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CONVERGENCE AND STABILITY OF THE ISHIKAWA ITERATION SCHEMES FOR QUASI-CONTRACTIVE MAPPINGS IN UNIFORMLY CONVEX BANACH SPACES

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Abstract: It is proved that certain Ishikawa iteration schemes with errors introduced by Xu (see *J. Math. Anal. Appl.*, **224** (1998), 91-101) can be used to approximate fixed points of quasi-contractive mappings in p-uniformly convex Banach spaces. A related result deals with the stability of the Ishikawa iteration scheme for quasi-contractive mappings in p-uniformly convex Banach spaces. Our results not only resolve affirmatively two open questions put forth by Rhoades [17] and Naimpally and Singh [13] and Osilike [16], respectively, in the more general setting, but also extend, improve and unify the corresponding results obtained by Chidume [4], Liu [14], [15], Osilike [16] and others.

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1. Introduction

Let X be a normed linear space with norm $\|\cdot\|$, T be an operator on X and

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$F(T) = \{x : Tx = x \in X\} \neq \phi$. Let x_0 be a point in X and $x_{n+1} = f(T, x_n)$ denote an iteration procedure which yields a sequence of point $\{x_n\}_{n=0}^{\infty}$ in X. Assume that $\{x_n\}_{n=0}^{\infty}$ converges strongly to $q \in F(T)$. Let $\{y_n\}_{n=0}^{\infty}$ be an arbitrary sequence in X, and set $\varepsilon_n = \|y_{n+1} - f(T, y_n)\|$. If $\lim_{n\to\infty} \varepsilon_n = 0$ implies that $\lim_{n\to\infty} y_n = q$, then the iteration procedure defined by $x_{n+1} = f(T, x_n)$ is said to be T-stable or stable with respect to T.

Let K be a nonempty convex subset of X and T be a mapping of K into itself. Recall that T is quasi-contractive if there exists a constant $r \in (0,1)$ such that

$$\|Tx - Ty\| \leq r \max\{\|x - y\|, \|x - Tx\|, \|y - Ty\|, \\ \|x - Ty\|, \|y - Tx\|\}, \quad \forall x, y \in K. \quad (1.1)$$

The following iteration schemes were introduced by Ishikawa [10], Mann [12] and Xu [25], respectively.

(i) For any given $x_0 \in K$ the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = (1 - a_n)x_n + a_n T y_n, \; y_n = (1 - b_n)x_n + b_n T x_n, \quad \forall n \geq 0,$$

is called the Ishikawa iteration sequence, where $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are real sequences in $[0,1]$ satisfying appropriate conditions.

(ii) In particular, if $b_n = 0$ for all $n \geq 0$, then the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_0 \in K, \; x_{n+1} = (1 - a_n)x_n + a_n T x_n, \quad \forall n \geq 0,$$

is called the Mann iteration sequence.

(iii) For any given $x_0 \in K$ the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = a_n x_n + b_n T y_n + c_n u_n, \; y_n = a'_n x_n + b'_n T x_n + c'_n v_n, \quad \forall n \geq 0,$$

where $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$ are arbitrary bounded sequences in K and $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$, $\{c_n\}_{n=0}^{\infty}$, $\{a'_n\}_{n=0}^{\infty}$, $\{b'_n\}_{n=0}^{\infty}$ and $\{c'_n\}_{n=0}^{\infty}$ are real sequences in $[0,1]$ such that $a_n + b_n + c_n = a'_n + b'_n + c'_n = 1$ for all $n \geq 0$ is called the Ishikawa iteration sequence with errors.

(iv) If, with the same notations and definitions as in (iii), $b'_n = c'_n = 0$ for all $n \geq 0$, then the sequence $\{x_n\}_{n=0}^{\infty}$ now defined by

$$x_0 \in K, \; x_{n+1} = a_n x_n + b_n T x_n + c_n u_n, \quad \forall n \geq 0,$$

is called the Mann iteration sequence with errors.

It is clear that the Ishikawa and Mann iterative processes are all special cases of the Ishikawa and Mann iterative processes with errors, respectively. Although the Mann iteration scheme can be obtained from the Ishikawa iteration scheme by setting $b_n = 0$, $\forall n \geq 0$, Rhoades [17] proved that the two iteration methods can exhibit different behaviors for different classes of nonlinear mappings.

In 1974, Ćirić [1] proved first both the existence of fixed point and convergence of Picards iteration for quasi-contractive mappings in complete metric spaces. Rhoades [18] showed that the quasi-contractive definition (1.1) is one of the most general contractive-type definitions for which Picards iteration yields a unique fixed point. In [17], Rhoades noted that the Mann iteration process can be used to approximate fixed points of quasi-contractive mappings in Hilbert spaces. Naimpally and Singh [13] and Rhoades [17] posed the following.

Question 1.1. Can the Mann iteration procedure be replaced by that of Ishikawa for quasi-contraction mapping of K into itself, where K is a compact convex subset of a Hilbert space?

This question was resolved in the affirmative by Ding [6], Chidume [2]-[4], Chidume and Osilike [5], Liu [14], [15], Xu [23] and Zhao [26] in Hilbert spaces, L_p (or l_p) spaces and Banach spaces, respectively.

A few stability results for certain classes of nonlinear mappings have been obtained by several researchers (see, e.g, [7]-[9], [15], [19]-[21]). As was shown by Harder [7] and Harder and Hicks [9], the study on the stability is both of theoretical and of numerical interest. Recently, Osilike [16] established the stability of certain Mann iteration procedure of quasi-contractive mappings in p-uniformly convex Banach spaces and put forth the following question:

Question 1.2. Is the Ishikawa iteration procedure T-stable for quasi-contractive maps in p-uniformly convex Banach spaces?

The purpose of this paper is to study Questions 1.1 and 1.2. In Section 3, we prove certain Ishikawa iteration schemes with errors converge strongly to fixed points for quasi-contractive mappings in p-uniformly convex Banach spaces. In Section 4, we show that the Ishikawa iteration scheme is T-stable for quasi-contractive mappings in p-uniformly convex Banach spaces. Our results not only give affirmative answers to the above open questions put forth by Rhoades [17] and Naimpally and Singh [13] and Osilike [16] in the more general setting, but also extend, improve and unify the corresponding results obtained by Chidume [4], Liu [14], [15] Osilike [16] and others.

2. Preliminaries

Let X be a Banach space and $p > 1$. The modulus of convexity of X is the function

$$\delta_X : [0, 2] \to [0, 1]$$

defined by

$$\delta_X(\varepsilon) = \inf\Big\{1 - \|\frac{1}{2}(x + y)\| : \|x\| \le 1, \|y\| \le 1, \|x - y\| \ge \varepsilon\Big\}.$$

X is called uniformly convex if $\delta_X(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$. X is said to be p-uniformly convex if there exists a constant $c > 0$ satisfying $\delta_X(\varepsilon) \ge c\varepsilon^p$ for all $\varepsilon \in (0, 2]$. It is well known that all Hilbert spaces, L_p (or l_p) spaces and the Sobolev spaces, W^p_m, $1 < p \le 2$, are 2-uniformly convex, while for $p \ge 2$, L_p (or l_p) and W^p_m spaces are p-uniformly convex.

Lemma 2.1. (see [24]) *Let X be a p-uniformly convex Banach space with $p > 1$. Then there exists a constant $\lambda_p > 0$ such that*

$$\|tx + (1 - t)y\|^p \le t\|x\|^p + (1 - t)\|y\|^p - \omega_p(t)\lambda_p\|x - y\|^p, \qquad (2.1)$$

for all $x, y \in X$ and $t \in (0, 1)$, where $\omega_p(t) = t(1 - t)^p + (1 - t)t^p$.

Lemma 2.2. (see [15]) *Suppose that $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are nonnegative sequences satisfying the following inequality:*

$$\alpha_{n+1} \le \gamma\alpha_n + \beta_n, \quad \forall n \ge 0,$$

where $\gamma \in [0, 1)$ and $\lim_{n\to\infty} \beta_n = 0$. Then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 2.3. (see [11]) *Let $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ be three non-negative real sequences satisfying the inequality*

$$\alpha_{n+1} \le (1 - \omega_n)\alpha_n + \beta_n + \gamma_n$$

for all $n \ge 0$, where $\{\omega_n\}_{n=0}^{\infty} \subset [0, 1]$, $\sum_{n=0}^{\infty} \omega_n = \infty$, $\beta_n = o(\omega_n)$ and $\sum_{n=0}^{\infty} \gamma_n < \infty$. Then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 2.4. *Suppose that x, y are in $[0, +\infty)$ and $p > 1$. Then*

$$(x + y)^p \le x^p + yp(x + y)^{p-1}. \qquad (2.2)$$

Proof. Set $f(x) = x^p$, $\forall x \ge 0$. It follows from Mean-Value Theorem that there exists $\xi \in [x, x + y]$ satisfying

$$(x + y)^p - x^p = f'(\xi)y = p\xi^{p-1}y \le yp(x + y)^{p-1},$$

which means that (2.2) holds. This completes the proof. □

For the remainder of this paper, r and λ_p denote the constants appearing in (1.1) and (2.1), respectively.

3. Convergence of the Ishikawa Iteration Sequences with Errors for Quasi-Contractive Mappings

Theorem 3.1. *Let K be a nonempty closed, bounded and convex subset of a p-uniformly convex Banach space X with $p > 1$ and $T : K \to K$ be a quasi-contractive mapping. Suppose that $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$ are arbitrary bounded sequences in K and $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$, $\{c_n\}_{n=0}^{\infty}$, $\{a'_n\}_{n=0}^{\infty}$, $\{b'_n\}_{n=0}^{\infty}$ and $\{c'_n\}_{n=0}^{\infty}$ are any sequences in $[0,1]$ satisfying*

$$a_n + b_n + c_n = a'_n + b'_n + c'_n = 1, \quad \forall n \geq 0; \tag{3.1}$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} b'_n = \lim_{n\to\infty} c'_n = 0; \tag{3.2}$$

$$\sum_{n=0}^{\infty} c_n < \infty, \quad \sum_{n=0}^{\infty} b_n = \infty. \tag{3.3}$$

$$r^p(1 - b'_n - c'_n) \leq [(1 - b_n - c_n)(b_n + c_n)^{p-1} + (1 - b_n - c_n)^p]\lambda_p, \quad \forall n \geq 0. \tag{3.4}$$

Then for any $x_0 \in K$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined iteratively by

$$\begin{aligned} y_n &= a'_n x_n + b'_n T x_n + c'_n v_n, \\ x_{n+1} &= a_n x_n + b_n T y_n + c_n u_n, \quad \forall n \geq 0, \end{aligned} \tag{3.5}$$

converges strongly to a unique fixed point of T.

Proof. It follows from Theorem 1 of Ćirić [1] or Theorem 11 of Rhoades [18] that T has a unique fixed point $q \in K$. Put $d_n = b_n + c_n$ and $d'_n = b'_n + c'_n$ for all $n \geq 0$. Since T is quasi-contractive, it follows that for all $x \in K$,

$$\begin{aligned} \|Tx - q\| &\leq r \max\{\|x - q\|, \|x - Tx\|, \|q - Tq\|, \|x - Tq\|, \|q - Tx\|\} \\ &= r \max\{\|x - q\|, \|x - Tx\|\}, \end{aligned} \tag{3.6}$$

which implies that

$$\|Tx - q\|^p \leq r^p[\|x - q\|^p + \|x - Tx\|^p] \tag{3.7}$$

for any $x \in K$. It follows from the boundedness of K, Lemmas 2.1 and 2.4 and (3.5) that there exists a constant N satisfying

$$p(\text{diam}K)^p < N,$$

$$\begin{aligned} \|y_n - Ty_n\|^p &= \|(1 - d'_n)(x_n - Ty_n) + d'_n(Tx_n - Ty_n) + c'_n(v_n - Tx_n)\|^p \\ &\leq \Big(\|(1 - d'_n)(x_n - Ty_n) + d'_n(Tx_n - Ty_n)\| + c'_n\|v_n - Tx_n\|\Big)^p \end{aligned}$$

$$\begin{aligned}&\le \|(1-d'_n)(x_n-Ty_n)+d'_n(Tx_n-Ty_n)\|^p+Nc'_n\\&\le (1-d'_n)\|x_n-Ty_n\|^p+d'_n\|Tx_n-Ty_n\|^p\\&\quad-\omega_p(1-d'_n)\lambda_p\|x_n-Tx_n\|^p+Nc'_n, \quad (3.8)\end{aligned}$$

$$\begin{aligned}\|y_n-q\|^p&=\|(1-d'_n)(x_n-q)+d'_n(Tx_n-q)+c'_n(v_n-Tx_n)\|^p\\&\le (1-d'_n)\|x_n-q\|^p+d'_n\|Tx_n-q\|^p\\&\quad-\omega_p(1-d'_n)\lambda_p\|x_n-Tx_n\|^p+Nc'_n, \quad (3.9)\end{aligned}$$

$$\begin{aligned}\|x_{n+1}-q\|^p&=\|(1-d_n)(x_n-q)+d_n(Ty_n-q)+c_n(u_n-Ty_n)\|^p\\&\le (1-d_n)\|x_n-q\|^p+d_n\|Ty_n-q\|^p\\&\quad-\omega_p(1-d_n)\lambda_p\|x_n-Ty_n\|^p+Nc_n, \quad (3.10)\end{aligned}$$

for all $n \ge 0$. Using (3.7)-(3.9), we have

$$\begin{aligned}\|Ty_n-q\|^p&\le r^p(\|y_n-q\|^p+\|y_n-Ty_n\|^p)\\&\le r^p(1-d'_n)\|x_n-q\|^p+r^pd'_n\|Tx_n-q\|^p+r^p(1-d'_n)\|x_n-Ty_n\|^p\\&\quad+r^pd'_n\|Tx_n-Ty_n\|^p+2Nr^pc'_n-2r^p\omega_p(1-d'_n)\lambda_p\|x_n-Tx_n\|^p, \quad (3.11)\end{aligned}$$

for all $n \ge 0$. From (3.1) (3.4), (3.10) and (3.11), we infer that

$$\begin{aligned}\|x_{n+1}-q\|^p&\le \Big(1-d_n+d_nr^p(1-d'_n)\Big)\|x_n-q\|^p+r^pd_nd'_n\|Tx_n-q\|^p\\&\quad+r^pd_nd'_n\|Tx_n-Ty_n\|^p+(2r^pd_nc'_n+c_n)N\\&\quad+\Big(r^p(1-d'_n)d_n-\omega_p(1-d_n)\lambda_p\Big)\|x_n-Ty_n\|^p-2d_nr^p\omega_p(1-d'_n)\lambda_p\|x_n-Tx_n\|^p\\&\le \Big(1-(1-r^p(1-d'_n))d_n\Big)\|x_n-q\|^p+2r^pNd_n(d'_n+c'_n)+Nc_n\\&\le \Big(1-(1-r^p)d_n\Big)\|x_n-q\|^p+4r^pNd_nd'_n+Nc_n, \quad (3.12)\end{aligned}$$

for all $n \ge 0$. Set $\alpha_n=\|x_n-q\|^p$, $\beta_n=4r^pNd_nd'_n$, $\gamma_n=Nc_n$ and $\omega_n=(1-r^p)d_n$ for all $n \ge 0$. Lemma 2.3 and (3.2) and (3.3) yield that $\alpha_n \to 0$ as $n \to \infty$. That is, $x_n \to q$ as $n \to \infty$. This completes the proof. □

Theorem 3.2. *Let K, X, T, p, $\{u_n\}_{n=0}^\infty$, $\{v_n\}_{n=0}^\infty$, $\{x_n\}_{n=0}^\infty$, $\{y_n\}_{n=0}^\infty$ be as in Theorem 3.1. Suppose that $\{a_n\}_{n=0}^\infty$, $\{b_n\}_{n=0}^\infty$, $\{c_n\}_{n=0}^\infty$, $\{a'_n\}_{n=0}^\infty$, $\{b'_n\}_{n=0}^\infty$ and $\{c'_n\}_{n=0}^\infty$ are any sequences in $[0,1]$ satisfying conditions (3.1), (3.2), (3.4) and*

$$b_n+c_n \ne 0, \quad \forall n \ge 0; \quad (3.13)$$

$$\lim_{n\to\infty}\frac{c_n}{b_n+c_n}=0; \quad (3.14)$$

$$\sum_{n=0}^{\infty} b_n = \infty. \tag{3.15}$$

Then $\{x_n\}_{n=0}^{\infty}$ *converges strongly to a unique fixed point of* T.

Proof. As in the proof of Theorem 3.1, we conclude that (3.12) holds. Set

$$\alpha_n = \|x_n - q\|^p, \ \beta_n = d_n\Big(4Nr^p d'_n + N\frac{c_n}{d_n}\Big),$$
$$\gamma_n = 0, \ \omega_n = (1-r^p)d_n, \quad \forall n \geq 0.$$

Then (3.12) can be rewritten as

$$\alpha_{n+1} \leq (1-\omega_n)\alpha_n + \beta_n + \gamma_n, \quad \forall n \geq 0.$$

It is easily seen that (3.1), (3.2), (3.13)-(3.15) imply that

$$\sum_{n=0}^{\infty} \omega_n = \infty, \ \omega_n \in (0,1], \ \beta_n = o(\omega_n), \quad \forall n \geq 0.$$

It follows from from Lemma 2.3 that $\alpha_n \to 0$ as $n \to \infty$. Hence $x_n \to q$ as $n \to \infty$. This completes the proof. □

Theorem 3.3. *Let* K, X, T, p, $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$, $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ *be as in Theorem 3.1. Suppose that* $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$, $\{c_n\}_{n=0}^{\infty}$, $\{a'_n\}_{n=0}^{\infty}$, $\{b'_n\}_{n=0}^{\infty}$ *and* $\{c'_n\}_{n=0}^{\infty}$ *are any sequences in* $[0,1]$ *satisfying conditions (3.1), (3.4), and*

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} b'_n = \lim_{n\to\infty} c'_n = 0; \tag{3.16}$$

$$b_n + c_n \geq \frac{1-s}{1-r^p}, \quad \forall n \geq 0, \tag{3.17}$$

where s *is a constant in* $(0,1)$. *Then* $\{x_n\}_{n=0}^{\infty}$ *converges strongly to a unique fixed point of* T.

Proof. It follows from (3.12) and (3.17) that

$$\begin{aligned}\|x_{n+1} - q\|^p &\leq [1-(1-r^p)d_n]\|x_n - q\|^p + 4r^pNd_nd'_n + Nc_n \\ &\leq s\|x_n - q\|^p + 4r^pNd_nd'_n + Nc_n, \end{aligned} \tag{3.18}$$

for all $n \geq 0$. In view of (3.16), (3.18) and Lemma 2.2, we conclude immediately that $x_n \to q$ as $n \to \infty$. This completes the proof. □

Remark 3.1. Theorems 3.1-3.3 give affirmative answers to Question 1.1 of Rhoades [17] and Naimpally and Singh [13] in the more general iteration schemes.

Remark 3.2. Theorems 3.1 and 3.3 extend, improve and unify Theorems 1 and 2 of Chidume [4], Theorem 1 of Liu [14] and the theorem of Liu [15] in the following ways:

(1) The Mann iteration scheme in [4] and the Ishikawa iteration scheme in

[4], [14], [15] are replaced by the Ishikawa iteration scheme with errors;

(2) The L_p (or l_p)spaces, $1 < p \leq 2$, in [4], Hilbert spaces in [14], [15] are replaced by p-uniformly convex Banach spaces, $1 < p$;

(3) The continuity and compactness assumptions on T and K, respectively, in [4], [14], are not needed;

(4) The condition $\alpha_n \leq \beta_n$, $\forall n \geq 0$, in [4], [14] are omitted.

Remark 3.3. The following examples demonstrate that Theorems 3.1 and 3.2 are independent.

Example 3.1. Let

$$a_n = 1 - (4+n)^{-\frac{1}{2}} - (2+n)^{-1},\ b_n = (4+n)^{-\frac{1}{2}},\ c_n = (2+n)^{-1}, \quad \forall n \geq 0,$$

$$a'_n = 1 - 2(2+n)^{-1},\ b'_n = c'_n = (2+n)^{-1}, \quad \forall n \geq 0.$$

Then conditions (3.1), (3.2), (3.13)-(3.15) are fulfilled. But

$$\sum_{n=0}^{\infty} c_n = \infty.$$

That is, condition (3.3) is not satisfied.

Example 3.2. Let $\{a'_n\}_{n=0}^{\infty}$, $\{b'_n\}_{n=0}^{\infty}$, $\{c'_n\}_{n=0}^{\infty}$ be as in Example 3.1. Set

$$a_{2n-1} = 1,\ b_{2n-1} = c_{2n-1} = 0, \quad \forall n \geq 1,$$

$$a_{2n} = 1 - \Big((2n+2)^{-1} - (2n^2+2)^{-1}\Big),\ b_{2n} = (2n+2)^{-1},$$

$$c_{2n} = (2n^2+2)^{-1}, \quad \forall n \geq 0.$$

Then conditions (3.1)-(3.3) hold. However, conditions (3.13) and (3.14) are not satisfied.

4. Stability of Ishikawa Iteration Procedure for Quasi-Contractive Mappings

Theorem 4.1. *Let X be a p-uniformly convex Banach space with $p > 1$ and $T : X \to X$ be a quasi-contractive mapping. Suppose that $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are real sequences in $[0,1]$ and s is constant in $(0,1)$ satisfying*

$$r^p \leq \min\{(1-a_n)a_n^{p-1} + (1-a_n)^p,$$
$$(1-b_n)b_n^{p-1} + (1-b_n)^p\}\lambda_p, \quad \forall n \geq 0, \tag{4.1}$$

$$1 - a_n(1-r^p) \leq s, \quad \forall n \geq 0. \tag{4.2}$$

For any $x_0 \in X$*, define the sequence* $\{x_n\}_{n=0}^{\infty}$ *by*

$$\begin{aligned} z_n &= (1-b_n)x_n + b_n T x_n, \quad \forall n \geq 0, \\ x_{n+1} &= (1-a_n)x_n + a_n T z_n, \quad \forall n \geq 0. \end{aligned} \tag{4.3}$$

Let $\{y_n\}_{n=0}^{\infty}$ *be any sequence in* X *and define* $\{\varepsilon_n\}_{n=0}^{\infty}$ *by*

$$\begin{aligned} w_n &= (1-b_n)y_n + b_n T y_n, \quad \forall n \geq 0, \\ \varepsilon_n &= \|y_{n+1} - (1-a_n)y_n - a_n T w_n\|, \quad \forall n \geq 0. \end{aligned} \tag{4.4}$$

Then:

(i) The sequence $\{x_n\}_{n=0}^{\infty}$ *converges strongly to the unique fixed point* q *of* T*;*

(ii) $\|y_{n+1} - q\| \leq \varepsilon_n + s^{\frac{1}{p}} \|y_n - q\|, \quad \forall n \geq 0;$

(iii) $\lim_{n\to\infty} y_n = q$ *if and only if* $\lim_{n\to\infty} \varepsilon_n = 0$.

Proof. The existence of a unique fixed point for T follows from Ćirić [1] or Rhoades [18]. Let q denote the fixed point of T. Observe that (4.2) implies that $a_n \geq \frac{1-s}{1-r^p} > 0$ for all $n \geq 0$. Hence $\sum_{n=0}^{\infty} a_n = \infty$. It follows from the Theorem of Xu [23] that (i) holds. Note that (4.4) means that

$$\|y_{n+1} - q\| \leq \varepsilon_n + \|(1-a_n)(y_n - q) + a_n(Tw_n - q)\| \tag{4.5}$$

for all $n \geq 0$. In view of Lemma 2.1, we have

$$\begin{aligned} &\|(1-a_n)(y_n - q) + a_n(Tw_n - q)\|^p \\ &\quad \leq (1-a_n)\|y_n - q\|^p + a_n\|Tw_n - q\|^p - \omega_p(1-a_n)\lambda_p\|y_n - Tw_n\|^p, \end{aligned} \tag{4.6}$$

for all $n \geq 0$. Using (1.1), we get that

$$\|Tx - q\| \leq r \max\{\|x - q\|, \|x - Tx\|\} \tag{4.7}$$

for all $x \in X$. In particular, we also have

$$\|Tw_n - q\| \leq r \max\{\|w_n - q\|, \|w_n - Tw_n\|\} \tag{4.8}$$

for all $n \geq 0$.

Suppose that $\|Tw_n - q\| \leq r\|w_n - q\|$ for some $n \geq 0$. It follows from Lemma 2.1 that

$$\begin{aligned} \|w_n - q\|^p &= \|(1-b_n)(y_n - q) + b_n(Ty_n - q)\|^p \\ &\leq (1-b_n)\|y_n - q\|^p + b_n\|Ty_n - q\|^p - \omega_p(1-b_n)\lambda_p\|y_n - Ty_n\|^p. \end{aligned} \tag{4.9}$$

In virtue of (4.7), we have

$$\|Ty_n - q\| \leq r \max\{\|y_n - q\|, \|y_n - Ty_n\|\}. \tag{4.10}$$

Hence (4.10) means that either $\|Ty_n - q\| \leq r\|y_n - q\|$ or $\|Ty_n - q\| \leq r\|y_n -$

$Ty_n\|$. If $\|Ty_n - q\| \le r\|y_n - q\|$, we conclude that by (4.9)

$$\|w_n - q\|^p \le (1 - b_n + b_n r^p)\|y_n - q\|^p - \omega_p(1 - b_n)\lambda_p\|y_n - Ty_n\|^p. \quad (4.11)$$

Thus (4.11) and (4.6) yield that

$$\begin{aligned}&\|(1 - a_n)(y_n - q) + a_n(Tw_n - q)\|^p \\ &\le \Big(1 - a_n + a_n r^p(1 - b_n + b_n r^p)\Big)\|y_n - q\|^p - a_n r^p \omega_p(1 - b_n)\lambda_p\|y_n - Ty_n\|^p \\ &\quad - \omega_p(1 - a_n)\lambda_p\|y_n - Tw_n\|^p \le \Big(1 - a_n + a_n r^p(1 - b_n + b_n r^p)\Big)\|y_n - q\|^p \\ &\le \Big(1 - a_n(1 - r^p)\Big)\|y_n - q\|^p. \quad (4.12)\end{aligned}$$

If $\|Ty_n - q\| \le r\|y_n - Ty_n\|$, we have from (4.9), (4.6) and (4.1)

$$\begin{aligned}&\|(1 - a_n)(y_n - q) + a_n(Tw_n - q)\|^p \le \Big(1 - a_n + a_n r^p(1 - b_n)\Big)\|y_n - q\|^p \\ &\quad + a_n r^p\Big(b_n r^p - \omega_p(1 - b_n)\lambda_p\Big)\|y_n - Ty_n\|^p - \omega_p(1 - a_n)\lambda_p\|y_n - Tw_n\|^p \\ &\le \Big(1 - a_n + a_n r^p(1 - b_n)\Big)\|y_n - q\|^p \le \Big(1 - a_n(1 - r^p)\Big)\|y_n - q\|^p. \quad (4.13)\end{aligned}$$

Suppose that $\|Tw_n - q\| \le r\|w_n - Tw_n\|$ for some $n \ge 0$. Then (4.6) reduce to

$$\begin{aligned}&\|(1 - a_n)(y_n - q) + a_n(Tw_n - q)\|^p \\ &\le (1 - a_n)\|y_n - q\|^p + a_n r^p\|w_n - Tw_n\|^p - \omega_p(1 - a_n)\lambda_p\|y_n - Tw_n\|^p. \quad (4.14)\end{aligned}$$

Lemma 2.1 ensures that

$$\begin{aligned}\|w_n - Tw_n\|^p &= \|(1 - b_n)(y_n - Tw_n) + b_n(Ty_n - Tw_n)\|^p \le (1 - b_n)\|y_n - Tw_n\|^p \\ &\quad + b_n\|Ty_n - Tw_n\|^p - \omega_p(1 - b_n)\lambda_p\|y_n - Ty_n\|^p. \quad (4.15)\end{aligned}$$

Observe that

$$\|Ty_n - Tw_n\| \le r\max\{\|y_n - Ty_n\|, \|w_n - Tw_n\|, \|y_n - Tw_n\|\},$$

which implies that either $\|Ty_n - Tw_n\| \le r\|y_n - Ty_n\|$, or $\|Ty_n - Tw_n\| \le r\|w_n - Tw_n\|$, or $\|Ty_n - Tw_n\| \le r\|y_n - Tw_n\|$. If $\|Ty_n - Tw_n\| \le r\|y_n - Ty_n\|$, by (4.14), (4.15) and (4.1) we infer that

$$\begin{aligned}&\|(1 - a_n)(y_n - q) + a_n(Tw_n - q)\|^p \le (1 - a_n)\|y_n - q\|^p \\ &\quad + \Big(a_n r^p(1 - b_n) - \omega_p(1 - a_n)\lambda_p\Big)\|y_n - Tw_n\|^p \\ &\quad + a_n r^p\Big(b_n r^p - \omega_p(1 - b_n)\lambda_p\Big)\|y_n - Ty_n\|^p \le (1 - a_n)\|y_n - q\|^p \\ &\quad + \Big(a_n r^p - \omega_p(1 - a_n)\lambda_p\Big)\|y_n - Tw_n\|^p \le (1 - a_n)\|y_n - q\|^p. \quad (4.16)\end{aligned}$$

If $\|Ty_n - Tw_n\| \le r\|y_n - Tw_n\|$, then (4.15) ensures that

$$\|w_n - Tw_n\|^p \le \frac{1-b_n}{1-r^p b_n}\|y_n - Tw_n\|^p - \omega_p(1-b_n)\frac{\lambda_p}{1-r^p b_n}\|y_n - Ty_n\|^p. \quad (4.17)$$

Substituting (4.17) into (4.14), by (4.1) we obtain that

$$\begin{aligned}&\|(1-a_n)(y_n-q)+a_n(Tw_n-q)\|^p \le (1-a_n)\|y_n-q\|^p + \\ &\Big(a_n r^p \frac{1-b_n}{1-r^p b_n} - \omega_p(1-a_n)\lambda_p\Big)\|y_n - Tw_n\|^p - a_n r^p \omega_p (1-b_n)\frac{\lambda_p}{1-r^p b_n}\|y_n - Ty_n\|^p \\ &\le (1-a_n)\|y_n-q\|^p + \Big(a_n r^p - \omega_p(1-a_n)\lambda_p\Big)\|y_n - Tw_n\|^p \\ &\le (1-a_n)\|y_n-q\|^p. \quad (4.18)\end{aligned}$$

If $\|Ty_n - Tw_n\| \le r\|y_n - Tw_n\|$, we conclude immediately from (4.14) and (4.15) that

$$\begin{aligned}&\|(1-a_n)(y_n-q)+a_n(Tw_n-q)\|^p \le (1-a_n)\|y_n-q\|^p \\ &+ \Big(a_n r^p(1-b_n+b_n r^p) - \omega_p(1-a_n)\lambda_p\Big)\|y_n - Tw_n\|^p \\ &- a_n r^p \omega_p(1-b_n)\lambda_p\|y_n - Ty_n\|^p \le (1-a_n)\|y_n-q\|^p \\ &+ \Big(a_n r^p - \omega_p(1-a_n)\lambda_p\Big)\|y_n - Tw_n\|^p \le (1-a_n)\|y_n-q\|^p. \quad (4.19)\end{aligned}$$

It follows from (4.2), (4.8), (4.12), (4.13), (4.16), (4.18) and (4.19) that

$$\begin{aligned}\|(1-a_n)(y_n-q)+a_n(Tw_n-q)\| &\le \Big(1-a_n(1-r^p)\Big)^{\frac{1}{p}}\|y_n-q\| \\ &\le s^{\frac{1}{p}}\|y_n-q\| \quad (4.20)\end{aligned}$$

for all $n \ge 0$. Therefore, (ii) follows from (4.5) and (4.20).

Suppose that $\lim_{n\to\infty} y_n = q$. Using (ii), we have

$$\begin{aligned}\varepsilon_n &= \|y_{n+1} - (1-a_n)y_n - a_n Tw_n\| \\ &\le \|y_{n+1} - q\| + \|(1-a_n)(y_n-q) + a_n(Tw_n - q)\| \\ &\le \|y_{n+1} - q\| + s^{\frac{1}{p}}\|y_n - q\| \to 0\end{aligned}$$

as $n \to \infty$. That is, $\lim_{n\to\infty}\varepsilon_n = 0$.

Suppose that $\lim_{n\to\infty}\varepsilon_n = 0$. It follows from (ii) and Lemma 2.2 that $\lim_{n\to\infty} y_n = q$. Hence (iii) holds. This completes the proof. □

Remark 4.1. Theorem 4.1 reveals that the Ishikawa iteration scheme is T-stable for quasi-contractive mappings in p-uniformly convex Banach spaces. Thus Theorem 4.1 resolves in the affirmative Question 1.2 raised by Osilike [16].

As a consequence of Theorem 4.1, we have the following

Corollary 4.1. *Let X be a p-uniformly convex Banach space with $p > 1$ and $T : X \to X$ be a quasi-contractive mapping. Suppose that $\{a_n\}_{n=0}^{\infty}$ is a real sequence in $[0, 1]$ and s is constant in $(0, 1)$ satisfying (4.2) and*

$$r^p \leq [(1 - a_n)a_n^{p-1} + (1 - a_n)^p]\lambda_p, \quad \forall n \geq 0. \tag{4.21}$$

For any $x_0 \in X$, define the sequence $\{x_n\}_{n=0}^{\infty}$ by

$$x_{n+1} = (1 - a_n)x_n + a_n T x_n, \quad \forall n \geq 0. \tag{4.22}$$

Let $\{y_n\}_{n=0}^{\infty}$ be any sequence in X and define $\{\varepsilon_n\}_{n=0}^{\infty}$ by

$$\varepsilon_n = \|y_{n+1} - (1 - a_n)y_n - a_n T y_n\|, \quad \forall n \geq 0. \tag{4.23}$$

Then:

(i) *The sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point q of T;*

(ii) $\|y_{n+1} - q\| \leq \varepsilon_n + s^{\frac{1}{p}}\|y_n - q\| \geq 0, \quad \forall n \geq 0;$

(iii) $\lim_{n\to\infty} y_n = q$ *if and only if* $\lim_{n\to\infty} \varepsilon_n = 0$.

Remark 4.2. Corollary 4.1 generalizes, improves and unifies Theorems 3 and 4 of Osilike [16].

Question 4.1. Are the Mann iteration sequence with errors and the Ishikawa iteration sequence with errors stable for quasi-contractive mappings in p-uniformly convex Banach spaces with $p > 1$?

Question 4.2. Can be Theorem 3.1, Theorem 3.2, Theorem 3.3 and Theorem 4.1 extended to uniformly convex Banach spaces?

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