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EVALUATION OF BETA-FUNCTION B-SPLINES, II: LOCAL BERNSTEIN BASES

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Abstract: This is the second paper in a sequence of three papers on the evaluation of *Beta-function B-splines* (BFBS), the first paper in the sequence being [4]. This sequence of papers studies explicit representations of BFBS yielding computationally efficient explicit formulae for evaluation of BFBS in terms of polynomial bases used in data interpolation, data fitting and geometric modelling, as well as in the design of multilevel constructions such as, e.g., multiwavelets. While in [4] an *interpolatory representation* of BFBS was developed in terms of *local monomial bases*, the objective of the present paper is to provide a *Bezier-type representation* in *local Bernstein bases*; the last of the three papers will be dedicated to a representation of BFBS in *global monomial bases*, suitable for use, e.g., in relevance to computing Fourier, Laplace and other transforms.

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1. Introduction

Expo-rational B-splines (ERBS) were introduced in [1] and were studied in detail in [6]. Applications of ERBS to Computer-Aided Geometric Design (CAGD) were considered in [1, 6, 10, 7, 8, 9].

Generalized expo-rational B-splines (GERBS) were introduced in [2], [5] as a generalization of ERBS [1], [6] including the polynomial simplified modifications of ERBS: Euler Beta-function B-splines (BFBS) [2], [5]. [3] contains a justification of the definition of BFBS and an exposition of the basic properties of BFBS.

Definition 1. ([3].) Let $t_k \in \mathbb{R}$ and $t_k < t_{k+1}$ for $k = 0, 1, 2, ..., n+1$. Consider the strictly increasing knot-vector $\{t_k\}_{k=0}^{n+1}$. A Beta-function B-spline (BFBS), associated with three strictly increasing knots t_{k-1}, t_k and t_{k+1}, $B_k(t) = B_k(i_{k-1}, i_k, i_{k+1}; t)$, $k = 1, \ldots, n$, is defined by

$$B_k(t) = \begin{cases} S_{k-1} \displaystyle\int_{t_{k-1}}^{t} \psi_{k-1}(s)ds, & \text{if} \quad t \in (t_{k-1}, t_k), \\ S_k \displaystyle\int_{t}^{t_{k+1}} \psi_k(s)ds, & \text{if} \quad t \in (t_k, t_{k+1}), \\ 1, & \text{if} \quad t = t_k, \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

with

$$S_k = \left[\int_{t_k}^{t_{k+1}} \psi_k(t)dt \right]^{-1}, \tag{2}$$

and

$$\psi_k(t) = C_k \frac{(t-t_k)^{i_k}(t_{k+1}-t)^{i_{k+1}}}{(t_{k+1}-t_k)^{i_k+i_{k+1}}}, \quad t \in [t_k, t_{k+1}], \tag{3}$$

where

$$C_k = \binom{i_k + i_{k+1}}{i_k}, \tag{4}$$

and

$$i_l > 0, \quad l = k-1, k, k+1. \tag{5}$$

Remark 1. The definition can be extended to B_k, $k = 0, n+1$, with obvious modifications.

As already discussed in [4], it is of considerable interest, both theoretical and computational, to find explicit representations of BFBS between the adjacent knots in terms of polynomial bases that are typically used in approximation theory, CAGD and operator calculus.

The article [4] was the first one in a sequence of three papers dedicated to the evaluation of BFBS between the knots in terms of expansions in polynomial bases. The present paper is the second paper in this sequence.

In [4] an interpolatory representation of BFBS was derived in terms of the *local monomial* bases

$$1,\ t-t_k,\ (t-t_k)^2,\ \ldots,\ (t-t_k)^{i_{k-1}+i_k}, \quad t \in (t_{k-1}, t_k), \tag{6}$$

and

$$1,\ t-t_k,\ (t-t_k)^2,\ \ldots,\ (t-t_k)^{i_k+i_{k+1}}, \quad t \in (t_k, t_{k+1}). \tag{7}$$

which coincide, modulo normalization, with the bases in the Taylor interpolation polynomials at the the central knot t_k of degree $i_{k-1} + i_k$ and $i_k + i_{k+1}$, respectively [4, Remark 1].

The objective of the present paper is to derive a different, Bezier-type, representation of BFBS in terms of the local Bernstein polynomials (in the sense that the Bernstein polynomials have been shifted and re-scaled for the segments t_{k-1}, t_k and t_k, t_{k+1}), so that the BFBS is being computed as a *polynomial Bezier curve* between each pair of neighbouring knots.

The third and last paper in this sequence will be addressing the derivation of a representation of BFBS in terms of *global monomial bases*, which can be useful in the context of using Fourier, Laplace, and some other integral transforms.

In the next brief Section 2 we shall formulate explicitly a technical result, proved in [4, Lemma 1], which is used in the proofs of the main results in all three papers of the present sequence. The main result is formulated and proved in Section 3. Finally, Section 4 contains some concluding remarks, including some orientation about the forthcoming third paper of the sequence.

2. Preliminaries

The following auxiliary result will be used in the proofs in Section 3.

Lemma 1. *([4, Lemma 1].) Formula (1) is equivalent to*

$$B_k(t) = \begin{cases} S_{k-1}d_{k-1}\displaystyle\int_{t_{k-1}}^{t} \varphi_{k-1}(\tau)d\tau,, & \textit{if} \quad t \in (t_{k-1}, t_k), \\ S_k d_k \displaystyle\int_{t}^{t_{k+1}} \varphi_k(\tau)d\tau,, & \textit{if} \quad t \in (t_k, t_{k+1}), \\ 1, & \textit{if} \quad t = t_k, \\ 0, & \textit{otherwise}, \end{cases} \tag{8}$$

where

$$d_{k-1} = \frac{(i_{k-1}+i_k)!}{i_{k-1}!i_k!}\frac{1}{(t_k - t_{k-1})^{i_{k-1}+i_k}}, \tag{9}$$

$$S_{k-1} = \frac{i_{k-1}+i_k+1}{t_k - t_{k-1}}, \tag{10}$$

$$d_k = \frac{(i_k+i_{k+1})!}{i_k!i_{k+1}!}\frac{1}{(t_{k+1} - t_k)^{i_k+i_{k+1}}}, \tag{11}$$

$$S_k = \frac{i_k+i_{k+1}+1}{t_{k+1} - t_k}, \tag{12}$$

$k = 0, \ldots, n.$

3. BFBS Evaluation in Terms of Local Bernstein Bases

Bernstein basis polynomials of degree $n \in \mathbb{N}$ ar usually being defined on the scaled interval $x \in [0, 1]$ as

$$b_{n,i}(x) = \binom{n}{i} x^i (1-x)^{n-i}, \quad i = 0, \ldots, n, \tag{13}$$

where $\binom{n}{i}$ is a binomial coefficient. In the present section we shall be deriving a representation of BFBS on the segments $[t_k, t_{k+1}]$, in terms of the *local Bernstein polynomials*, shifted and scaled onto $t \in [t_k, t_{k+1}]$:

$$b_{n,i}(t_k, t_{k+1}; t) = \frac{1}{(t_{k+1} - t_k)^n} \binom{n}{i} (t - t_k)^i (t_{k+1} - t)^{n-i}, \quad i = 0, \ldots, n, \tag{14}$$

for appropriate value of n, depending on i_k and i_{k+1}, $k = 0, \dots, n$. Our main result is the following theorem.

Theorem 1. *Under the conditions of Definition 1, let $k = 1, \dots, n$.*

(i) If $t \in (t_{k-1}, t_k)$, then,

$$B_k(t) = \sum_{l=0}^{i_k} \binom{i_{k-1} + i_k + 1}{l} \frac{(t_k - t)^l (t - t_{k-1})^{i_{k-1}+i_k+1-l}}{(t_k - t_{k-1})^{i_{k-1}+i_k+1}} = \tag{15}$$

$$= \sum_{l=0}^{i_k} b_{i_{k-1}+i_k+1,l}(t_{k-1}, t_k; t). \tag{16}$$

(ii) If $t \in (t_k, t_{k+1})$, then,

$$B_k(t) = \sum_{l=0}^{i_k} \binom{i_k + i_{k+1} + 1}{l} \frac{(t - t_k)^l (t_{k+1} - t)^{i_k+i_{k+1}+1-l}}{(t_{k+1} - t_k)^{i_k+i_{k+1}+1}} = \tag{17}$$

$$= \sum_{l=0}^{i_k} b_{i_k+i_{k+1}+1,l}(t_k, t_{k+1}; t). \tag{18}$$

Proof. Case (ii): $t \in (t_k, t_{k+1})$.
Applying Lemma 1, we get

$$B_k(t) = S_k d_k \int_t^{t_{k+1}} \varphi_k(\tau) d\tau.$$

Consecutively integrating by parts, we obtain the following chain of equalities

$$\int_t^{t_{k+1}} \varphi_k(\tau) d\tau =$$

$$= \int_t^{t_{k+1}} (\tau - t_k)^{i_k} (t_{k+1} - \tau)^{i_{k+1}} d\tau =$$

$$= \frac{(-1)}{i_{k+1} + 1} \int_t^{t_{k+1}} (\tau - t_k)^{i_k} d(t_{k+1} - \tau)^{i_{k+1}+1} =$$

$$= \frac{1}{i_{k+1}+1}(t-t_k)^{i_k}(t_{k+1}-t)^{i_{k+1}+1}+$$

$$+\frac{i_k}{i_{k+1}+1}\int\limits_t^{t_{k+1}}(\tau-t_k)^{i_k-1}(t_{k+1}-\tau)^{i_{k+1}+1}d\tau =$$

$$= \frac{1}{i_{k+1}+1}(t-t_k)^{i_k}(t_{k+1}-t)^{i_{k+1}+1}+$$

$$+\frac{i_k}{(i_{k+1}+1)}\frac{(-1)}{(i_{k+1}+2)}\int\limits_t^{t_{k+1}}(\tau-t_k)^{i_k-1}d(t_{k+1}-\tau)^{i_{k+1}+2} =$$

$$= \frac{1}{i_{k+1}+1}(t-t_k)^{i_k}(t_{k+1}-t)^{i_{k+1}+1}+$$

$$+\frac{i_k}{(i_{k+1}+1)(i_{k+1}+2)}(t-t_k)^{i_k-1}(t_{k+1}-t)^{i_{k+1}+2}+$$

$$+\frac{i_k}{(i_{k+1}+1)(i_{k+1}+2)}\int\limits_t^{t_{k+1}}(t_{k+1}-\tau)^{i_{k+1}+2}d(\tau-t_k)^{i_k-1} =$$

$$= \frac{1}{i_{k+1}+1}(t-t_k)^{i_k}(t_{k+1}-t)^{i_{k+1}+1}+$$

$$+\frac{i_k}{(i_{k+1}+1)(i_{k+1}+2)}(t-t_k)^{i_k-1}(t_{k+1}-t)^{i_{k+1}+2}+$$

$$\dots$$

$$+\frac{i_k(i_k-1)\dots 2}{(i_{k+1}+1)(i_{k+1}+2)\dots(i_{k+1}+i_k)}(t-t_k)(t_{k+1}-t)^{i_{k+1}+i_k}+$$

$$+(-1)\frac{i_k(i_k-1)\dots 1}{(i_{k+1}+1)(i_{k+1}+2)\dots(i_{k+1}+i_k)}\int\limits_t^{t_{k+1}}(t_{k+1}-\tau)^{i_{k+1}+i_k}d\tau =$$

$$= \frac{1}{i_{k+1}+1}(t-t_k)^{i_k}(t_{k+1}-t)^{i_{k+1}+1}+$$

$$+\frac{i_k}{(i_{k+1}+1)(i_{k+1}+2)}(t-t_k)^{i_k-1}(t_{k+1}-t)^{i_{k+1}+2}+$$

$$\dots$$

$$+\frac{i_k(i_k-1)\dots 2}{(i_{k+1}+1)(i_{k+1}+2)\dots(i_{k+1}+i_k)}(t-t_k)(t_{k+1}-t)^{i_{k+1}+i_k}+$$

$$+\frac{i_k(i_k-1)\dots 1}{(i_{k+1}+1)(i_{k+1}+2)\dots(i_{k+1}+i_k+1)}(t_{k+1}-t)^{i_{k+1}+i_k+1}=$$

$$=\sum_{j=0}^{i_k}\frac{i_k!i_{k+1}!}{(i_k-j)!(i_{k+1}+1+j)!}(t-t_k)^{i_k-j}(t_{k+1}-t)^{i_{k+1}+1+j}$$

Taking in consideration also the coefficients S_k and d_k, Lemma 1 yields

$$\begin{aligned}
B_k(t) &= S_k d_k \sum_{j=0}^{i_k}\frac{i_k!i_{k+1}!}{(i_k-j)!(i_{k+1}+1+j)!}(t-t_k)^{i_k-j}(t_{k+1}-t)^{i_{k+1}+1+j}\\
&=\sum_{j=0}^{i_k} S_k d_k\frac{i_k!i_{k+1}!}{(i_k-j)!(i_{k+1}+1+j)!}(t-t_k)^{i_k-j}(t_{k+1}-t)^{i_{k+1}+1+j}\\
&=\sum_{j=0}^{i_k}\frac{(i_k+i_{k+1}+1)!}{i_k!i_{k+1}!}\frac{1}{(t_{k+1}-t_k)^{i_k+i_{k+1}+1}}\frac{i_k!i_{k+1}!}{(i_k-j)!(i_{k+1}+1+j)!}\\
&\qquad\times(t-t_k)^{i_k-j}(t_{k+1}-t)^{i_{k+1}+1+j}\\
&=\sum_{j=0}^{i_k}\frac{(i_k+i_{k+1}+1)!}{(i_k-j)!(i_{k+1}+1+j)!}\frac{(t-t_k)^{i_k-j}(t_{k+1}-t)^{i_{k+1}+1+j}}{(t_{k+1}-t_k)^{i_k+i_{k+1}+1}}\\
&=\sum_{j=0}^{i_k}\binom{i_k+i_{k+1}+1}{i_k-j}\frac{(t-t_k)^{i_k-j}(t_{k+1}-t)^{i_{k+1}+1+j}}{(t_{k+1}-t_k)^{i_k+i_{k+1}+1}}.
\end{aligned}$$

Now change index j, as follows:

$$\begin{aligned}
&i_k-j=l, \qquad l=i_k-0,\dots,i_k-j, \quad j=0,\dots,i_k,\\
&j=i_k-l.
\end{aligned}$$

and obtain the following as a final expression for $B_k(t)$

$$B_k(t)=\sum_{l=0}^{i_k}\binom{i_k+i_{k+1}+1}{l}\frac{(t-t_k)^{l}(t_{k+1}-t)^{i_k+i_{k+1}+1-l}}{(t_{k+1}-t_k)^{i_k+i_{k+1}+1}}, \tag{19}$$

i.e., (17) is proved. Formula (18) now follows from (17) and (14).

Case (i): $t\in(t_{k-1},t_k)$.

The proof is analogous to that of Case (ii). For completeness (see also Section 4), we shall provide the full details of the proof of this case, too.

By Lemma 1,

$$B_k(t) = S_{k-1} d_{k-1} \int_{t_{k-1}}^{t} \varphi_{k-1}(\tau) d\tau.$$

Repeated integration by parts yields the following equality chain

$$\int_{t_{k-1}}^{t} \varphi_{k-1}(\tau) d\tau =$$

$$= \int_{t_{k-1}}^{t} (\tau - t_{k-1})^{i_{k-1}} (t_k - \tau)^{i_k} d\tau =$$

$$= \frac{1}{i_{k-1}+1} \int_{t_{k-1}}^{t} (t_k - \tau)^{i_k} d(\tau - t_{k-1})^{i_{k-1}+1} =$$

$$= \frac{1}{i_{k-1}+1} (t_k - t)^{i_k} (t - t_{k-1})^{i_{k-1}+1} +$$

$$+ \frac{i_k}{i_{k-1}+1} \int_{t_{k-1}}^{t} (t_k - \tau)^{i_k - 1} (\tau - t_{k-1})^{i_{k-1}+1} d\tau =$$

$$= \frac{1}{i_{k-1}+1} (t_k - t)^{i_k} (t - t_{k-1})^{i_{k-1}+1} +$$

$$+ \frac{i_k}{(i_{k-1}+1)(i_{k-1}+2)} \int_{t_{k-1}}^{t} (t_k - \tau)^{i_k - 1} d(\tau - t_{k-1})^{i_{k-1}+2} =$$

$$= \frac{1}{i_{k-1}+1} (t_k - t)^{i_k} (t - t_{k-1})^{i_{k-1}+1} +$$

$$+ \frac{i_k}{(i_{k-1}+1)(i_{k-1}+2)} (t_k - t)^{i_k - 1} (t - t_{k-1})^{i_{k-1}+2} +$$

$$+ (-1) \frac{i_k}{(i_{k-1}+1)(i_{k-1}+2)} \int_{t_{k-1}}^{t} (\tau - t_{k-1})^{i_{k-1}+2} d(t_k - \tau)^{i_k - 1} =$$

$$= \frac{1}{i_{k-1}+1} (t_k - t)^{i_k} (t - t_{k-1})^{i_{k-1}+1} +$$

$$\frac{i_k}{(i_{k-1}+1)(i_{k-1}+2)}(t_k-t)^{i_k-1}(t-t_{k-1})^{i_{k-1}+2}+$$
$$\dots$$
$$+\frac{i_k(i_k-1)\dots 2}{(i_{k-1}+1)(i_{k-1}+2)\dots(i_{k-1}+i_k)}(t_k-t)(t-t_{k-1})^{i_{k-1}+i_k}+$$
$$+\frac{i_k(i_k-1)\dots 1}{(i_{k-1}+1)(i_{k-1}+2)\dots(i_{k-1}+i_k)}\int\limits_{t_{k-1}}^{t}(\tau-t_{k-1})^{i_{k-1}+i_k}d\tau=$$
$$=\frac{1}{i_{k-1}+1}(t_k-t)^{i_k}(t-t_{k-1})^{i_{k-1}+1}+$$
$$+\frac{i_k}{(i_{k-1}+1)(i_{k-1}+2)}(t_k-t)^{i_k-1}(t-t_{k-1})^{i_{k-1}+2}$$
$$\dots$$
$$+\frac{i_k(i_k-1)\dots 2}{(i_{k-1}+1)(i_{k-1}+2)\dots(i_{k-1}+i_k)}(t_k-t)(t-t_{k-1})^{i_{k-1}+i_k}+$$
$$+\frac{i_k(i_k-1)\dots 1}{(i_{k-1}+1)(i_{k-1}+2)\dots(i_{k-1}+i_k+1)}(t-t_{k-1})^{i_{k-1}+i_k+1}=$$
$$=\sum_{j=0}^{i_k}\frac{i_k!i_{k-1}!}{(i_k-j)!(i_{k-1}+1+j)!}(t_k-t)^{i_k-j}(t-t_{k-1})^{i_{k-1}+1+j}$$

Next, we take into account the coefficients S_{k-1} and d_{k-1}. By Lemma 1,

$$B_k(t)=S_{k-1}d_{k-1}\int\limits_{t_{k-1}}^{t}\varphi_{k-1}(\tau)d\tau$$
$$=S_{k-1}d_{k-1}\sum_{j=0}^{i_k}\frac{i_k!i_{k-1}!}{(i_k-j)!(i_{k-1}+1+j)!}(t_k-t)^{i_k-j}(t-t_{k-1})^{i_{k-1}+1+j}$$
$$=\sum_{j=0}^{i_k}S_{k-1}d_{k-1}\frac{i_k!i_{k-1}!}{(i_k-j)!(i_{k-1}+1+j)!}(t_k-t)^{i_k-j}(t-t_{k-1})^{i_{k-1}+1+j}$$
$$=\sum_{j=0}^{i_k}\frac{(i_{k-1}+i_k+1)!}{i_k!i_{k-1}!}\frac{1}{(t_{k+1}-t_k)^{i_{k-1}+i_k+1}}\frac{i_k!i_{k-1}!}{(i_k-j)!(i_{k-1}+1+j)!}$$
$$\times(t_k-t)^{i_k-j}(t-t_{k-1})^{i_{k-1}+1+j}$$

$$= \sum_{j=0}^{i_k} \frac{(i_{k-1}+i_k+1)!}{(i_k-j)!(i_{k-1}+1+j)!} \frac{(t_k-t)^{i_k-j}(t-t_{k-1})^{i_{k-1}+1+j}}{(t_k-t_{k-1})^{i_{k-1}+i_k+1}}$$

$$= \sum_{j=0}^{i_k} \binom{i_{k-1}+i_k+1}{i_k-j} \frac{(t_k-t)^{i_k-j}(t-t_{k-1})^{i_{k-1}+1+j}}{(t_k-t_{k-1})^{i_{k-1}+i_k+1}}.$$

After a change of the index j

$$i_k - j = l, \qquad l = i_k - 0, \ldots, i_k - j, \quad l = 0, \ldots, i_k,$$
$$j = i_k - l,$$

the final expression for $B_k(t)$ is

$$B_k(t) = \sum_{l=0}^{i_k} \binom{i_{k-1}+i_k+1}{l} \frac{(t_k-t)^l (t-t_{k-1})^{i_{k-1}+i_k+1-l}}{(t_k-t_{k-1})^{i_{k-1}+i_k+1}}, \tag{20}$$

hence, (15) is proved. Formula (16) now follows from (15) and (14). □

4. Concluding Remarks

Remark 2. The Bezier-type representation developed in this paper and, in particular, formulae (16) and (18), provide a lucid insight into the way in which the value of BFBS is being formed. The computation is both order-preserving and convex, which ensures stability and high robustness with respect to computational errors.

Remark 3. As already mentioned in [4, Section 4], it is possible to obtain a local representation of BFBS between neighbouring knots in terms of any polynomial basis spanning the polynomials which are of the same degree, by using the transformation matrix for change between the two bases. In particular, it is possible, starting from the Bezier representation developed here, and using the transformation matrix for change between the local Bernstein basis and the local monomial basis, to obtain the interpolatory representation of BFBS developed in [4], and vice versa.

Remark 4. In the setting of Remark 3, the chains of equalities leading to each of the two representations reveal important linear-algebraic properties of the transformation matrix between the Bezier and the monomial bases (as well as of the inverse transformation matrix between these bases). We intend to return on exploring this topic in subsequent research, after having studied also the definition of BFBS in the multivariate case, especially, in the case of BFBS on triangulations.

Remark 5. The objective of the third paper of this sequence will be to derive a representation of BFBS in terms of *global* monomial bases, which would be a helpful representation when computing Fourier, Laplace and some other transforms.

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