MODULUS HYPERINVARIANT CLOSED IDEALS FOR QUASINILPOTENT OPERATORS WITH MODULUS ON $l_p$-SPACES

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Abstract: In this paper, it is proved that every non-zero continuous operator with modulus on an $l_p$-space whose modulus is quasinilpotent at a non-zero positive vector has a non-trivial modulus hyperinvariant closed ideal.

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In 1954, N. Aronszajn and K.T. Smith [5] showed that every compact operator on a Banach space has a non-trivial invariant closed subspace.

But it was not until 1986 that people solved the invariant closed ideal problem for a special kind of compact operators. To be more precise, in 1986, B. de Pagter [11] proved the long standing conjecture that every positive quasinilpotent compact operator on a Banach lattice has a non-trivial invariant closed ideal.

In 1993, Y.A. Abramovich, C.D. Aliprantis and O. Burkinshaw [2] showed the following theorem.

**Theorem A.** Let $S : l_p \rightarrow l_p(1 \leq p < \infty)$ be a continuous operator with modulus. If there exists a non-zero positive operator $T : l_p \rightarrow l_p$ such that:

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(1) \( T \) commutes with the modulus of \( S \), and
(2) \( T \) is quasinilpotent at a non-zero positive vector.

Then \( S \) has a non-trivial invariant closed subspace.

In this paper, using the Abramovich-Aliprantis-Burkinshaw technique based on the idea from [1], [2], [3] and so on, we obtain an invariant closed ideal theorem for a large class of operators. To be more precise, we show that every non-zero continuous operator on an \( l_p \)-space whose modulus is quasinilpotent at a non-zero positive vector has a non-trivial modulus hyperinvariant closed ideal. In particular, all continuous operators on an \( l_p \)-space whose modulus commute with a non-zero positive operator \( T \) that is quasinilpotent at a non-zero positive vector have a common non-trivial invariant closed ideal (hence they have a common non-trivial invariant closed subspace).

For a convenience of the reader, we first recall some basic notions and facts from [1], [2], [4], [11], and others. For the notation and terminology not explained in the text we refer to [2] and any standard book in this area.

Following [2], a continuous operator \( T \) on a Banach space \( X \) is said to be quasinilpotent at a vector \( x \in X \) if \( \lim_{n \to \infty} \|T^n x\|^{1/n} = 0 \).

Let \( E \) be a Banach lattice. A linear subspace \( I \) of \( E \) is said to be an (order) ideal whenever \( |x| \leq |y| \) and \( y \in I \) imply that \( x \in I \). The ideal generated by a non-empty subset \( F \) of \( E \) is defined by \( I_F = \{ x \in E ; \text{there are } x_1, \ldots, x_n \in F \text{ and } \lambda_1, \ldots, \lambda_n > 0 \text{ with } |x| \leq \sum_{k=1}^{n} \lambda_k |x_k| \} \). In particular, the ideal generated by a singleton \( \{ y \} \) is given by
\[
I_y = \{ x \in E ; \text{there is } \lambda > 0 \text{ such that } |x| \leq \lambda |y| \}.
\]
A positive vector \( y \in E \) is called a quasi-interior point in \( E \) whenever \( I_y \) is norm dense in \( E \), that is, \( I_y = E \).

If \( T \) and \( B \) are continuous operators on a Banach lattice \( E \) with \( B \) positive, then \( T \) is said to be dominated by \( B \) whenever \( |T x| \leq B(|x|) \) holds for all \( x \in E \).

In addition, the proof of our main theorem will use the following lemma which is due to de Pagter [11].

**Lemma 1.** (see [11], Lemma 1) Let \( E \) be a Banach lattice. If \( E \) admits a quasi-interior point and vectors \( x, y \in E \) with \( 0 \leq y \leq |x| \), then there is a sequence \( \{ T_n \} \) of continuous operators on \( E \) such that \( \|T_n x - y\| \to 0 \) as \( n \to \infty \), and \( |T_n z| \leq |z| \) for every \( n = 1, 2, \cdots \) and all \( z \in E \).

From now on, we only deal with the classical \( l_p \)-spaces over complex numbers, where \( 1 \leq p < \infty \). A vector \( x = (x_1, x_2, \cdots, x_n, \cdots) \) in an \( l_p \)-space is said
to be positive, in symbols $x \geq 0$, if its components are non-negative real numbers. The absolute value $|x|$ of $x$ is the vector $|x| = (|x_1|, |x_2|, \cdots, |x_n|, \cdots)$. The symbol $e_n$ will denote the vector in an $l_p$-space whose $n$-th component is one and every other zero. It is well known there is a sequence $\{f_n\}$ in the dual space of the $l_p$-space such that $f_n(e_m) = \delta_{nm}$.

It is well known that every positive operator on an $l_p$-space is automatically continuous ([4], Theorem 12.3).

A continuous operator $T : l_p \rightarrow l_p$ with matrix $[t_{ij}]$ has modulus whenever the matrix $||t_{ij}||$ of absolute values also defines a continuous operator on the $l_p$-space. In this case, the operator defined by the matrix $||t_{ij}||$ is called the modulus of $T$ and is denoted by $|T|$. If a continuous operator $T$ on an $l_p$-space has a modulus, then we have $|T|x \leq |T||x|$ for each $x \in l_p$.

If $T$ is a continuous operator with modulus on an $l_p$-space, then $\{T\}'_M$ denotes the set of all continuous operators $C$ with modulus on the $l_p$-space such that $|T||C| = |C||T|$.

Moreover, we say that a continuous operator $T$ with modulus on an $l_p$-space has a non-trivial modulus hyperinvariant closed ideal if there exists a non-trivial closed ideal $M$ of $l_p$ such that $M$ is invariant under $\{T\}'_M$.

It is clear that every invariant closed ideal is necessarily an invariant closed subspace, but the converse is not true.

Now we are in a position to give the main result.

**Theorem 1.** Let $T$ be a non-zero continuous operator with modulus on an $l_p$-space. If the modulus of $T$ is quasinilpotent at a non-zero positive vector $x_0 \in l_p$, then $T$ has a non-trivial modulus hyperinvariant closed ideal.

**Proof.** Since $T$ is non-zero, it is clear that

$$I_{\text{ran} \cdot T} = \{x \in l_p : \text{there exists } y \geq 0 \text{ such that } |x| \leq |T|y\}$$

is a non-zero closed ideal. Let $S \in \{T\}'_M$, that is $|S||T| = |T||S|$. If $x \in I_{\text{ran} \cdot T}$, then there exists $x_n \in \{x \in l_p : \text{there exists } y \geq 0 \text{ such that } |x| \leq |T|y\}$ such that $x_n \rightarrow x$ as $n \rightarrow \infty$. Thus there are $y_n \geq 0 (n = 1, 2, \ldots)$ such that $|x_n| \leq |T|y_n$. Therefore we have

$$|Sx_n| \leq |S||x_n| \leq |S||T|y_n = |T||S|y_n,$$

and so $Sx_n \in I_{\text{ran} \cdot T}$. Hence $Sx \in I_{\text{ran} \cdot T}$. Consequently $I_{\text{ran} \cdot T}$ is a modulus, hyperinvariant closed ideal for $T$. If $I_{\text{ran} \cdot T} \neq l_p$, then $I_{\text{ran} \cdot T}$ is a non-trivial modulus hyperinvariant closed ideal for $T$. So, assume that $I_{\text{ran} \cdot T} = l_p$. Since the $l_p$-space is separable, there is a countable subset $\{z_1, z_2, \ldots, z_n, \cdots\}$ of the range of $T$ consisting of non-zero vectors that is norm dense in $\text{ran} \cdot T$. Let
$z = \sum_{n=1}^{\infty} \frac{|z_n|}{\|z_n\|}$. Then $z > 0$ and $\{z_1, z_2, \ldots, z_n, \ldots\} \subset \mathcal{I}_z$, where $\mathcal{I}_z$ denotes the ideal generated by $z$, and so $\text{ran}T \subset \mathcal{T}_z$. Consequently we have $l_p = \mathcal{I} \text{ran}T \subset \mathcal{T}_z$. This implies $\mathcal{T}_z = l_p$. Thus $l_p$ admits a quasi-interior point.

Since $x_0 > 0$, there is an appropriate scalar $\lambda > 0$ and a positive integer $n_0$ such that $\lambda x_0 \geq e_{n_0} > 0$. Since $T$ is quasi-nilpotent at $x_0$, it is also quasi-nilpotent at $\lambda x_0$. Let $\mathcal{A}$ be the algebra of all continuous operators on the $l_p$-space such that each $A \in \mathcal{A}$ is dominated by some operator in the form of $\sum_{j=1}^{n} |C_j||T|^j$ with $C_j \in \{T\}_M$.

(1). If $Ae_{n_0} = 0$ for all $A \in \mathcal{A}$, then $\mathcal{N}_A = \{x; A(|x|) = 0 \text{ for all } A \in \mathcal{A}\}$ is a non-zero closed ideal in the $l_p$-space, and $0 \neq T \in \mathcal{A}$ implies $\mathcal{N}_A \neq l_p$. It only remains to show that $\mathcal{N}_A$ is invariant under all operators in the modulus commutant $\{T\}_M$ of $T$. To this end, take $x \in \mathcal{N}_A$ and $C \in \{T\}_M$. For any $A \in \mathcal{A}$, it follows from the definition of $\mathcal{A}$ that there are operators $C_1, C_2, \ldots, C_n \in \{T\}_M$ such that $|Ay| \leq \sum_{j=1}^{n} |C_j||T|^j(|y|)$ for all $y \in l_p$. Thus we have $|A(Cx)| \leq \sum_{j=1}^{n} |C_j||T|^j(|Cx|) \leq \sum_{j=1}^{n} |C_j||C||T|^j(|x|)$. Since $|C_j||C| \in \{T\}_M$, it follows that $\sum_{j=1}^{n} |C_j||C||T|^j \in \mathcal{A}$. Thus for $x \in \mathcal{N}_A$ we obtain $\sum_{j=1}^{n} |C_j||C||T|^j(|x|) = 0$. Consequently we have $|A(Cx)| = 0$. This implies $A(Cx) = 0$ for all $A \in \mathcal{A}$, and so $Cx \in \mathcal{N}_A$. Consequently $\mathcal{N}_A$ is a non-trivial modulus hyperinvariant closed ideal for $T$.

(2). If there is an operator $A_0 \in \mathcal{A}$ such that $A_0 e_{n_0} \neq 0$, then $M = \{Ae_{n_0}; A \in \mathcal{A}\}$ is a non-zero linear subspace in the $l_p$-space. We now prove that $\mathcal{M}$ is an ideal in the $l_p$-space. Let $\mathcal{I}_M$ denote the ideal generated by $M$, and take $x \in \mathcal{I}_M$. Then there are $A_1, A_2, \ldots, A_n \in \mathcal{A}$ such that $|x| \leq \sum_{k=1}^{n} |A_k e_{n_0}|$, and so $|\text{Re}| \leq \sum_{k=1}^{n} |A_k e_{n_0}|$, $|\text{Im}| \leq \sum_{k=1}^{n} |A_k e_{n_0}|$, where $\text{Re}$ and $\text{Im}$ denote the real part and imaginary part of $x$ respectively. Hence $(\text{Re})^+ \leq \sum_{k=1}^{n} |A_k e_{n_0}|$. Thus we can write $(\text{Re})^+ = \sum_{k=1}^{n} x_k$ with $0 \leq x_k \leq |A_k e_{n_0}|$ for each $k$. Since $l_p$ admits a quasi-interior point, it follows from Lemma 1 that for each $k = 1, 2, \ldots, n$, there is a sequence $\{T_{k,m}\}_{m=1}^{\infty}$ of continuous operators on the $l_p$-space such that $\|T_{k,m} A_k e_{n_0} - x_k\| \longrightarrow 0$ as $m \longrightarrow \infty$, and $|T_{k,m} y| \leq |y|$ for every $m = 1, 2, \ldots$ and all $y \in l_p$. Since $A_k \in \mathcal{A}$, it follows that there is $C_{k_j} \in \{T\}_M$ such that

$$|T_{k,m} A_k y| \leq |A_k y| \leq \sum_{j=1}^{n} |C_{k_j}||T|^j(|y|)$$

for all $y \in l_p$, and so $T_{k,m} A_k \in \mathcal{A}$. It follows from above that $x_1 \in \mathcal{M}$ for each $k$, and so $(\text{Re})^+ \in \mathcal{M}$. Similarly, $(\text{Re})^- \in \mathcal{M}, (\text{Im})^+ \in \mathcal{M}, (\text{Im})^- \in \mathcal{M}$ and so $x = (\text{Re})^+ - (\text{Re})^- + i[(\text{Im})^+ - (\text{Im})^-] \in \mathcal{M}$. This shows that
$M \subset \mathcal{I}_M \subset \overline{M}$, which implies that $\overline{M} = \overline{\mathcal{I}_M}$. Since the closure of an ideal is an ideal, we may conclude that $\overline{M}$ is an ideal.

Next we prove that $M$ is invariant under all operators in the modulus commutant $\{T\}'_M$ of $T$. To this end, take $z \in M$ and $C \in \{T\}'_M$. Then there is an operator $A \in \mathcal{A}$ such that $z = Ae_{n_0}$. It follows from the definition of $\mathcal{A}$ that there exist operators $C_1, C_2, \ldots, C_n \in \{T\}'_M$ such that $|Au| \leq \sum^n_{j=1} |C_j||T|^j(|u|)$ for all $u \in l_p$. This implies $|CAu| \leq \sum^n_{j=1} |C_j||T|^j(|u|)$ for all $u \in l_p$. Since $|C||C_j| \in \{T\}'_M$, we have $CA \in \mathcal{A}, Cz = CAe_{n_0} \in M$.

We now show that $\overline{M} \neq l_p$. Let $P$ denote the natural projection from the $l_p$-space onto the linear subspace generated by $e_{n_0}$. It is clear that $0 \leq Pv \leq v$ holds whenever $0 \leq v \in l_p$. We claim that
\begin{equation}
P|C||T|^j e_{n_0} = 0
\end{equation}
for each $j \geq 0$ and each $C \in \{T\}'_M$. To this end, we write $P|C||T|^j e_{n_0} = ae_{n_0}$ for some $a \geq 0$. Since $P$ is a positive operator and the composition of positive operators is also a positive operator, it follows that the estimate
\begin{equation}
0 \leq a^k e_{n_0} = (P|C||T|^j)^k e_{n_0} \leq (|C||T|^j)^k e_{n_0} = |C|^k|T|^j(\lambda x_0)
\end{equation}
holds for every positive integer $k$. Since $f_{n_0}$ is a positive functional on the $l_p$-space, it follows from (2) that
\begin{equation}
0 \leq a^k = f_{n_0}(a^k e_{n_0}) \leq f_{n_0}(|C|^k|T|^j(\lambda x_0)).
\end{equation}
Since the modulus of $T$ is finitely quasinilpotent at $\lambda x_0$, it follows from (3) that $0 \leq a \leq \|f_{n_0}\|^{|1/k||C|^k|T|^j(\lambda x_0)|} = \|f_{n_0}\|^{|1/k||C||T|^j(\lambda x_0)|} \to 0$ as $k \to \infty$, from which it follows that $a = 0$.

For every $w \in M$, the definition of $M$ implies that there is an operator $A \in \mathcal{A}$ such that $w = Ae_{n_0}$. Thus by the definition of $\mathcal{A}$ there are operators $C_1, C_2, \cdots, C_n \in \{T\}'_M$ such that $|Ae_{n_0}| \leq \sum^n_{j=1} |C_j||T|^j e_{n_0}$. Thus by (1) we obtain $P(w) = P(Ae_{n_0}) \leq \sum^n_{j=1} P|C_j||T|^j e_{n_0} = 0$.

Hence it is easy to see that $f_{n_0}(w) = f_{n_0}(Pw) = 0$ for every $w \in M$. Consequently $f_{n_0}(w) = 0$ for every $w \in \overline{M}$. Observing that $f_{n_0}(e_{n_0}) = 1$, we obtain $\overline{M} \neq l_p$.

From above we conclude that $\overline{M}$ is a non-trivial modulus hyperinvariant closed ideal for $T$, and this completes the proof of Theorem 1.

**Corollary 1.** Let $T$ be a non-zero positive operator on an $l_p$-space. If $T$ is quasinilpotent at a non-zero positive vector $x_0 \in l_p$, then $T$ has a non-trivial modulus hyperinvariant closed ideal.

In other words, all continuous operators on an $l_p$-space whose modulus
commute with a non-zero positive operator $T$ that is quasinilpotent at a non-zero positive vector have a common non-trivial invariant closed ideal.

Since every invariant closed ideal is necessarily the invariant closed subspace, we derive Theorem A from Corollary 1.

**Remark 1.** It is also worth mentioning that C.J. Read [12] presented a quasinilpotent continuous operator $T$ on $l_1$ without non-trivial invariant closed subspaces.

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**References**


