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ON A NEW IMPROVEMENT OF VAN DER CORPUT'S INEQUALITY

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Abstract: A new improvement of Van Der Corput's inequality is made in this paper.

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1. Introduction

In his paper, Van Der Corput (1936) proved the following inequality which now known as the on Van Der Corput's inequality. If

$$\sum_{n=1}^{\infty} (n+1)\, a_n < \infty,$$

then

$$\sum_{n=1}^{\infty} \left(\prod_{k=1}^{n} a_k^{1/k} \right)^{S_n} < e^{1+\gamma} \sum_{n=1}^{\infty} (n+1)\, a_n, \tag{1.1}$$

where $a_n > 0$, $n = 1, 2, 3, \cdots$, $S_n = \sum_{k=1}^{n} \frac{1}{k}$, and $\gamma = 0.57721566 \cdots$ is the Euler

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constant. The constant factor $\exp(1+\gamma)$ in (1.1) is the best possible.

Recently, several authors made some generalization and improvement of Van Der Corput's inequality (1.1). Hu (2003) proved

$$\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n} a_k^{1/k}\right)^{S_n} < e^{1+\gamma}\sum_{n=1}^{\infty}\left(n-\frac{\ln n}{4}\right)a_n, \tag{1.2}$$

where $\exp(1+\gamma)$ is the best possible.

Cao et al (2006) made an extenstion of Van Der Corput's inequality (1.1) and proved that

$$\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n} a_k^{1/k}\right)^{1/S_n} < e^{1+\gamma}\sum_{n=1}^{\infty} n\left(1-\frac{\ln n}{3n-1/4}\right)a_n. \tag{1.3}$$

On the other hand, Yang (2007) provided an extension of Van Der Corput's inequality as follows:

$$\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n} a_k^{1/k}\right)^{1/S_n} < e^{1+\gamma}\sum_{n=1}^{\infty}\left(n-\frac{\ln n}{3}\right)a_n. \tag{1.4}$$

Niu et al (2006a, 2006b) made two more refinements of (1.1) as follows:

$$\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n} a_k^{1/k}\right)^{1/S_n}$$
$$< e^{1+\gamma}\sum_{n=1}^{\infty}\exp\left[-\frac{6(6n+1)\gamma-9}{(6n+1)(12n+11)}\right] n\left(1-\frac{\ln n}{2n+\ln n+11/6}\right)a_n, \tag{1.5}$$

where $a_n \geq 0$ for $n \in N$ and

$$\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n} a_k^{1/k}\right)^{1/S_n} < e^{1+\gamma}\sum_{n=1}^{\infty} e^{-\frac{3\gamma}{6n+4}}\, n\left(1-\frac{\ln n}{2n+\ln n+4/3}\right)a_n, \tag{1.6}$$

where $a_n \geq 0$ such that

$$0 < \sum_{n=1}^{\infty} n\left(1-\frac{\ln n}{2n+\ln n+4/3}\right)a_n < \infty. \tag{1.7}$$

The major objective of this paper to refine Van Der Corput's inequality further as follows:

$$\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n} a_k^{1/k}\right)^{1/S_n} < \sum_{n=1}^{\infty}\exp\left(1+\gamma-\frac{c}{n}\right)\left(n-\frac{\ln n}{2}\right)a_n, \tag{1.8}$$

where $a_n > 0$, $S_n = \sum_{k=1}^{n} \frac{1}{k}$, $(n = 1, 2, 3, \cdots)$, $c = (1 + \gamma - \ln 3)$ and

$$\sum_{n=1}^{\infty} \left(n - \frac{1}{2}\ln n\right) a_n < \infty.$$

Inequality (1.8) sharpens inequalities (1.1)-(1.5) and (1.6), and hence, (1.8) is superior to above results.

2. Lemmas

In order to prove the main result (1.8), the following lemmas are needed.

Lemma 1. (Franel's Inequality, see Polya and Szego (1972)) *If* $n \geq 1$, $n \in N$, $S_n = \sum_{k=1}^{n} \frac{1}{k}$ *is the harmonic series, then*

$$\ln n + \gamma + \frac{1}{2n} - \frac{1}{8n^2} < S_n < \ln n + \gamma + \frac{1}{2n}. \tag{2.1}$$

Lemma 2. (see Yang (1999)) *Let* $x > 0$. *Then*

$$\left(1 + \frac{1}{x}\right)^x < e\left(1 - \frac{1}{2x+2}\right). \tag{2.2}$$

Lemma 3. *Let* $n \geq 1$, $n \in N$, $S_n = \sum_{k=1}^{n} \frac{1}{k}$, $c = (1 + \gamma - \ln 3) \approx 0.4786033.$ *Then*

$$\left[\frac{(n+1)\,S_n + 1}{nS_n}\right]^{nS_n} \leq \exp\left(1 + \gamma - \frac{c}{n}\right)\left(n - \frac{\ln n}{2}\right). \tag{2.3}$$

Equality holds if and only if $n = 1$.

Proof. If $n = 1$, equality in (2.3) holds. If $n = 2, 3, 4, 5, 6, 7, 8$, inequality (2.3) is equivalent to

$$\left(\frac{n+1}{n} + \frac{1}{nS_n}\right)^{nS_n} \leq \exp\left(1 + \gamma - \frac{c}{n}\right)\left(n - \frac{\ln n}{2}\right). \tag{2.4}$$

A simple calculation shows that inequality (2.4) holds.

Let $n \geq 9$. We easily obtain

$$\frac{\ln n}{n} \leq \frac{\ln 9}{9}, \qquad \frac{\ln^2 n}{n} \leq \frac{\ln^2 9}{9}, \quad \frac{\ln n}{n^2} \leq \frac{\ln 9}{9^2},$$

$$\frac{\ln^3 n}{n^2} \le \frac{\ln^3 9}{9^2}, \qquad \frac{\ln^2 n}{n^2} \le \frac{\ln^2 9}{9^2}, \quad \frac{\ln^2 n}{n^3} \le \frac{\ln^2 9}{9^3},$$

and

$$\frac{\ln^3 n}{n} \le \frac{\ln^3 20}{20}.$$

Thus,

$$\begin{aligned}
&0.62\ln n + 0.78065 - \frac{2.48834}{n} - \frac{3.03443}{n}\ln n - \frac{1.16492\ln^2 n}{n} \\
&\quad - \frac{1.45721}{2n^2} - \frac{\ln n}{2n^2} - \frac{2\ln^3 n}{7n^2} - \frac{2\ln^3 n}{7n} - \frac{0.593491\ln^2 n}{n^2} - \frac{\ln^2 n}{7n^3} \ge \\
&0.62\ln 9 + 0.78065 - \frac{2.48834}{9} - \frac{3.03443}{9}\times\ln 9 - \frac{1.16492\times\ln^2 9}{9} - \frac{1.45721}{2\times 9^2} - \frac{\ln 9}{2\times 9^2} \\
&\quad - \frac{2\times\ln^3 9}{7\times 9^2} - \frac{2\times\ln^3 20}{7\times 20} - \frac{0.593491\times\ln^2 9}{9^2} - \frac{\ln^2 9}{7\times 9^3} \\
&> 1.362279 + 0.78065 - 0.276483 - 0.740814 - 0.624889 - 0.0089952 \\
&\quad - 0.013564 - 0.037418 - 0.384071 - 0.035374 - 0.0009461 > 0. \qquad (2.5)
\end{aligned}$$

It follows from direct calculation that

$$\begin{aligned}
&1.57721 - 1.38\gamma > 0.78065, \\
&-\frac{1.38}{2n} - \frac{1.45721}{n}\gamma - \frac{0.95721}{n} > -\frac{2.48834}{n}, \\
&-\frac{1.45721}{n}\ln n - \frac{\ln n}{n}\gamma - \frac{\ln n}{n} > -\frac{3.03443}{n}\ln n, \\
&-\frac{2\gamma\ln^2 n}{7n} - \frac{\ln^2 n}{n} > -\frac{1.16492\ln^2 n}{n}
\end{aligned} \qquad (2.6)$$

and

$$-\frac{2\gamma\ln^2 n}{7n^2} - \frac{\ln^2 n}{7n^2} - \frac{2\ln^2 n}{7n^2} > -\frac{0.59349\ln^2 n}{n^2}. \qquad (2.7)$$

Inequalities (2.5), (2.6) and (2.7) imply

$$\begin{aligned}
&0.62\ln n + 1.57721 - 1.38\gamma - \frac{1.38}{2n} - \frac{1.45721}{n}\gamma - \frac{0.95721}{n} \\
&\quad - \frac{1.45721}{n}\ln n - \frac{\ln n}{n}\gamma - \frac{\ln n}{n} - \frac{1.45721}{2n^2} - \frac{\ln^2 n}{n} - \frac{\ln n}{2n^2} \\
&\quad - \frac{2\ln^3 n}{7n^2} - \frac{2\ln^3 n}{7n} - \frac{2\gamma\ln^2 n}{7n^2} - \frac{2\gamma\ln^2 n}{7n} - \frac{\ln^2 n}{7n^3} - \frac{\ln^2 n}{7n^2} - \frac{2\ln^2 n}{7n^2} \ge 0
\end{aligned}$$

and

$$\left[-1.38-\frac{1.45721}{n}-\frac{\ln n}{n}-\frac{2(n+1)\ln^2 n}{7n^2}\right]\left(\ln n+\gamma+\frac{1}{2n}\right)$$
$$+1.57721-\frac{0.95721}{n}+2\ln n-\frac{\ln n}{n}-\frac{2\ln^2 n}{7n^2}\geq 0.$$

In view of $\gamma-2c-1>-1.38$, $1+\gamma>1.57721$, and $-\frac{2c}{n}\geq-\frac{0.95721}{n}$, it turns out that

$$\left[\gamma-2c-1-\frac{1}{2n}-\frac{2c}{n}-\frac{\ln n}{n}-\frac{2(n+1)\ln^2 n}{7n^2}\right]\left(\ln n+\gamma+\frac{1}{2n}\right)$$
$$+\left(1+2\gamma-\frac{2c}{n}\right)+2\ln n-\frac{\ln n}{n}-\frac{2\ln^2 n}{7n^2}\geq 0$$

and

$$\left[\ln n+\gamma-2c-1-\frac{1}{2n}-\frac{2c}{n}-(n+1)\left(\frac{\ln n}{n}+\frac{2\ln^2 n}{7n^2}\right)\right]\left(\ln n+\gamma+\frac{1}{2n}\right)$$
$$+\left(1+2\gamma-\frac{2c}{n}\right)+2\ln n-\frac{\ln n}{n}-\frac{2\ln^2 n}{7n^2}\geq 0.$$

For $n\geq 9$, we know $x=\dfrac{\ln\, n}{2n}<0.125$ and $\ln(1-x)>-x-\dfrac{4}{7}x^2$. Hence, it follows that

$$\left[\ln n+\gamma-2c-1-\frac{1}{2n}-\frac{2c}{n}+2(n+1)\ln\left(1-\frac{\ln n}{2n}\right)\right]\left(\ln n+\gamma+\frac{1}{2n}\right)$$
$$+\left(1+2\gamma-\frac{2c}{n}\right)+2\ln n+2\ln\left(1-\frac{\ln n}{2n}\right)\geq 0$$

and

$$-(2n+1)\left(\ln n+\gamma+\frac{1}{2n}\right)^2$$
$$+\left[2\gamma(n+1)-\frac{2c(n+1)}{n}+2(n+1)\ln\left(n-\frac{\ln n}{2}\right)\right]$$
$$\times\left(\ln n+\gamma+\frac{1}{2n}\right)^2+\left(1+2\gamma-\frac{2c}{n}\right)+2\ln\left(n-\frac{\ln n}{2}\right)\geq 0. \quad (2.8)$$

We consider the following function

$$f:x\to-(2n+1)x^2+\left[2\gamma(n+1)-\frac{2c(n+1)}{n}+2(n+1)\ln\left(n-\frac{\ln n}{2}\right)\right]x$$
$$+\left(1+2\gamma-\frac{2c}{n}\right)+2\ln\left(n-\frac{\ln n}{2}\right),$$

For symmetric axis of the function f, the following is true

$$x = -\frac{2\gamma(n+1) - \frac{2c(n+1)}{n} + 2(n+1)\ln\left(n - \frac{\ln n}{2}\right)}{2(2n+1)} < \ln n + \gamma + \frac{1}{2n} - \frac{1}{8n^2}.$$

In view of Lemma 1, we obtain

$$f(S_n) \geq f\left(\ln n + \gamma + \frac{1}{2n}\right). \tag{2.9}$$

Combining (2.8) and (2.9) gives

$$-(2n+1)S_n^2 + \left[2\gamma(n+1) - \frac{2c(n+1)}{n} + 2(n+1)\ln\left(n - \frac{\ln n}{2}\right)\right]S_n + \left(1 + 2\gamma - \frac{2c}{n}\right) + 2\ln\left(n - \frac{\ln n}{2}\right) \geq 0$$

and

$$(S_n+1)\left[1 - \frac{S_n+1}{2(n+1)S_n+2}\right] \leq \left(1 + \gamma - \frac{c}{n}\right) + \ln\left(n - \frac{\ln n}{2}\right).$$

If $x = \dfrac{S_n+1}{2(n+1)S_n+2} < 1$, then $\ln(1-x) \leq -x$. It follows that

$$(S_n+1)\left[1 + \ln\left(1 - \frac{S_n+1}{2(n+1)S_n+2}\right)\right] \leq \left(1 + \gamma - \frac{c}{n}\right) + \ln\left(n - \frac{\ln n}{2}\right),$$

and

$$\left[e\left(1 - \frac{1}{2\frac{nS_n}{S_n+1} + 2}\right)\right]^{S_n+1} \leq \exp\left(1 + \gamma - \frac{c}{n}\right)\left(n - \frac{\ln n}{2}\right).$$

In view of Lemma 2, we find

$$\left(1 + \frac{S_n+1}{nS_n}\right)^{\frac{nS_n}{S_n+1}(S_n+1)} \leq \exp\left(1 + \gamma - \frac{c}{n}\right)\left(n - \frac{\ln n}{2}\right),$$

so that inequality (2.3) holds on. □

3. Main Result

Theorem 3.1. *If* $a_n > 0$, $S_n = \sum_{k=1}^{n} \frac{1}{k}\,(n = 1, 2, \cdots)$, $c = (1 + \gamma - \ln 3)$, $\gamma = 0.57721566\cdots$ *is the Euler constant, and* $\sum_{n=1}^{\infty} \left(n - \frac{\ln n}{2}\right) a_n < +\infty$, *then*

$$\sum_{n=1}^{\infty} \left(\prod_{k=1}^{n} a_k^{1/k} \right)^{1/S_n} < \sum_{n=1}^{\infty} \exp\left(1 + \gamma - \frac{c}{n}\right) \left(n - \frac{\ln n}{2}\right) a_n. \tag{3.1}$$

Proof. Suppose $S_0 = 1$ and $\lambda_n = \frac{[(n+1)\,S_{n+1}]^{nS_n}}{(nS_n)^{nS_{n-1}}}$, where $n = 1, 2, \cdots$, we have

$$\prod_{k=1}^{n} \lambda_k^{1/k} = \prod_{k=1}^{n} \frac{[(k+1)\,S_{k+1}]^{S_k}}{(kS_k)^{S_{k+1}}} = [(n+1)\,S_{k+1}]^{S_n}.$$

According to the arithmetic and geometric means inequality, we obtain

$$\sum_{n=1}^{\infty} \left[\prod_{k=1}^{n} a_k^{1/k} \right]^{1/S_n} = \sum_{n=1}^{\infty} \left[\prod_{k=1}^{n} (\lambda_k a_k)^{1/k} \right]^{1/S_n} \left[\prod_{k=1}^{n} \lambda_k^{1/k} \right]^{-1/S_n}$$

$$\leq \sum_{n=1}^{\infty} \sum_{k=1}^{n} \left(\frac{\lambda_k a_k}{kS_n} \right) \cdot \left[\prod_{k=1}^{n} \lambda_k^{1/k} \right]^{-1/S_n} = \sum_{n=1}^{\infty} \sum_{k=1}^{n} \left(\frac{\lambda_k a_k}{kS_n} \right) \cdot \left[((n+1)\,S_{n+1})^{S_n} \right]^{-1/S_n}$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{(n+1)\,S_{n+1} S_n} \sum_{n=1}^{n} \frac{\lambda_k a_k}{k} \right)$$

$$= \frac{1}{2S_2S_1}(\lambda_1 a_1) + \frac{1}{3S_3S_2}\left(\lambda_1 a_1 + \frac{\lambda_2 a_2}{2}\right) + \frac{1}{4S_4S_3}\left(\lambda_1 a_1 + \frac{\lambda_2 a_2}{2} + \frac{\lambda_3 a_3}{3}\right) + \cdots$$

$$= \sum_{k=1}^{\infty} \left(\frac{\lambda_k a_k}{k} \sum_{n=k}^{\infty} \frac{1}{(n+1)\,S_{n+1} S_n} \right) = \sum_{k=1}^{\infty} \left(\frac{\lambda_k a_k}{k} \sum_{n=k}^{\infty} \left(\frac{1}{S_n} - \frac{1}{S_{n+1}} \right) \right)$$

$$= \sum_{k=1}^{\infty} \frac{\lambda_k a_k}{kS_k} = \sum_{k=1}^{\infty} \frac{[(k+1)\,S_{k+1}]^{kS_k}}{(kS_k)^{kS_{k+1}+1}} a_k = \sum_{k=1}^{\infty} \left[\frac{1 + (k+1)\,S_k}{kS_k} \right]^{kS_k} a_k,$$

which is, by Lemma 3,

$$\leq \sum_{k=1}^{\infty} \exp\left(1 + \gamma - \frac{c}{k}\right) \left(k - \frac{1}{2}\ln k\right) a_k.$$

This completes the proof of Theorem 3.1. □

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