

IRREDUCIBLE REPRESENTATIONS OF  
THE CPT GROUPS IN QED

B. Carballo Pérez<sup>1</sup>, M. Socolovsky<sup>2</sup> §

Instituto de Ciencias Nucleares  
Universidad Nacional Autónoma de México  
Circuito Exterior, Ciudad Universitaria, 04510, México D.F., MÉXICO

<sup>1</sup>e-mail: brendacp@nucleares.unam.mx

<sup>2</sup>e-mail: socolovs@nucleares.unam.mx

**Abstract:** We construct the inequivalent irreducible representations (IIR's) of the *CPT* groups of the Dirac field operator  $\hat{\psi}$  and the electromagnetic quantum potential  $\hat{A}_\mu$ . The results are valid both for free and interacting (QED) fields. Also, and for the sake of completeness, we construct the IIR's of the *CPT* group of the Dirac equation.

**AMS Subject Classification:** 20C30, 20C35

**Key Words:** *CPT* groups, Dirac-Maxwell fields, irreducible representations

1. Introduction

The *CPT* group  $G_{\hat{\Theta}}(\hat{\psi})$  of the Dirac quantum field  $\hat{\psi}$  is isomorphic to the direct product of the quaternion group  $Q$  and the cyclic group of two elements  $\mathbb{Z}_2$  (see [1] and [2]):

$$G_{\hat{\Theta}}(\hat{\psi}) \cong Q \times \mathbb{Z}_2. \quad (1)$$

$\hat{\Theta}$  is the product  $\hat{C} * \hat{P} * \hat{T}$  of the three operators:  $\hat{C}$  (charge conjugation),  $\hat{P}$

---

Received: December 11, 2009

© 2010 Academic Publications

§Correspondence author

(space inversion),  $\hat{T}$  (time reversal); the operation  $\hat{A} * \hat{B}$ , where  $\hat{A}$  and  $\hat{B}$  are any of the operators  $\hat{C}$ ,  $\hat{P}$ ,  $\hat{T}$ , is given by  $(\hat{A} * \hat{B}) \cdot \hat{\psi} = (\hat{A}\hat{B})^\dagger \hat{\psi} (\hat{A}\hat{B})$ .  $G_{\hat{\Theta}}(\hat{\psi})$  is one of the nine non abelian groups of a total of fourteen groups with sixteen elements [3]; only three of them have three generators.

The isomorphism  $G_{\hat{\Theta}}(\hat{\psi}) \rightarrow Q \times \mathbb{Z}_2$  is given by the relations (2):

$$\begin{aligned}
 1 &\mapsto (1, 1) & -1 &\mapsto (-1, 1) \\
 \hat{C} &\mapsto (1, -1) & -\hat{C} &\mapsto (-1, -1) \\
 \hat{P} &\mapsto (\iota, 1) & -\hat{P} &\mapsto (-\iota, 1) \\
 \hat{T} &\mapsto (\gamma, 1) & -\hat{T} &\mapsto (-\gamma, 1) \\
 \hat{C} * \hat{P} &\mapsto (\iota, -1) & -\hat{C} * \hat{P} &\mapsto (-\iota, -1) \\
 \hat{C} * \hat{T} &\mapsto (\gamma, -1) & -\hat{C} * \hat{T} &\mapsto (-\gamma, -1) \\
 \hat{P} * \hat{T} &\mapsto (\kappa, 1) & -\hat{P} * \hat{T} &\mapsto (-\kappa, 1) \\
 \hat{\Theta} &\mapsto (\kappa, -1) & -\hat{\Theta} &\mapsto (-\kappa, -1);
 \end{aligned} \tag{2}$$

where  $\iota$ ,  $\gamma$ ,  $\kappa$  are the three imaginary units defining the quaternion numbers.

On the other hand, the action of the  $\hat{C}$ ,  $\hat{P}$  and  $\hat{T}$  operators on the Maxwell electromagnetic 4-potential  $\hat{A}_\mu$  is given by (3) (see [4] and [5]), with the Minkowski space-time metric  $diag(1, -1, -1, -1)$ :

$$\begin{aligned}
 \hat{P}\hat{A}^\mu(\mathbf{x}, t)\hat{P}^{-1} &= \hat{A}_\mu(-\mathbf{x}, t), \\
 \hat{C}\hat{A}^\mu(\mathbf{x}, t)\hat{C}^{-1} &= -\hat{A}^\mu(\mathbf{x}, t), \\
 \hat{T}\hat{A}^\mu(\mathbf{x}, t)\hat{T}^{-1} &= \hat{A}_\mu(\mathbf{x}, -t).
 \end{aligned} \tag{3}$$

This leads to the *CPT* group of the electromagnetic field operator,  $G_{\hat{\Theta}}(\hat{A}_\mu)$ , which is an abelian group of eight elements with three generators, isomorphic to  $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 = \mathbb{Z}_2^3$ . Geometrically,  $\mathbb{Z}_2^3$  is isomorphic to the group  $D_{2h}$ , the symmetry group of the parallelepiped.

The isomorphism

$$G_{\hat{\Theta}}(\hat{A}_\mu) \rightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \tag{4}$$

is given by

$$\begin{aligned}
 1 &\mapsto (e_1, e_2, e_3) \\
 \hat{C} &\mapsto (a_1, e_2, e_3) \\
 \hat{P} &\mapsto (e_1, a_2, e_3) \\
 \hat{T} &\mapsto (e_1, a_2, a_3) \\
 \hat{P} * \hat{T} &\mapsto (e_1, a_2, a_3)
 \end{aligned}$$

$$\begin{aligned}
 \hat{C} * \hat{P} &\mapsto (a_1, a_2, e_3) \\
 \hat{C} * \hat{T} &\mapsto (a_1, e_2, a_3) \\
 \hat{\Theta} &\mapsto (a_1, a_2, a_3);
 \end{aligned}
 \tag{5}$$

where  $e_i$  and  $a_i$  (for  $i = 1, 2, 3$ ) represent, respectively, the identity and the “minus 1” of each of the  $\mathbb{Z}_2$  groups.

As the *CPT* transformation properties of the interacting  $\hat{\psi} - \hat{A}_\mu$  fields are the same as for the free fields (see [5] and [6]), it is then clear that the complete *CPT* group for QED,  $G_{\hat{\Theta}}(QED)$ , is the direct product of the two above mentioned groups,  $G_{\hat{\Theta}}(\hat{\psi})$  and  $G_{\hat{\Theta}}(\hat{A}_\mu)$ , i.e.,

$$G_{\hat{\Theta}}(QED) = G_{\hat{\Theta}}(\hat{\psi}) \times G_{\hat{\Theta}}(\hat{A}_\mu),
 \tag{6}$$

which turns out to be a group of order  $|G_{\hat{\Theta}}(QED)| = 16 \times 8 = 128$ .

Thus,

$$G_{\hat{\Theta}}(QED) \cong (Q \times \mathbb{Z}_2) \times \mathbb{Z}_2^3.
 \tag{7}$$

Since, as will be shown below,  $G_{\hat{\Theta}}(\hat{\psi})$  has ten inequivalent irreducible representations (IIR’s), while  $G_{\hat{\Theta}}(\hat{A}_\mu)$  has only eight IIR’s, then, the total number of IIR’s of  $G_{\hat{\Theta}}(QED)$  is eighty: sixty four of them 1-dimensional and sixteen 2-dimensional.

It is interesting at this point to make a comment about the geometrical content of  $G_{\hat{\Theta}}(QED)$ , which is a set of points in spheres. In fact, each factor  $\mathbb{Z}_2$  is a 0-sphere  $S^0$ , while  $Q$  can be thought as a subset of eight points in the 3-sphere  $S^3$ . Thus, topologically

$$G_{\hat{\Theta}}(QED) \subset SU(2) \times (U(1))^4.
 \tag{8}$$

In Section 2 we construct by tensor products the ten IIR’s and character tables of  $G_{\hat{\Theta}}(\hat{\psi})$ . In Section 3 we construct the eight IIR’s of  $G_{\hat{\Theta}}(\hat{A}_\mu)$ ; in this case, since all representations are 1-dimensional, the characters coincide with the representations. In Section 4 we construct the IIR’s of the *CPT* group corresponding to the Dirac equation for the wave function  $\psi$ ,  $G_{\hat{\Theta}}^{(2)}(\psi)$  in [1], and compare the result with the case for the Dirac field operator  $\hat{\psi}$ . Finally, in Section 5, we discuss the relation between the fermionic” and bosonic” *CPT* groups and comment about the relation of these groups with some approaches to quantum paradoxes.

Ch Q	[1]	[-1]	2[l]	2[γ]	2[κ]
χ <sub>1</sub>	1	1	1	1	1
χ <sub>2</sub>	1	1	1	-1	-1
χ <sub>3</sub>	1	1	-1	1	-1
χ <sub>4</sub>	1	1	-1	-1	1
χ <sub>5</sub>	2	-2	0	0	0

Table 1: Characters table of Q

### 2. IIR's of $G_{\hat{\Theta}}(\hat{\psi})$

The irreducible representations (irrep's) of the direct product of two groups,  $G \times H$ , are the tensor products of the irrep's of each of the factors, namely the set  $\{r_G \otimes r_H\}^1$  for all  $r'_G$ s, irrep's of  $G$  and all  $r'_H$ s, irrep's of  $H$  [7]. If  $G$  is the quaternion group, which, as is well known, has five IIR's  $\varphi_\mu$  (with  $\varphi_1, \dots, \varphi_4$  1-dimensional and  $\varphi_5$  2-dimensional)[8] and characters table given by table 1; and  $H$  is  $\mathbb{Z}_2$  (with IIR's  $\psi_1$  and  $\psi_2$ ), then  $G_{\hat{\Theta}}(\hat{\psi})$  has ten IIR's:

$$\phi_\alpha = \varphi_\alpha \otimes \psi_1, \quad \phi_{\alpha+4} = \varphi_\alpha \otimes \psi_2, \quad \alpha = 1, 2, 3, 4, \tag{9}$$

$$\phi_9 = \varphi_5 \otimes \psi_1, \quad \phi_{10} = \varphi_5 \otimes \psi_2. \tag{10}$$

$\phi_1, \dots, \phi_8$  are 1-dimensional, while  $\phi_9$  and  $\phi_{10}$  are 2-dimensional.

The corresponding characters are the functions

$$\kappa_\alpha = \chi_\alpha \varphi_1 : Q \times \mathbb{Z}_2 \rightarrow \mathbb{C}, \quad \kappa_\alpha(q, h) = \chi_\alpha(q) \varphi_1(h), \tag{11}$$

$$\kappa_{\alpha+4} = \chi_\alpha \varphi_2 : Q \times \mathbb{Z}_2 \rightarrow \mathbb{C}, \quad \kappa_{\alpha+4}(q, h) = \chi_\alpha(q) \varphi_2(h), \tag{12}$$

$$\kappa_9 = \chi_5 \varphi_1 : Q \times \mathbb{Z}_2 \rightarrow \mathbb{C}, \quad \kappa_9(q, h) = \chi_5(q) \varphi_1(h), \tag{13}$$

$$\kappa_{10} = \chi_5 \varphi_2 : Q \times \mathbb{Z}_2 \rightarrow \mathbb{C}, \quad \kappa_{10}(q, h) = \chi_5(q) \varphi_2(h). \tag{14}$$

These representations and characters (for conjugate classes) are explicitly given by the tables 2 and 3, respectively, where  $\lambda_i = tr\phi_i$ , and the conjugate classes are given by the relations (15):

$$\begin{aligned} [\hat{J}] &= [(1, e)] = \{(1, e)\}, \\ [\hat{C}] &= [(1, a)] = \{(1, a)\}, \end{aligned}$$

---

<sup>1</sup> $\otimes = \otimes_{\mathbb{C}}$  is the tensor product over the complex numbers

IIR's $G_{\hat{\Theta}}(\hat{\psi})$	$\hat{I}$ (1, e)	$\hat{C}$ (1, a)	$-\hat{I}$ (-1, e)	$-\hat{C}$ (-1, a)	$\hat{P}$ ( $\iota$ , e)	$\hat{C} * \hat{P}$ ( $\iota$ , a)
$\phi_1$	1	1	1	1	1	1
$\phi_2$	1	-1	1	-1	1	-1
$\phi_3$	1	1	1	1	1	1
$\phi_4$	1	-1	1	-1	1	-1
$\phi_5$	1	1	1	1	-1	-1
$\phi_6$	1	-1	1	-1	-1	1
$\phi_7$	1	1	1	1	-1	-1
$\phi_8$	1	-1	1	-1	-1	1
$\phi_9$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$	$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$
$\phi_{10}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$	$\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$

$-\hat{P}$ (- $\iota$ , e)	$-\hat{C} * \hat{P}$ (- $\iota$ , a)	$\hat{T}$ ( $\gamma$ , e)	$\hat{C} * \hat{T}$ ( $\gamma$ , a)	$-\hat{T}$ (- $\gamma$ , e)	$-\hat{C} * \hat{T}$ (- $\gamma$ , a)	$\hat{P} * \hat{T}$ ( $\kappa$ , e)	$\hat{\Theta}$ ( $\kappa$ , a)	$-\hat{P} * \hat{T}$ (- $\kappa$ , e)	$-\hat{\Theta}$ (- $\kappa$ , a)
1	1	1	1	1	1	1	1	1	1
1	-1	1	-1	1	-1	1	-1	1	-1
1	1	-1	-1	-1	-1	-1	-1	-1	-1
1	-1	-1	1	-1	1	-1	1	-1	1
-1	-1	1	1	1	1	-1	-1	-1	-1
-1	1	1	-1	1	-1	-1	1	-1	1
-1	-1	-1	-1	-1	-1	1	1	1	1
-1	1	-1	1	-1	1	1	-1	1	-1
$\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$	$\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$
$\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$	$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$

Table 2: IIR's table of  $G_{\hat{\Theta}}(\hat{\psi})$

$$\begin{aligned}
 [-\hat{I}] &= [(-1, e)] = \{(-1, e)\}, \\
 [-\hat{C}] &= [(-1, a)] = \{(-1, a)\}, \\
 [\hat{P}] &= [(\iota, e)] = \{(\iota, e), (-\iota, e)\}, \\
 [\hat{C} * \hat{P}] &= [(\iota, a)] = \{(\iota, a), (-\iota, a)\}, \\
 [\hat{T}] &= [(\gamma, e)] = \{(\gamma, e), (-\gamma, e)\}, \\
 [\hat{C} * \hat{T}] &= [(\gamma, a)] = \{(\gamma, a), (-\gamma, a)\}, \\
 [\hat{P} * \hat{T}] &= [(\kappa, e)] = \{(\kappa, e), (-\kappa, e)\}, \\
 [\hat{\Theta}] &= [(\kappa, a)] = \{(\kappa, a), (-\kappa, a)\};
 \end{aligned}
 \tag{15}$$

where  $e$  and  $a$  are the elements of  $\mathbb{Z}_2$ .

The character table of  $G_{\hat{\Theta}}(\hat{\psi})$  is the same as the character table of  $D_{4h}$ , the group of symmetries of a prism with square base, though  $Q \times \mathbb{Z}_2 \not\cong D_{4h}$ .

Ch $G_{\hat{\Theta}}(\hat{\psi})$	$[(1, e)]$	$[(1, a)]$	$[(-1, e)]$	$[(-1, a)]$	$2[(\iota, e)]$	$2[(\iota, a)]$	$2[(\gamma, e)]$	$2[(\gamma, a)]$	$2[(\kappa, e)]$	$2[(\kappa, a)]$
$\lambda_1$	1	1	1	1	1	1	1	1	1	1
$\lambda_2$	1	-1	1	-1	1	-1	1	-1	1	-1
$\lambda_3$	1	1	1	1	1	1	-1	-1	-1	-1
$\lambda_4$	1	-1	1	-1	1	-1	-1	1	-1	1
$\lambda_5$	1	1	1	1	-1	-1	1	1	-1	-1
$\lambda_6$	1	-1	1	-1	-1	1	1	-1	-1	1
$\lambda_7$	1	1	1	1	-1	-1	-1	-1	1	1
$\lambda_8$	1	-1	1	-1	-1	1	-1	1	1	-1
$\lambda_9$	2	2	-2	-2	0	0	0	0	0	0
$\lambda_{10}$	2	-2	-2	2	0	0	0	0	0	0

Table 3: Characters table of  $G_{\hat{\Theta}}(\hat{\psi})$

IIR's (Ch)	$\hat{I}$	$\hat{C}$	$\hat{P}$	$\hat{T}$	$\hat{P} * \hat{T}$	$\hat{C} * \hat{P}$	$\hat{C} * \hat{T}$	$\hat{\Theta}$
$G_{\hat{\Theta}}(\hat{A}_\mu)$	$(e_1, e_2, e_3)$	$(a_1, e_2, e_3)$	$(e_1, a_2, e_3)$	$(e_1, e_2, a_3)$	$(e_1, a_2, a_3)$	$(a_1, a_2, e_3)$	$(a_1, e_2, a_3)$	$(a_1, a_2, a_3)$
$\phi_{111} = \Phi_1$	1	1	1	1	1	1	1	1
$\phi_{211} = \Phi_2$	1	-1	1	1	1	-1	-1	-1
$\phi_{121} = \Phi_3$	1	1	-1	1	-1	-1	1	-1
$\phi_{112} = \Phi_4$	1	1	1	-1	-1	1	-1	-1
$\phi_{221} = \Phi_5$	1	-1	-1	1	-1	1	-1	1
$\phi_{212} = \Phi_6$	1	-1	1	-1	-1	-1	1	1
$\phi_{122} = \Phi_7$	1	1	-1	-1	1	-1	-1	1
$\phi_{222} = \Phi_8$	1	-1	-1	-1	1	1	1	-1

Table 4: IIR's (characters) table of  $G_{\hat{\Theta}}(\hat{A}_\mu)$

### 3. IIR's of $G_{\hat{\Theta}}(\hat{A}_\mu)$

Since  $G_{\hat{\Theta}}(\hat{A}_\mu) \cong \mathbb{Z}_2^3$  is an abelian group, all its irrep's are 1-dimensional. Since  $|G_{\hat{\Theta}}(\hat{A}_\mu)| = 8$ , then  $G_{\hat{\Theta}}(\hat{A}_\mu)$  has eight IIR's, which can be identified with the corresponding characters. Using the isomorphism between  $G_{\hat{\Theta}}(\hat{A}_\mu)$  and  $\mathbb{Z}_2^3$  (eq. (5)), and the character table of  $\mathbb{Z}_2$ , we obtain the table of representations ( $\phi_{ijk}$ ) or characters ( $\chi_{ijk}$ ) for  $G_{\hat{\Theta}}(\hat{A}_\mu)$ , where  $\phi_{ijk} = \chi_{ijk} = \psi_i \psi_j \psi_k$ . See table 4.

### 4. IIR's of $G_{\Theta}^{(2)}(\psi)$

In [1] it was shown that at the level of the Dirac equation, for the 4-spinor  $\psi$ , exists two *CPT* groups,  $G_{\Theta}^{(1)}(\psi)$  and  $G_{\Theta}^{(2)}(\psi)$ , whose elements are constructed with products of Dirac  $\gamma$ -matrices. Only  $G_{\Theta}^{(2)}(\psi)$ , which turns out isomorphic to the semi-direct product  $D_4 \rtimes \mathbb{Z}_2$ , where  $D_4 \equiv DH_8$  is the dihedral group of eight elements (the group of symmetries of the square), is compatible with  $G_{\hat{\Theta}}(\psi)$ .

It is then of interest to construct the IIR's of  $G_{\Theta}^{(2)}(\psi)$ , both for completeness and also for comparison with the IIR's of the operator group.

The action of  $\mathbb{Z}_2 \cong \{1, -1\}$  on  $D_4$  as a subgroup of  $S_4$  (the symmetric group of 4 elements) is given by

$$\begin{aligned}
 \lambda : \mathbb{Z}_2 &\rightarrow \text{Aut}(D_4), \\
 1 &\mapsto \lambda(1) = Id_{D_4}, \\
 \lambda(-1)(I) &= I, \\
 \lambda(-1)(1234) &= (1234), \\
 \lambda(-1)(24) &= (13), \\
 \lambda(-1)(13) &= (24), \\
 \lambda(-1)((12)(34)) &= (14)(23), \\
 \lambda(-1)((14)(23)) &= (12)(34), \\
 \lambda(-1)((13)(24)) &= (13)(24), \\
 \lambda(-1)(1432) &= (1432).
 \end{aligned}
 \tag{16}$$

Here

$$D_4 = \{I; (1234), (1432); (13)(24); (12)(34), (14)(23); (24), (13)\},
 \tag{17}$$

where between semi-colons we have enclosed the five conjugation classes.

Then, the composition law in  $D_4 \rtimes \mathbb{Z}_2$  is

$$(g', h')(g, h) = (g' \lambda(h')(g), h' h) :
 \tag{18}$$

$$(g', 1)(g, h) = (g' g, h)
 \tag{19}$$

$$(g', -1)(g, h) = (g' \lambda(-1)(g), -h),
 \tag{20}$$

where, as a set,

$$D_4 \times \mathbb{Z}_2 = \{(I, 1), (I, -1), (-I, 1), (-I, -1), (P, 1), (P, -1), (-P, 1),$$

$$(-P, -1), (CT, 1), (CT, -1), (-CT, 1), (-CT, -1), (\Theta, 1),$$

$$(\Theta, -1), (-\Theta, 1), (-\Theta, -1)\}, \quad (21)$$

with inverses

$$(g, h)^{-1} = (\lambda(h)(g^{-1}), h). \quad (22)$$

From eq. (55) in [1], there is the isomorphism

$$D_4 \rightarrow \{I, -I, P, -P, CT, -CT, \Theta, -\Theta\} \quad (23)$$

given by

$$I \mapsto I,$$

$$(1234) \mapsto P,$$

$$(1432) \mapsto -P,$$

$$(13)(24) \mapsto -I,$$

$$(12)(34) \mapsto \Theta,$$

$$(14)(23) \mapsto -\Theta,$$

$$(24) \mapsto -CT,$$

$$(13) \mapsto CT \quad (24)$$

while from eq. (60) in [1] and eq. (17), the isomorphism between  $G_{\Theta}^{(2)}$  and  $D_4 \rtimes \mathbb{Z}_2$ ,

$$G_{\Theta}^{(2)} \rightarrow D_4 \rtimes \mathbb{Z}_2, \quad (25)$$

is given by:

$$I \mapsto (I, 1), \quad -I \mapsto (-I, 1),$$

$$C \mapsto (-\Theta, -1), \quad -C \mapsto (\Theta, -1),$$

$$P \mapsto (P, 1), \quad -P \mapsto (-P, 1),$$

$$T \mapsto (P, -1), \quad -T \mapsto (-P, -1),$$

$$CP \mapsto (CT, -1), \quad -CP \mapsto (-CT, -1),$$

$$CT \mapsto (CT, 1), \quad -CT \mapsto (-CT, 1),$$

$$PT \mapsto (-I, -1), \quad -PT \mapsto (I, -1),$$

$$\Theta \mapsto (\Theta, 1), \quad -\Theta \mapsto (-\Theta, 1). \quad (26)$$

It is easy to verify that  $G_{\Theta}^{(2)}$  has ten conjugation classes, namely

$$[I] = \{I\},$$



$$\begin{aligned}
 [-I] &= \{ -I \}, \\
 [C] &= \{ C, -C \}, \\
 [T] &= \{ T \}, \\
 [-T] &= \{ -T \}, \\
 [P] &= \{ P, -P \}, \\
 [CP] &= \{ CP, -CP \}, \\
 [CT] &= \{ CT, -CT \}, \\
 [PT] &= \{ PT, -PT \}, \\
 [\Theta] &= \{ \Theta, -\Theta \}.
 \end{aligned}
 \tag{27}$$

Then, being a finite group,  $G_{\Theta}^{(2)}$  has as many IIR's as conjugation classes. Since the sum of the squares of the dimensions of these representations must be 10, a simple calculation leads to the existence of eight 1-dimensional irrep's  $\varphi_k, k = 1, \dots, 8$ , and two 2-dimensional irrep's,  $\varphi_9$  and  $\varphi_{10}$ .

$\varphi_1$  is the trivial representation  $g \mapsto 1$ , for all  $g \in G_{\Theta}^{(2)}$ . The three 1-dimensional irrep's  $\varphi_2, \varphi_3$  and  $\varphi_4$  are obtained taking into account:

1.  $D_4, C_4 \times \mathbb{Z}_2$  and  $Q$  are invariant subgroups of  $D_4 \rtimes \mathbb{Z}_2$ .
2. The well known theorem by which if  $H$  is an invariant subgroup of  $G$  and  $\varphi : G/H \rightarrow K < GL(V)$  is a representation of  $G/H$  over the vector space  $V$ , then  $\varphi \circ p$  is a degenerate representation of  $G$ , where  $p : G \rightarrow G/H$  is the canonical projection  $p(g) = \langle g \rangle = gH$  [9]. Thus, we have the commutative diagram:

$$\begin{array}{ccc}
 & K & \\
 \varphi \circ p \nearrow & & \searrow \varphi \\
 G & \xrightarrow{p} & G/H
 \end{array}
 \tag{28}$$

$D_4$  is given by the r.h.s. of eq. (23) and  $C_4 = \langle \{P\} \rangle$ , so

$$C_4 \times \mathbb{Z}_2 = \{ (I, 1), (I, -1), (-I, 1), (-I, -1), (P, 1), (P, -1), (-P, 1), (-P, -1) \},
 \tag{29}$$

and

$$Q \cong \langle \{C, P\} \rangle = \{ I, C, P, CP, -I - C, -P, -CP \}.
 \tag{30}$$

In diagram (28), we choose  $G = D_4 \rtimes \mathbb{Z}_2$  and  $H$  equal to  $D_4, C_4 \times \mathbb{Z}_2$  and  $Q$ , respectively. Then

$$\frac{D_4 \rtimes \mathbb{Z}_2}{H} \cong \mathbb{Z}_2 = \{ e, a \}
 \tag{31}$$

with

$$\begin{aligned}
 e &= D_4, & a &= \{C, T, CP, PT, -C, -T, -CP, -PT\}, \\
 e &= C_4 \times \mathbb{Z}_2, & a &= \{C, CP, CT, \Theta, -C, -CP, -CT, -\Theta\}, \\
 e &= Q, & a &= \{T, CT, PT, \Theta, -T, -CT, -PT, -\Theta\},
 \end{aligned} \tag{32}$$

respectively. The elements in the  $e$ 's are represented by 1, while the elements in the  $a$ 's are represented by  $-1$  (see table 5).

As is well known,  $D_4$  has five IIR's (one 2-dimensional and four 1-dimensional). The 2-dimensional irrep is given by

$$\begin{aligned}
 I &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1_{2 \times 2} \equiv I \\
 (13)(24) &\mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -1_{2 \times 2} \equiv -I \\
 (1234) &\mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma_2 \equiv P \\
 (1432) &\mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma_2 \equiv -P \\
 (12)(34) &\mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -\sigma_3 \equiv \Theta \\
 (14)(23) &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3 \equiv -\Theta \\
 (24) &\mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1 \equiv -CT \\
 (13) &\mapsto \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = -\sigma_1 \equiv CT,
 \end{aligned} \tag{33}$$

where in the last identification we used the relations (24).

With the choice  $C = i\sigma_1$ , we obtain,

$$T = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} = iI, \quad CP = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\sigma_3, \quad PT = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_2, \tag{34}$$

which completes the 2-dimensional irrep  $\varphi_9$  of  $G_\Theta^{(2)}$ ; while with the choice  $C = -i\sigma_1$ , we obtain,

$$T = \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix} = -iI, \quad CP = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -i\sigma_3, \quad PT = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = -\sigma_2, \tag{35}$$

IIR's $G_{\Theta}^{(2)}(\psi)$	$I$	$C$	$-I$	$-C$	$CP$	$-P$
$\varphi_1$	1	1	1	1	1	1
$\varphi_2$	1	-1	1	-1	1	-1
$\varphi_3$	1	-1	1	-1	1	-1
$\varphi_4$	1	1	1	1	1	-1
$\varphi_5$	1	-1	1	-1	1	-1
$\varphi_6$	1	-1	1	-1	1	-1
$\varphi_7$	1	1	1	1	1	-1
$\varphi_8$	1	-	1	-	1	1
$\varphi_9$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$
$\varphi_{10}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$

  

IIR's $G_{\Theta}^{(2)}(\psi)$	$-CP$	$T$	$CT$	$-T$	$-CT$	$PT$	$\Theta$	$-PT$	$-\Theta$
$\varphi_1$	1	1	1	1	1	1	1	1	1
$\varphi_2$	1	-1	1	-1	1	-1	1	-1	1
$\varphi_3$	1	1	-1	1	-1	1	-1	1	-1
$\varphi_4$	1	-1	-1	-1	-1	-1	-1	-1	-1
$\varphi_5$	1	-1	1	-1	1	1	-1	1	-1
$\varphi_6$	1	1	-1	1	-1	-1	1	-1	1
$\varphi_7$	-1	-1	-1	-1	-1	1	1	1	1
$\varphi_8$	-1	1	1	1	1	-1	-1	-1	-1
$\varphi_9$	$\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$	$\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
$\varphi_{10}$	$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$	$\begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Table 5: IIR's table of  $G_{\Theta}^{(2)}(\psi)$

which completes the 2-dimensional irrep  $\varphi_{10}$  of  $G_{\Theta}^{(2)}$ . It can be easily verified that  $\varphi_9$  and  $\varphi_{10}$  are inequivalent, i.e., it does not exist a matrix  $S$  such that  $SMS^\dagger = M'$  (or  $SMS^{-1} = M'$ ) for  $M \in \varphi_9$  and  $M' \in \varphi_{10}$ .

The remaining four 1-dimensional IIR's,  $\varphi_5, \varphi_6, \varphi_7$  and  $\varphi_8$ , are obtained from the orthogonality between the columns of the characters table for conjugation classes (completeness relation). Then, the orthogonality between the rows of the complete characters table is verified.

The IIR's and characters for  $G_{\Theta}^{(2)}(\psi) \cong D_4 \rtimes \mathbb{Z}_2$  are summarized in tables 5 and 6. Comparing with the tables for  $G_{\hat{\Theta}}(\hat{\psi})$ , we see that the 1-dimensional irrep's coincide:

$$\begin{aligned}
 \phi_1 &= \varphi_1, \phi_2 = \varphi_3, \phi_3 = \varphi_4, \phi_4 = \varphi_2, \\
 \phi_5 &= \varphi_8, \phi_6 = \varphi_6, \phi_7 = \varphi_7, \phi_8 = \varphi_5.
 \end{aligned}
 \tag{36}$$

However,  $\phi_9$  is not equivalent to either  $\varphi_9$  or  $\varphi_{10}$  and the same holds for  $\phi_{10}$ .

Ch $G_{\Theta}^{(2)}(\psi)$	[I]	[-I]	2[C]	[T]	[-T]	2[P]	2[CP]	2[CT]	2[PT]	2[Θ]
$\chi_1$	1	1	1	1	1	1	1	1	1	1
$\chi_2$	1	1	-1	-1	-1	1	-1	1	-1	1
$\chi_3$	1	1	-1	1	1	1	-1	-1	1	-1
$\chi_4$	1	1	1	-1	-1	1	1	-1	-1	-1
$\chi_5$	1	1	-1	-1	-1	-1	1	1	1	-1
$\chi_6$	1	1	-1	1	1	-1	1	-1	-1	1
$\chi_7$	1	1	1	-1	-1	-1	-1	-1	1	1
$\chi_8$	1	1	1	1	1	-1	-1	1	-1	-1
$\chi_9$	2	-2	0	$2i$	$-2i$	0	0	0	0	0
$\chi_{10}$	2	-2	0	$-2i$	$2i$	0	0	0	0	0

Table 6: Characters table of  $G_{\Theta}^{(2)}(\psi)$

### 5. Final Comments

It is of interest to ask the question if, from the point of view of group theory, the behaviour of the photon field  $\hat{A}_\mu$  under the  $C, P, T$  transformation is independent of the behaviour of the electron-positron field  $\hat{\psi}$ . The answer to this question is affirmative. In fact, in  $G_{\hat{\Theta}}(\hat{\psi})$  there is only one  $\mathbb{Z}_2$  subgroup, namely that generated by  $\hat{C} : \mathbb{Z}_2 \cong \{\hat{I}, \hat{C}\}$ . Then,  $G_{\hat{\Theta}}(\hat{A}_\mu) \cong \mathbb{Z}_2^3$  is not a subgroup of  $G_{\hat{\Theta}}(\hat{\psi})$ :

$$G_{\hat{\Theta}}(\hat{A}_\mu) \not\subset G_{\hat{\Theta}}(\hat{\psi}). \tag{37}$$

The same happens in relation to  $G_{\Theta}^{(2)}(\psi)$ . In this group there are three  $\mathbb{Z}_2$ -subgroups:

$$\mathbb{Z}_2^{(1)} = \{I, CT\}, \mathbb{Z}_2^{(2)} = \{I, PT\}, \mathbb{Z}_2^{(3)} = \{I, \Theta\}. \tag{38}$$

However, the table of  $\mathbb{Z}_2^3$  has eight 1's, while the table of  $G_{\Theta}^{(2)}(\psi)$  has only seven 1's. So

$$G_{\hat{\Theta}}(\hat{A}_\mu) \not\subset G_{\Theta}^{(2)}. \tag{39}$$

Finally, in relation with the posible relevance of the  $CPT$  group structures in other areas of physics, besides field theory, we call the attention on the recently found relation between the groups  $G_{\hat{\Theta}}(\hat{\psi}) \cong Q \times \mathbb{Z}_2$ ,  $G_{\Theta}^{(2)}(\psi) \cong D_4 \rtimes \mathbb{Z}_2$  and  $G_{\hat{\Theta}}(\hat{A}_\mu) \cong \mathbb{Z}_2^3$ , and fundamental theorems in the context of quantum paradoxes, like the Kochen-Specker theorem [10].

### Acknowledgments

This work was partially support by the project PAPIIT IN 118609-2, DGAPA-UNAM, México. B. Carballo Pérez also acknowledge financial support from CONACyT, México. The authors thank Luis Perissinotti for useful comments.

### References

- [1] M. Socolovsky, The CPT group of the Dirac field, *International Journal of Theoretical Physics*, **43** (2004), 1941-1967; *ArXiv*: math-ph/0404038.
- [2] B. Carballo Pérez, M. Socolovsky, Charge conjugation from space-time inversion, *International Journal of Theoretical Physics*, **48** (2009), 1712-1716; *ArXiv*: hep-th/08110842.
- [3] D. Asche, *An Introduction to Groups*, Hilger, Bristol (1989), p. 86.
- [4] W. Greiner, J. Reinhardt, *Field Quantization*, Springer (1993).
- [5] J.A. de Azcárraga, *P, C, T,  $\Theta$  in Quantum Field Theory*, Lectures at GIFT, 7/75 (1975).
- [6] J.D. Bjorken, S.D. Drell, *Relativistic Quantum Fields*, McGraw-Hill, New York (1965), p. 111.
- [7] S. Sternberg, *Group Theory and Physics*, Cambridge University Press, Cambridge (1994), p. 75.
- [8] M. Hamermesh, *Group Theory and its Application to Physical Problems*, Dover (1989), p. 145.
- [9] Wu-Ki Tung, *Group Theory in Physics*, World Scientific, New Jersey (2003), p. 31-32.
- [10] M. Planat, Three-qubit entanglement embeddings of CPT and Dirac groups within  $E_8$  Weyl group, *ArXiv*: 0906.1063 v1 (2009) [quant-ph].

278