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A GEOMETRIC CONDITION FOR THE EXISTENCE OF THE HOMOCLINIC ORBITS OF LIÉNARD SYSTEMS

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Abstract: In this paper, a geometric condition in order to prove the existence of homoclinic orbits of Liénard systems is stated. As a result, the necessary and sufficient condition for the existence of the homoclinic orbit of the system will be given in the simple form. Our idea is to apply curves with some invariance to the existence of the homoclinic orbit. It will be shown that the positions of these curves on the phase space play an important role in the existence of homoclinic orbits of the system. The results will be applied to many examples.

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1. Introduction

In this paper, the existence of the homoclinic orbit of a Liénard system:

$$\begin{cases} \dot{x} = y - F(x), \\ \dot{y} = -g(x), \end{cases} \tag{L}$$

is discussed, where $F(x)$ and $g(x)$ are continuous on an open interval which contains the origin. We assume that the functions $F(x)$ and $g(x)$ satisfy smoothness conditions for the uniqueness of solutions of initial value problems.

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In system (L), a trajectory is said to be a "homoclinic orbit" if its α- and ω-limit sets are the origin. Investigating the existence of homoclinic orbits of system (L) is great significance and its orbits play an important role in nonlinear dynamical systems.

We gave (see [2]) the global existence theorem for the homoclinic orbits of system (L). It has been proved by using the existence of the invariant curves of the system. We also want to discuss the existence of the homoclinic orbits by applying this idea. As a result, the necessary and sufficient condition in order that system (L) has homoclinic orbits is given by using the existence of a curve with some invariance. It is simply proved under some geometric conditions.

In [1], the similar condition to our condition [C2] in the theorem has been given partially by setting the function $F(x) - \varphi(x) = h(x)$. As is shown in [1; Corollaries 2.9 and 2.10], we remark that the boundedness of the solutions and the non-existence of the non-trivial closed orbits for system (L) are supposed. In [5], the condition for the existence of the homoclinic orbits (the system has property (Z_1^+)) has been studied by the form of an integration. The explicite condition given by it is complicate. So it is difficult to apply this method to a concrete system. Using the polynomials as the invariant curve in our method, we will see that the verdict of the existence of the homoclinic orbits is easy (see Examples 1 and 2). As it is seen in the corollaries of this paper, in virture of the geometric condition in our result, it is possible to give the following:

• whether the system has the homoclinic orbits locally or globally (Corollary 1),

• the interval $|x| < \delta$ on x-axis that the homoclinic orbits exist,

• the condition in order that the system has homoclinic orbits, but no limit cycles (Corollary 3).

Moreover, it is concluded that the polynomial system (L) has no homoclinic orbits if $\deg g = 1$ (see Corollary 2).

In Section 2, the theorem is easily proved by using "a negative invariant curve" of the system. In Section 3, our result is applied to some system (L) with two parameters. Finally, it shall be shown that the theorem is rewritten by using the functions in [4].

Throughout this paper, we also assume the condition $F(0) = 0$ and $xg(x) > 0$ for $x \neq 0$, which guarantee that the origin is the unique equilibrium point of system (L) and $\mathrm{Index}(0, 0) = +1$. Then, if system (L) has a homoclinic orbit, then $F(x)$ is necessarily positive (or negative) in the neighbourhood of the origin. For this reason, taking into account the vector field of system (L), we see that no homoclinic orbit crosses the x-axis. Hence, when a homoclinic orbit appears in the upper (respectively, lower) half-plane, all the other homoclinic orbits must exist in the same half-plane.

If system (L) has a homoclinic orbit, then all trajectories of (L) in the region are enclosed by the union the homoclinic orbit and the origin are also homoclinic orbits.

For the details see [1].

Consider a function $\varphi(x)$ with the condition:

[C1] $\varphi \in C^1$, $\varphi(0) = 0$ and $xF(x)\varphi^{'}(x) > 0$ for $0 < |x| < \delta$.

Our main results are the following

Theorem. *system* (L) *has homoclinic orbits if and only if there exists a function* $\varphi(x)$ *with* [C1] *such that*

[C2] $(F(x) > \varphi(x) > 0$ *or* $F(x) < \varphi(x) < 0)$ *and*

$$\varphi^{'}(x)[F(x) - \varphi(x)] \geq g(x),$$

for $0 < |x| < \delta$.

Corollary 1. *system* (L) *has homoclinic orbits globally if and only if there exists a function* $\varphi(x)$ *with* [C1] *such that the condition* [C2] *holds for* $\delta = +\infty$.

We say that, if the orbit of system (L) starting from an arbitrary point belongs to the domain $\{(x, y) \mid y > F(x) > 0$ (or $y < F(x) < 0)\}$ is always homoclinic, then the system has homoclinic orbits globally. Thus, we have a remark below.

Remark. If system (L) has homoclinic orbits globally, then there exists no limit cycles.

In [2], we have given a necessary and sufficient condition in order that system (L) has homoclinic orbits globally. To prove this fact, the existence of the invariant curves of system (L) was used. It has been given under the condition $\varphi^{'}(x)[F(x) - \varphi(x)] = g(x)$ for all x, instead of [C2]. Thus, the above corollary is an exact improvement of the result of [2].

Corollary 2. *Let* F *and* g *be polynomials. If* $\deg g = 1$, *then system* (L) *has no homoclinic orbits.*

In facts, if $\deg g = 1$, we see that the value of $\varphi^{'}(x)[F(x) - \varphi(x)] - g(x)$ must be negative at the neighborfood of positive number x. Thus, we see from the theorem that system (L) has no homoclinic orbits.

We will introduce an example to understand the above theorem.

Example 1. Consider system (L) with $F(x) = 0.2x^2(x^2 - 4)(x - 3)$ and

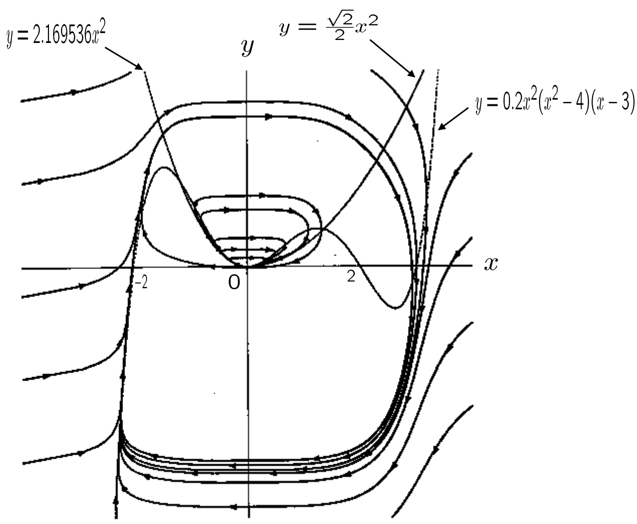

Figure 1: system (L) with both homoclinic orbits and a unique limit cycle

$g(x) = x^3$. Then the system has both homoclinic orbits locally and a unique limit cycle. See Figure 1.

Take the supplement function $\varphi(x) = Ax^2$ with

$$A = \begin{cases} a^* = \sqrt{2}/2 & (0 < x \leq 0.846473), \\ b^* = 2.169536 & (-1 \leq x < 0), \end{cases}$$

where a^* is a minimal point of the function $2\alpha + 1/\alpha - 4.8$ and b^* is a solution of the equation $2\alpha^2 - 4.8\alpha + 1 = 0$. Then, we can calculate two inequalities:

$$\varphi'(x)[F(x)-\varphi(x)]-g(x) = 2a^*x^3\left(0.2x^3 - 0.6x^2 - 0.8x + 2.4 - a^* - \frac{1}{2a^*}\right) \geq 0,$$

for $0 < x \leq 0.846473$ and

$$\varphi'(x)[F(x) - \varphi(x)] - g(x) = 0.4b^*x^4(x+1)(x-4) \leqq 0,$$

for $-1 \leq x < 0$.

Thus, since the conditions [C1] and [C2] for $\delta = 0.846473$ are satisfied, we conclude from the theorem that the system has homoclinic orbits locally. For the existence of a unique limit cycle of the system, see the method used in [3].

2. Proof of Theorem

Let $C^+ = \{(x, y) \mid x \in [0, \infty)$ and $y = F(x)\}$ and $C^- = \{(x, y) \mid x \in (-\infty, 0]$ and $y = F(x)\}$. We also denote $\gamma^+(P)$ and $\gamma^-(P)$ for the positive orbit and negative orbit of (L) starting at a point $P \in \mathbb{R}^2$, respectively. If an orbit of system (L) crosses the y-axis, then its tangent is horizontal at the intersection point. Also, if an orbit of (L) meets the curve C^+ or C^-, then its tangent is vertical at the point of intersection. Thus, system (L) has a homoclinic orbit if and only if there exists a point $P \in C^+$ or C^- such that both $\gamma^+(P)$ and $\gamma^-(P)$ approach the origin.

Suppose that a solution orbit $(x(t), y(t))$ of (L) defined on $t \in I = [t_1, t_2]$ meets the curve $y = \varphi(x)$ at $t^* \in I$. We say that, if $dy(t^*)/dx(t^*) \leqq \varphi'(x(t^*))$ for all $t^* \in I$, then the system has a negative invariant curve $y = \varphi(x)$ on I.

We may assume without loss of generality that P is in a neighborhood of the origin. First, we will consider the case $F(x) > 0$ and $P \in \{(x, y) \mid x \in [0, \infty)$ and $y > F(x)\}$.

Sufficiency. Suppose that the conditions [C1] and [C2] are satisfied. Then the system has a negative invariance curve $y = \varphi(x)$ for $0 < x < \delta$. Since P is in a neighborhood of the origin, $\gamma^+(P)$ rotates in a clockwise direction about the origin, the orbit crosses the curve $y = F(x)$ vertically and must stay in the region $\{(x, y) \mid x \in [0, \infty)$ and $y < F(x)\}$. Hence, it approaches the origin without crossing the curve $y = \varphi(x)$. Similarly, $\gamma^-(P)$ must approach the origin.

Necessity. Suppose that $\gamma^+(P)$ approaches the origin through only the region $\{(x, y) \mid 0 < x < \delta$ and $0 < y < F(x)\}$. Then its corresponding solution $y(x)$ of system (L) satisfies $y(x) \searrow 0$ as $x \to +0$. So letting $y(x) = \varphi(x)$ for $0 < x < \delta$, we get $\varphi(x) = y(x) < F(x)$ and

$$\frac{dy}{dx} = \frac{g(x)}{F(x) - y(x)} = \frac{g(x)}{F(x) - \varphi(x)} = \varphi'(x) \quad (> 0).$$

Hence, we have the function $\varphi(x)$ satisfying the conditions [C1] and [C2] for all orbits $\gamma^+(P)$.

We enumerate analogous results in the case that a point P is in the second quadratic. Also the discussion for the case $F(x) < 0$ is essentially the same. So we detail the proofs. □

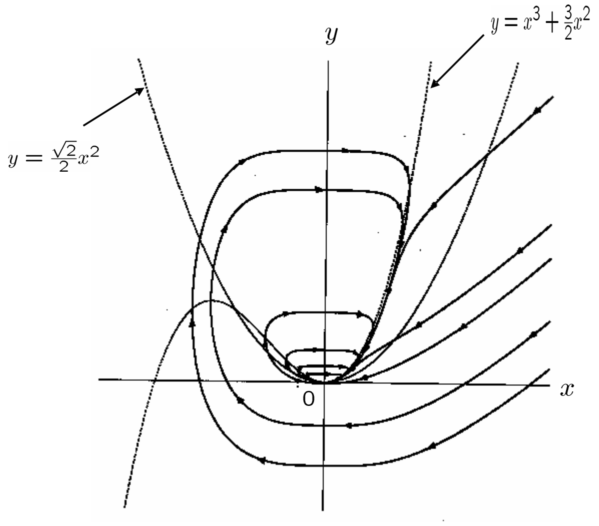

Figure 2: system (L) with $\lambda = 3$ and $k = 1$

3. Application to System (L) with Two Parameters

We will apply the theorem to a concrete system with two parameters. Also, by drawing the phase portraits of the system, it will be shown that the existence of the negative invariant curves of the system is deeply relation to both the existence of the homoclonic orbits and the non-existence of the limit cycle.

Example 2. Consider (L) with $F(x) = (\lambda/3)x^3 + (\lambda/2)x^2$ and $g(x) = kx^3$ $(\lambda, k > 0)$. This system is a generalization of the system($\lambda = 1$) in [1]. Using the theorem, we see easily that the system has homoclinic orbits if and only if $\lambda^2 - 8k > 0$. Then we can take the function $\varphi(x) = \alpha x^2$ with $(\lambda - \sqrt{\lambda^2 - 8k})/4 < \alpha < (\lambda + \sqrt{\lambda^2 - 8k})/4$.

We will concretely consider the case $\lambda = 3$ and $k = 1$ for the above system. Take the supplement function $\varphi(x) = (\sqrt{2}/2)x^2$. Then we have

$$\varphi'(x)[F(x) - \varphi(x)] - g(x) = \sqrt{2}x^3\left(x + \frac{3 - 2\sqrt{2}}{2}\right) \geq 0,$$

for $x > (-3 + 2\sqrt{2})/2$.

Then, the system satisfies the conditions [C1] and [C2] for $x > (-3+2\sqrt{2})/2$. Thus, we conclude from the theorem that the system has homoclinic orbits locally, but no limit cycles. See Figure 2.

From this example, we have the following result for the non-existence of the limit cycles of system (L).

Corollary 3. *If the conditions* [C1] *and* [C2] *hold for* $x_1 < x < +\infty$ $(x_1 < 0)$ *or* $-\infty < x < x_2$ $(x_2 > 0)$, *then system* (L) *has homoclinic orbits locally, but no limit cycles.*

Finally, we want to state on the relation to the result in [4]. By using the transformation $y = z + \varphi(x)$ to system (L), a new Liénard system is given as follows.

$$\begin{cases} \dot{x} = z - E(x), \\ \dot{y} = -\varphi^{'}(x)\{z - H(x)\}, \end{cases}$$

where the functions $E(x)$ and $H(x)$ are defined as $E(x) = F(x) - \varphi(x)$ and $\varphi^{'}(x)H(x) = \varphi^{'}(x)E(x) - g(x)$.

In [4], by using these functions $\varphi(x)$, $E(x)$ and $H(x)$, the non-existence of limit cycles and the existence of heteroclinic orbits for system (L) have been discussed. In view of the results, rewritting the theorem by these function, we have the following

Corollary 4. *Suppose that the function* $\varphi(x)$ *satisfies the condition* [C1]. *Then system* (L) *has homoclinic orbits if and only if* $E(x) > 0$ *and* $H(x) \geqq 0$ *for* $0 < |x| < \delta$. *Also, if* $E(x) > 0$ *and* $H(x) \geqq 0$ *for* $x > 0$ *or* $x < 0$, *then system* (L) *has no limit cycles.*

Remark

A part of these results has been published at the Poster Competition of ICM (Madrid) on August 22-30 of 2006.

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