-\int_0^t (A - \dot{L}/L)dt} \left(c_0 - \frac{\ln L}{\theta} \right) dt \right]$$

$$= e^{At} L^{-1} \left[k_0 - \int_0^t e^{-At} L \left(c_0 - \frac{\ln L}{\theta} \right) dt \right]. \qquad \square$$

Corollary 1. *The function L is monotone increasing from 1 to $L_\infty = a/b$, and c is monotone decreasing from c_0 to $c_0 - (\ln L_\infty)/\theta$. Moreover, we have*

$$\lim_{t\to\infty} k = \frac{1}{A}\left(c_0 - \frac{\ln L_\infty}{\theta} \right), \quad \textit{where } c_0 = \frac{k_0 + \int_0^\infty \frac{e^{-At} L \ln L}{\theta} dt}{\int_0^\infty e^{-At} L dt}.$$

Proof. The first part of the statement follows from (10) noting that $\dot{L} > 0$, $\dot{c} = -\dot{L}/(\theta L) < 0$, and using the fact that $1 \le L \le L_\infty$. Next, substituting for c and k into the transversality condition (7), we obtain

$$\lim_{t\to\infty} e^{-\rho t} e^{-\theta c} k = 0 \Leftrightarrow k_0 = \int_0^\infty e^{-At} L \left(c_0 - \frac{\ln L}{\theta} \right) dt \Leftrightarrow c_0 = \frac{k_0 + \int_0^\infty \frac{e^{-At} L \ln L}{\theta} dt}{\int_0^\infty e^{-At} L dt}.$$

The final part of the statement is now an application of Hopital's rule:

$$\lim_{t\to\infty} k = \frac{1}{L_\infty} \lim_{t\to\infty} \frac{k_0 - \int_0^t e^{-At} L \left(c_0 - \frac{\ln L}{\theta} \right) dt}{e^{-At}} = \frac{1}{A}\left(c_0 - \frac{\ln L_\infty}{\theta} \right). \qquad \square$$

4. Conclusion

In this paper, we have considered a modified version of the Ramsey growth model, obtained by assuming an exponential utility, and a logistic law formulation for the evolution of population. Within this setup, the model is shown to be described by a non-linear dynamical system with no unique or isolated steady state equilibrium. The original non-linear system is seen to be structurally unstable and, consequently, the linear approximation method cannot be used to study its dynamic properties. However, we have been able to determine the general form of the model's solution, and to investigate its long-run behavior.

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