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SEMICONJUGATE FACTORIZATION OF NON-AUTONOMOUS HIGHER ORDER DIFFERENCE EQUATIONS

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Abstract: We introduce time-dependent form symmetries to obtain semiconjugate factorizations and reductions in orders in a larger set of non-autonomous difference equations than previously considered. We show that there is a substantial class of equations having this feature that includes the general (non-autonomous, non-homogeneous) linear equation with variable coefficients in an arbitrary algebraic field.

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1. Introduction

In previous studies (e.g., [3], [4], [5] or [6]) of semiconjugate factorizations of higher order difference equations of type

$$x_{n+1} = f_n(x_n, x_{n-1}, \ldots, x_{n-k}) \tag{1}$$

the form symmetry linking the higher dimensional unfolding map of the original equation to that of the lower dimensional factor was assumed to be independent of n. This method worked for non-homogeneous linear equations with constant coefficients but did not apply to linear equations with variable coefficients.

The main goal of this article is to extend the aforementioned factorization method to *time-dependent* form symmetries that may depend explicitly on the discrete independent variable n. This extension is significant as it covers all non-autonomous equations of type (1). In particular, the extended method may be applied to general (non-autonomous, non-homogeneous) linear equations

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over arbitrary algebraic fields to show that such equations admit semiconjugate factorizations via eigensequences (i.e., solutions of an associated discrete Riccati difference equation of lower order).

2. Semiconjugate Relation and Factorization

For ease of reference we state some of the basic concepts and notation here; additional background material for this article is available in [5]. For related results and examples on factorizations and reductions in orders of difference equations that are beyond the scope of [5], we refer to [2].

The number k in (1) is a fixed positive integer and $k+1$ represents the *order* of the difference equation (1). The underlying space of variables x_n is a group G and $f_n : G^{k+1} \to G$ is a given function for each $n \geq 1$. If $f_n = f$ does not explicitly depend on n then (1) is said to be *autonomous*; it is *non-autonomous* otherwise. A (forward) *solution* of equation (1) is a sequence $\{x_n\}_{n=-k}^{\infty}$ that is recursively generated by (1) from a set of $k+1$ initial values $x_0, x_{-1}, \ldots, x_{-k} \in G$.

Each f_n is "unfolded" by the associated vector map $F_n : G^{k+1} \to G^{k+1}$ that are defined as

$$F_n(u_0, \ldots, u_k) = [f_n(u_0, \ldots, u_k), u_0, \ldots, u_{k-1}], \quad u_j \in G \text{ for } j = 0, 1, \ldots, k. \tag{2}$$

Let F_n be the unfolding on G^{k+1} of f_n for each n. Then (1) is equivalent to

$$X_{n+1} = F_n(X_n), \quad X_n = (x_n, \ldots, x_{n-k}). \tag{3}$$

We are interested in deriving a lower dimensional equation

$$Y_{n+1} = \Phi_n(Y_n), \quad Y_n = (y_n, \ldots, y_{n-m+1}), \ m \leq k \tag{4}$$

for (3). If there exists a sequence of maps $H_n : G^{k+1} \to G^m$ such that for every solution $\{X_n\}$ of (3)

$$Y_n = H_n(X_n), \quad n = 0, 1, 2, \ldots \tag{5}$$

is a solution of (4) then

$$\Phi_n(H_n(X_n)) = \Phi_n(Y_n) = Y_{n+1} = H_{n+1}(X_{n+1}) = H_{n+1}(F_n(X_n)).$$

Therefore, (5) is satisfied for all solutions of (3) and (4) if and only if the sequence $\{H_n\}$ of maps satisfies the following equality for all n

$$H_{n+1} \circ F_n = \Phi_n \circ H_n. \tag{6}$$

If the mappings H_n are independent of n, i.e., $H_n = H$ for all n then equation (6) reduces to the time-independent semiconjugate relation of previous studies.

Definition 1. Let $k \geq 1$, $1 \leq m \leq k$. If there is a sequence of surjective maps $H_n : G^{k+1} \to G^m$ such that equation (6) is satisfied for a given pair of function sequences $\{F_n\}$ and $\{\Phi_n\}$ then we say that F_n is *semiconjugate* to Φ_n for each n and refer to the sequence $\{H_n\}$ as a *(time-dependent) form symmetry* of equation (3) or equivalently, of equation (1). Since $m < k+1$, the form symmetry $\{H_n\}$ is *order-reducing*.

The following result extends its time-independent analog in [5].

Lemma 2. *Let $k \geq 1$, $1 \leq m \leq k$, let $h_n : G^{k-m+1} \to G$ for $n \geq -m+1$ be a sequence of functions on a given non-trivial group G and define the functions $H_n : G^{k+1} \to G^m$ by*

$$\begin{aligned} H_n(u_0, \ldots, u_k) \\ = [u_0 * h_n(u_1, \ldots, u_{k+1-m}), \ldots, u_{m-1} * h_{n-m+1}(u_m, \ldots, u_k)]. \quad (7) \end{aligned}$$

Then the following statements are true:

(a) The function H_n defined by (7) is surjective for each fixed $n \geq 0$.

(b) If $\{H_n\}$ is an order-reducing form symmetry then the difference equation (1) is equivalent to the system of equations

$$t_{n+1} = \phi_n(t_n, \ldots, t_{n-m+1}), \quad (8)$$

$$x_{n+1} = t_{n+1} * h_{n+1}(x_n, \ldots, x_{n-k+m})^{-1} \quad (9)$$

whose orders m and $k+1-m$ respectively, add up to the order of (1).

(c) The map $\Phi_n : G^m \to G^m$ in (6) is the unfolding of equation (8) for each $n \geq 0$; i.e., each Φ_n is of scalar type.

Proof. (a) Let n be a fixed non-negative integer and for $j = 0, \ldots, m-1$ denote the j-th coordinate function of H_n by

$$\eta_{j+1}(u_0, \ldots, u_k) = u_j * h_{n-j}(u_{j+1}, \ldots, u_{j+k+1-m}) \quad (10)$$

Now choose an arbitrary point $(v_1, \ldots, v_m) \in G^m$ and define

$$\begin{aligned} u_{m-1} &= v_m * h_{n-m+1}(u_m, u_{m+1} \ldots, u_k)^{-1}, \quad (11) \\ u_m &= u_{m+1} = \ldots u_k = \bar{u} \end{aligned}$$

where $\bar{u}$ is a fixed element of G, e.g., the identity. Then

$$\begin{aligned} v_m &= u_{m-1} * h_{n-m+1}(\bar{u}, \bar{u} \ldots, \bar{u}) \\ &= u_{m-1} * h_{n-m+1}(u_m, u_{m+1} \ldots, u_k) \end{aligned}$$

$$= \eta_m(u_0,\ldots,u_k)$$
$$= \eta_m(u_0,\ldots,u_{m-2},\underbrace{v_m * h_{n-m+1}(\bar{u},\bar{u}\ldots,\bar{u})^{-1}}_{u_{m-1}},\bar{u}\ldots,\bar{u}).$$

for any selection of elements $u_0,\ldots,u_{m-2}\in G$. Using the same idea, define

$$u_{m-2} = v_{m-1} * h_{n-m+2}(u_{m-1},\bar{u}\ldots,\bar{u})^{-1}$$

with u_{m-1} defined by (11) so as to get

$$\begin{aligned} v_{m-1} &= u_{m-2} * h_{n-m+2}(u_{m-1},\bar{u}\ldots,\bar{u}) \\ &= u_{m-2} * h_{n-m+2}(u_{m-1},u_m\ldots,u_{k-1}) \\ &= \eta_{m-1}(u_0,\ldots,u_k) \\ &= \eta_{m-1}(u_0,\ldots,u_{m-3},\underbrace{v_{m-1} * h_{n-m+2}(u_{m-1},\bar{u}\ldots,\bar{u})^{-1}}_{u_{m-2}},u_{m-1},\bar{u}\ldots,\bar{u}) \end{aligned}$$

for any choice of $u_0,\ldots,u_{m-3}\in G$. Continuing in this way, by induction we obtain elements $u_{m-1},\ldots,u_0\in G$ such that

$$v_i = \eta_i(u_0,\ldots,u_{m-1},\bar{u}\ldots,\bar{u}),\quad i=1,\ldots,m.$$

Therefore, $H_n(u_0,\ldots,u_{m-1},\bar{u}\ldots,\bar{u}) = (v_1,\ldots,v_m)$ and it follows that H_n is onto G^m.

(b) To show that the SC factorization system consisting of equations (8) and (9) is equivalent to equation (1) we show that: (i) each solution $\{x_n\}$ of (1) uniquely generates a solution of the system (8) and (9) and conversely (ii) each solution $\{(t_n,y_n)\}$ of the system (8) and (9) correseponds uniquely to a solution $\{x_n\}$ of (1). To establish (i) let $\{x_n\}$ be the unique solution of (1) corresponding to a given set of initial values $x_0,\ldots x_{-k}\in G$. Define the sequence

$$t_n = x_n * h_n(x_{n-1},\ldots,x_{n-k+m-1}) \tag{12}$$

for $n\geq -m+1$. Then for each $n\geq 0$ if H_n is defined by (7) it follows from the semiconjugate relation (6) that

$$\begin{aligned} x_{n+1} &= f_n(x_n,\ldots,x_{n-k}) \\ &= \phi_n(x_n * h_n(x_{n-1},\ldots,x_{n-k+m-1}),\ldots, \\ &\qquad x_{n-m+1} * h_{n-m+1}(x_{n-m},\ldots,x_{n-k})) * [h_{n+1}(x_n,\ldots,x_{n-k+m})]^{-1} \\ &= \phi_n(t_n,\ldots,t_{n-m+1}) * [h_{n+1}(x_n,\ldots,x_{n-k+m})]^{-1}. \end{aligned}$$

Therefore, $\phi_n(t_n,\ldots,t_{n-m+1}) = x_{n+1} * h_{n+1}(x_n,\ldots,x_{n-k+m}) = t_{n+1}$ so that $\{t_n\}$ is the unique solution of the factor equation (8) with initial values

$$t_{-j} = x_{-j} * h_{-j}(x_{-j-1},\ldots,x_{-j-k+m-1}),\quad j=0,\ldots,m-1.$$

Further, since $x_{n+1} = t_{n+1} * [h_{n+1}(x_n,\ldots,x_{n-k+m})]^{-1}$ for $n\geq 0$ by (12),

$\{x_n\}$ is the unique solution of the cofactor equation (9) with initial values $y_{-i} = x_{-i}$ for $i = 0, 1, \ldots, k - m$ and with the values t_n obtained above.

To establish (ii) let $\{(t_n, y_n)\}$ be a solution of the factor-cofactor system with initial values

$$t_0, \ldots, t_{-m+1}, y_{-m}, \ldots y_{-k} \in G.$$

Note that these numbers determine $y_{-m+1}, \ldots, y_0$ through the cofactor equation

$$y_{-j} = t_{-j} * [h_{-j}(y_{-j-1}, \ldots, y_{-j-1-k+m})]^{-1}, \quad j = 0, \ldots, m - 1. \tag{13}$$

Now for $n \geq 0$ we obtain

$$\begin{aligned} y_{n+1} &= t_{n+1} * [h_{n+1}(y_n, \ldots, y_{n-k+m})]^{-1} \\ &= \phi_n(t_n, \ldots, t_{n-m+1}) * [h_{n+1}(y_n, \ldots, y_{n-k+m})]^{-1} \\ &= \phi_n(y_n * h_n(y_{n-1}, \ldots, y_{n-k+m-1}), \ldots, \\ &\qquad y_{n-m+1} * h_{n-m+1}(y_{n-m}, \ldots, y_{n-k})) * h_{n+1}(y_n, \ldots, y_{n-k+m})^{-1} \\ &= f_n(y_n, \ldots, y_{n-k}) \end{aligned}$$

Thus $\{y_n\}$ is the unique solution of equation (1) that is generated by the initial values (13) and $y_{-m}, \ldots y_{-k}$. This completes the proof of (b).

(c) We show that each coordinate function $\phi_{j,n}$ is the projection into coordinate $j - 1$ for $j > 1$. From the definition of H_n in (7) and the semiconjugate relation (6) we infer that

$$\begin{aligned} H_{n+1}(F_n(u_0, \ldots, u_k)) &= H_{n+1}(f_n(u_0, \ldots, u_k), u_0, \ldots, u_{k-1}) \\ &= (f_n(u_0, \ldots, u_k) * h_{n+1}(u_0, \ldots, u_{k-m}), \\ &\quad u_0 * h_n(u_1, \ldots, u_{k-m+1}), \ldots, \\ &\quad u_{m-2} * h_{n-m+2}(u_{m-1}, \ldots, u_{k-1})). \end{aligned}$$

Matching the corresponding component functions in the above equality for $j \geq 2$ yields

$$\begin{aligned} \phi_{j,n}(u_0 * h_n(u_1, \ldots, u_{k+1-m}), \ldots, u_{m-1} * h_{n-m+1}(u_m, \ldots, u_k)) = \\ u_{j-2} * h(u_{j-1}, u_j \ldots, u_{j+k-m-1}) \end{aligned}$$

which shows that $\phi_{j,n}$ maps its j-th coordinate to its $(j - 1)$-st. Therefore, for each n and every $(t_1, \ldots, t_m) \in H_n(G^{k+1})$ we have

$$\Phi_n(t_1, \ldots, t_m) = [\phi_n(t_1, \ldots, t_m), t_1, \ldots, t_{m-1}]$$

i.e., $\Phi_n|_{H_n(G^{k+1})}$ is of scalar type. Since by Part (a) $H_n(G^{k+1}) = G^m$ for every n, it follows that Φ_n is of scalar type. □

The pair of equations (8) and (9) in Theorem 2 is uncoupled in the sense that

(8) is independent of (9). Such a pair forms a triangular system as defined in [1] and [7]. In the next definition we use convenient and suggestive terminology to describe these equations.

Definition 3. equation (8) is a *factor* of equation (1) since it is derived from the semiconjugate factor Φ_n. equation (9) that links the factor to the original equation is a *cofactor* of equation (1). We refer to the system of equations (8) and (9) as a *semiconjugate (SC) factorization* of equation (1). Note that *orders m and $k+1-m$ of (8) and (9) respectively, add up to the order of (1). We refer to the system of equations (8) and (9) as a type-$(m, k+1-m)$ order reduction* of equation *(1).*

3. Invertible-Map Criterion

In [4] a useful necessary and sufficient condition is obtained by which to determine whether the difference equation (1) has order-reducing form symmetries (not time-dependent). In this section we extend this useful idea to the time-dependent case.

Consider the following special case of (7) with $m = k$

$$H_n(u_0, u_1, \ldots, u_k) = [u_0 * h_n(u_1), u_1 * h_{n-1}(u_2), \ldots, u_{k-1} * h_{n-k+1}(u_k)] \quad (14)$$

with $h_n : G \to G$ being a sequence of surjective self-maps of the underlying group G for $n \geq -k+1$. If (1) has the form symmetry (14) then it admits a type-$(k, 1)$ order-reduction and its SC factorization is

$$t_{n+1} = \phi_n(t_n, \ldots, t_{n-k+1}), \quad (15)$$

$$x_{n+1} = t_{n+1} * h_{n+1}(x_n)^{-1}. \quad (16)$$

The initial values of the factor equation (15) are given in terms of the initial values of (1) as follows

$$t_{-j} = x_{-j} * h_{-j}(x_{-j+1}), \quad j = 0, 1, \ldots, k-1.$$

Theorem 4. *(Time-dependent invertible map criterion) Assume that $h_n : G \to G$ is a sequence of bijections of G for $n \geq -k+1$. For arbitrary elements $u_0, v_1, \ldots, v_k \in G$ and every $n \geq 0$ define $\zeta_{0,n}(u_0) \equiv u_0$ and for $j = 1, \ldots, k$,*

$$\zeta_{j,n}(u_0, v_1, \ldots, v_j) = h^{-1}_{n-j+1}(\zeta_{j-1,n}(u_0, v_1, \ldots, v_{j-1})^{-1} * v_j). \quad (17)$$

with the usual distinction observed between map inversion and group inversion. Then equation (1) has the form symmetry $\{H_n\}$ defined by (14) if and only if *the quantity*

$$f_n(\zeta_{0,n}, \zeta_{1,n}(u_0, v_1), \ldots, \zeta_{k,n}(u_0, v_1, \ldots, v_k)) * h_{n+1}(u_0) \quad (18)$$

is independent of u_0 *for every* $n \geq 0$.

In this case equation (1) has a SC factorization whose factor functions in (15) are given by

$$\phi_n(v_1, \ldots, v_k) = f_n(\zeta_{0,n}, \zeta_{1,n}(u_0, v_1), \ldots, \zeta_{k,n}(u_0, v_1, \ldots, v_k)) * h_{n+1}(u_0). \quad (19)$$

Proof. Assume first that (18) is independent of u_0 for all $v_1, \ldots, v_k$ so that the functions

$$\phi_n(v_1, \ldots, v_k) = f_n(\zeta_{0,n}, \zeta_{1,n}, \ldots, \zeta_{k,n}) * h_{n+1}(u_0) \quad (20)$$

are well defined. Next, if H_n is given by (14) then for all $u_0, u_1, \ldots, u_k$

$$\phi_n(H_n(u_0, u_1, \ldots, u_k)) = \phi_n(u_0 * h_n(u_1), u_1 * h_{n-1}(u_2), \ldots, u_{k-1} * h_{n-k+1}(u_k)).$$

Now, by (17) for each n and all u_0, u_1

$$\zeta_{1,n}(u_0, u_0 * h_n(u_1)) = h_n^{-1}(u_0^{-1} * u_0 * h_n(u_1)) = u_1.$$

Similarly, for each n and all u_0, u_1, u_2

$$\begin{aligned}\zeta_{2,n}(u_0, u_0 * h_n(u_1), u_1 * h_{n-1}(u_2)) &= h_{n-1}^{-1}(\zeta_{1,n}(u_0, u_0 * h_n(u_1))^{-1} * \\ &\qquad u_1 * h_{n-1}(u_2)) \\ &= u_2.\end{aligned}$$

Suppose by way of induction that $\zeta_{l,n}(u_0 * h_n(u_1), \ldots, u_{l-1} * h_{n-l+1}(u_k)) = u_l$ for $1 \leq l < j$. Then

$$\zeta_{j,n}(u_0 * h_n(u_1), \ldots, u_{j-1} * h_{n-j+1}(u_j)) = h_{n-j+1}^{-1}(u_{j-1}^{-1} * u_{j-1} * h_{n-j+1}(u_j)) = u_j.$$

Thus by (20)

$$\phi_n(H_n(u_0, u_1, \ldots, u_k)) = f_n(u_0, \ldots, u_k) * h_{n+1}(u_0)$$

Now if F_n and Φ_n are the unfoldings of f_n and ϕ_n respectively, then

$$\begin{aligned}H_{n+1}(F_n(u_0, \ldots, u_k)) &= [f_n(u_0, \ldots, u_k) * h_{n+1}(u_0), u_0 * h_n(u_1), \\ &\qquad \ldots, u_{k-2} * h_{n-k+2}(u_{k-1})] \\ &= [\phi_n(H_n(u_0, u_1, \ldots, u_k)), u_0 * h_n(u_1), \\ &\qquad \ldots, u_{k-2} * h_{n-k+2}(u_{k-1})] \\ &= \Phi_n(H_n(u_0, \ldots, u_k))\end{aligned}$$

and it follows that $\{H_n\}$ is a semiconjugate form symmetry for equation (1). The existence of a SC factorization with factor functions defined by (19) now follows from Lemma 2.

Conversely, if $\{H_n\}$ as given by (14) is a time-dependent form symmetry of equation (1) then the semiconjugate relation implies that for arbitrary $u_0, \ldots, u_k$ in G there are functions ϕ_n such that

$$f_n(u_0, \ldots, u_k) * h_{n+1}(u_0) = \phi_n(u_0 * h_n(u_1), \ldots, u_{k-1} * h_{n-k+1}(u_k)). \tag{21}$$

For every $u_0, v_1, \ldots, v_k$ in G and with functions $\zeta_{j,n}$ as defined above, note that

$$\zeta_{j-1,n}(u_0, v_1, \ldots, v_{j-1}) * h_{n-j+1}(\zeta_{j,n}(u_0, v_1, \ldots, v_j)) = v_j, \quad j = 1, 2, \ldots, k.$$

Therefore, abbreviating $\zeta_{j,n}(u_0, v_1, \ldots, v_j)$ by $\zeta_{j,n}$ we have

$$\begin{aligned} f_n(\zeta_{0,n}, \zeta_{1,n}, \ldots, \zeta_{k,n}) * h_{n+1}(u_0) &= \phi_n(\zeta_{0,n} * h_n(\zeta_{1,n}), \zeta_{1,n} * h_{n-1}(\zeta_{2,n}), \\ &\qquad \ldots, \zeta_{k-1,n} * h_{n-k+1}(\zeta_{k,n})) \\ &= \phi_n(v_1, \ldots, v_k) \end{aligned}$$

which is independent of u_0. □

Recall that an algebraic field $\mathcal{F} = (\mathcal{F}, +, \cdot)$ is, in particular, a commutative group with respect to addition. Further, its set of nonzero elements $\mathcal{F}\backslash\{0\}$ is a commutative group under multiplication. A simple yet important type of form symmetry may be defined on a field.

Definition 5. Let $\mathcal{F}$ be a non-trivial field and $\{\alpha_n\}$ a sequence of elements of $\mathcal{F}$ such that $\alpha_n \in \mathcal{F}\backslash\{0\}$ for all $n \geq -k+1$. A *(time-dependent) linear form symmetry* is defined as the following special case of (14) with $h_n(u) = -\alpha_{n-1}u$

$$[u_0 - \alpha_{n-1}u_1, u_1 - \alpha_{n-2}u_2, \ldots, u_{k-1} - \alpha_{n-k}u_k]. \tag{22}$$

The sequence $\{\alpha_n\}$ of nonzero elements in $\mathcal{F}$ may be called the *eigensequence* of the linear form symmetry. If equation (1) has a linear form symmetry then we say that $\{\alpha_n\}$ is an eigensequence of (1).

The use of the term "eigen" which is borrowed from the theory of linear equations is apt here for two reasons. First, the sequence $\{\alpha_n\}$ characterizes the linear form symmetry (22) completely and secondly, in the next section we find that linear difference equations are indeed among those equations that have linear form symmetries.

The existence of a linear form symmetry implies a type-$(k, 1)$ order reduction for equation (1) and a SC factorization with cofactor equation (16)

$$x_{n+1} = t_{n+1} + \alpha_n x_n. \tag{23}$$

The following necessary and sufficient condition for the existence of a time-dependent linear form symmetry is an immediate consequence of Theorem 4.

Corollary 6. *Equation (1) has a time-dependent linear form symmetry of type (22) with an eigensequence* $\{\alpha_n\}$ *in a non-trivial field* $\mathcal{F}$ *if and only if the quantity*

$$f_n(u_0, \zeta_{1,n}(u_0, v_1), \ldots, \zeta_{k,n}(u_0, v_1, \ldots, v_k)) - \alpha_n u_0 \tag{24}$$

is independent of u_0 *for all* $n \geq 0$ *with the functions* $\zeta_{j,n}$ *for* $j = 1, \ldots, k$ *given by*

$$\zeta_{j,n}(u_0, v_1, \ldots, v_j) = \frac{\zeta_{j-1,n}(u_0, v_1, \ldots, v_{j-1}) - v_j}{\alpha_{n-j}}$$
$$= \frac{1}{\prod_{i=1}^{j} \alpha_{n-i}} \left(u_0 - \sum_{i=1}^{j} v_i \prod_{p=1}^{i} \alpha_{n-p} \right).$$

4. Factorization of Linear Equations

We now show that linear difference equations are among difference equations that have the linear form symmetry. The following application of Corollary 6 and Theorem 4 gives the semiconjugate factorization for non-autonomous and non-homogeneous linear difference equations.

Corollary 7. *(The general linear equation) Let* $\{a_{i,n}\}$, $i = 1, \ldots, k$ *and* $\{b_n\}$ *be given sequences in a non-trivial field* $\mathcal{F}$ *such that* $a_{k,n} \neq 0$ *for all* $n \geq 0$. *The non-homogeneous linear equation of order* $k + 1$

$$x_{n+1} = a_{0,n}x_n + a_{1,n}x_{n-1} + \cdots + a_{k,n}x_{n-k} + b_n \tag{25}$$

has a linear form symmetry with eigensequence $\{\alpha_n\}$ *for every solution* $\{\alpha_n\}$ *in* $\mathcal{F}$ *of the following Riccati equation of order* k

$$\alpha_n = a_{0,n} + \frac{a_{1,n}}{\alpha_{n-1}} + \frac{a_{2,n}}{\alpha_{n-1}\alpha_{n-2}} + \cdots + \frac{a_{k,n}}{\alpha_{n-1}\cdots\alpha_{n-k}} \tag{26}$$

The corresponding SC factorization of (25) is

$$t_{n+1} = b_n - \sum_{i=1}^{k} \sum_{j=i}^{k} \frac{a_{j,n}}{\alpha_{n-i}\cdots\alpha_{n-j}} t_{n-i+1} \tag{27}$$

$$x_{n+1} = \alpha_n x_n + t_{n+1} \tag{28}$$

Proof. By Corollary 6 it is only necessary to determine a sequence $\{\alpha_n\}$ of nonzero elements of $\mathcal{F}$ such that for each n the quantity (24) is independent of u_0 for the following function

$$f_n(u_0, \ldots, u_k) = a_{1,n}u_0 + a_{2,n}u_1 + \cdots + a_{k,n}u_k + b_n.$$

For arbitrary $u_0, v_1, \ldots, v_k \in \mathcal{F}$ and $j = 0, 1, \ldots, k$ define $\zeta_{j,n}(u_0, v_1, \ldots, v_j)$ as in Corollary 6. Then the expression (24) is

$$-\alpha_n u_0 + b_n + a_{1,n}u_0 + a_{2,n}\zeta_{1,n}(u_0, v_1) + \cdots + a_{k,n}\zeta_{k,n}(u_0, v_1, \ldots, v_k) =$$

$$b_n + \left[\sum_{j=1}^{k} \frac{a_{j,n}}{\prod_{i=1}^{j} \alpha_{n-i}} - \alpha_n\right] u_0 - \sum_{j=1}^{k} a_{j,n} \sum_{i=1}^{j} \frac{v_i}{\prod_{p=i}^{j} \alpha_{n-p}}$$

The above quantity is independent of u_0 if and only if the coefficient of u_0 is zero for all n; i.e., if $\{\alpha_n\}$ is a solution of the Riccati difference equation

$$\alpha_n = \sum_{j=1}^{k} \frac{a_{j,n}}{\prod_{i=1}^{j} \alpha_{n-i}}$$

which is equation (26). It follows that equation (25) has a linear form symmetry of type (22) with eigensequence $\{\alpha_n\}$ for each solution $\{\alpha_n\}$ of the Riccati equation. For the corresponding SC factorization of (25), the cofactor equation is simply (23) while the factor equation is obtained using the above calculations and equation (19) of Theorem 4 as follows

$$\begin{aligned} t_{n+1} &= b_n - \sum_{j=1}^{k} a_{j,n} \sum_{i=1}^{j} \frac{t_{n-i+1}}{\prod_{p=i}^{j} \alpha_{n-p}} \\ &= b_n - \sum_{i=1}^{k} \sum_{j=i}^{k} \frac{a_{j,n}}{\alpha_{n-i} \cdots \alpha_{n-j}} t_{n-i+1}. \end{aligned}$$

This completes the proof. □

Corollary 7 states that *any* solution of the Riccati equation (26) gives a form symmetry and a SC factorization of (25) as specified above. The next example illustrates Corollary 7.

Example 8. *Consider the second-order difference equation*

$$x_{n+1} = (-1)^{n+1} x_n + x_{n-1} + b_n \tag{29}$$

where b_n, x_0, x_{-1} are in a field $\mathcal{F}$ which we may take to be any one of the familiar fields $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$. The associated Riccati equation of (29) is

$$\alpha_n = (-1)^{n+1} + \frac{1}{\alpha_{n-1}}. \tag{30}$$

Straightforward calculation shows that if $\alpha_0 \neq 0, -1$ then

$$\alpha_1 = \frac{\alpha_0 + 1}{\alpha_0},\ \alpha_2 = -\frac{1}{\alpha_0 + 1},\ \alpha_3 = -\alpha_0,\ \alpha_4 = -\frac{\alpha_0 + 1}{\alpha_0},\ \alpha_5 = \frac{1}{\alpha_0 + 1},\ \alpha_6 = \alpha_0.$$

It follows that all solutions of the Riccati equation (30) with initial value *outside the singularity set* $\{0, -1\}$ *are eigensequences in* $\mathcal{F}$ *of period 6. The SC factorization of the linear equation (29) is now obtained by Corollary 7 as*

$$t_{n+1} = -\frac{1}{\alpha_{n-1}} t_n + b_n = [(-1)^{n+1} - \alpha_n] t_n + b_n,$$
$$x_{n+1} = \alpha_n x_n + t_{n+1}.$$

The next result is concerned with the case of constant coefficients. It is an immediate consequence of Corollary 7.

Corollary 9. *Let* $\{b_n\}$ *be a given sequence in a non-trivial field* $\mathcal{F}$ *and let* $\{a_i\}$, $i = 1, \ldots, k$ *be constants in* $\mathcal{F}$ *such that* $a_k \neq 0$.

(a) The non-homogeneous linear equation of order $k + 1$

$$x_{n+1} = a_0 x_n + a_1 x_{n-1} + \cdots + a_k x_{n-k} + b_n \tag{31}$$

has a linear form symmetry with eigensequence $\{\alpha_n\}$ *for every solution* $\{\alpha_n\}$ *in* $\mathcal{F}$ *of the following autonomous Riccati equation of order* k

$$\alpha_n = a_0 + \frac{a_1}{\alpha_{n-1}} + \frac{a_2}{\alpha_{n-1}\alpha_{n-2}} + \cdots + \frac{a_k}{\alpha_{n-1} \cdots \alpha_{n-k}}. \tag{32}$$

(b) Every fixed point of (32) in $\mathcal{F}$ *is a nonzero root of the characteristic polynomial of (31), i.e.,*

$$\lambda^{k+1} - a_0 \lambda^k - a_1 \lambda^{k-1} - \cdots - a_{k-1}\lambda - a_k \tag{33}$$

and thus, an eigenvalue of the homogeneous part of (31) in $\mathcal{F}$. *As constant solutions of (32) such eigenvalues are constant eigensequences of (31).*

Example 10. *Consider the autonomous second-order linear difference equation*

$$x_{n+1} = x_n - x_{n-1}. \tag{34}$$

This equation has two complex eigenvalues $\alpha_\pm = (1 \pm i\sqrt{3})/2$ *that are roots of its characteristic polynomial* $\lambda^2 - \lambda + 1$. *Thus, (34) has no constant eigensequences in* $\mathbb{R}$ *but it does have non-constant real eigensequences since the Riccati equation*

$$\alpha_n = 1 - \frac{1}{\alpha_{n-1}}$$

with the initial value $\alpha_0 = 2$ *has a solution* $\{2, \frac{1}{2}, -1, 2, \frac{1}{2}, -1, \ldots\}$ *of period three in* $\mathbb{R}$ *with a corresponding real SC factorization*

$$t_{n+1} = -\frac{1}{\alpha_{n-1}} t_n = (\alpha_n - 1) t_n, \quad x_{n+1} = \alpha_n x_n + t_{n+1}.$$

It is worth noting that real solutions of (34) can be generated from the above factorization by direct iteration.

The occurrence of Riccati difference equation in Corollary 7 may seem less surprising if we recall some basic facts from [5]. In particular, the homogeneous part of (25) is a homogeneous equation of degree one relative to the multiplicative group $\mathcal{F}\backslash\{0\}$. Therefore, it has an inversion form symmetry and the factor equation of its SC factorization is none other than the Riccati equation (26). Using this fact it is possible to restate Corollary 7 without explicit reference to the Riccati equation as follows.

Corollary 11. *Assume that the homogeneous part of equation (25) has a solution $\{y_n\}$ in the field $\mathcal{F}$ such that $y_n \neq 0$ for all n. Then $\{y_{n+1}/y_n\}$ is an eigensequence of (25) whose SC factorization is given by the pair of equations (27) and (28).*

Proof. It is given that $\{y_n\}$ satisfies the homogeneous part of (25), i.e.,

$$y_{n+1} = a_{0,n}y_n + a_{1,n}y_{n-1} + a_{2,n}y_{n-2} + \cdots + a_{k,n}y_{n-k}.$$

Since $y_n \neq 0$ for all n, we may divide the above equation by y_n to obtain

$$\begin{aligned}\frac{y_{n+1}}{y_n} &= a_{0,n} + a_{1,n}\frac{y_{n-1}}{y_n} + a_{2,n}\frac{y_{n-2}}{y_n} + \cdots + a_{k,n}\frac{y_{n-k}}{y_n} \\ &= a_{0,n} + a_{1,n}\frac{y_{n-1}}{y_n} + a_{2,n}\frac{y_{n-2}}{y_{n-1}}\frac{y_{n-1}}{y_n} + \cdots + a_{k,n}\frac{y_{n-k}}{y_{n-k+1}}\cdots\frac{y_{n-1}}{y_n}.\end{aligned}$$

Now defining $\alpha_n = y_{n+1}/y_n$ for all n and substituting these terms in the last equation above yields the Riccati equation (26). Thus $\{y_{n+1}/y_n\}$ is an eigensequence of (25) in $\mathcal{F}\backslash\{0\}$, as claimed. The SC factorization is obtained as in the proof of Corollary 7. □

Corollary 12. *In equation (25) let $\{a_{i,n}\}$, $i = 1, \ldots, k$ and $\{b_n\}$ be sequences of real numbers with $a_{i,n} \geq 0$ for all i, n and $a_{k,n} > 0$ for all n. Then (25) has an eigensequence $\{y_{n+1}/y_n\}$ and a SC factorization in $\mathbb{R}$ given by the pair of equations (27) and (28).*

Proof. If we choose $y_{-j} = 1$ for $j = 0, \ldots, k$ then the corresponding solution $\{y_n\}$ of the homogeneous part of (25) is a sequence of positive real numbers. Now an application of Corollary 11 completes the proof. □

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