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EFFECT OF DEPENDENCE OF COMPONENTS ON THE SYSTEM PERFORMANCE

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Abstract: We generally think about engineering system whose component life times are independent and consider its life length, probably the most meaningful effectiveness measure, to measure the system's quality. When there is a question of selection of the best system from a collection of N systems, placed into service, it is quite common to collect the data on their life lengths (or times-to-failure) for the purpose of assessing the overall system performance. In the life-testing experiment, prior to deployment of the system or sale of the system, such data are very useful and critically analyzed. However, the assumption of independence of component lifetimes in the system is a very restrictive assumption, and the components may be dependent (in many ways) rather than being independent. The systems become more complex under the dependent set-up and the dependence of components is difficult to describe even for simple systems. It is essential to study the effect of dependence on system performance for better reliability design and its analysis. It is possible that we might underestimate or overestimate the system performance if we ignore the effect of dependence. In this paper we consider different multi-component coherent systems, and make an attempt to explore the effect of dependence on

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the system performance.

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1. Introduction

In reliability engineering system reliability can be studied at the structural level by the way of establishing a relationship between the system reliability and the reliability of the associated components. We generally consider the engineered systems whose components have independent lifetimes for the purpose of studying system reliabilities and for developing any new theories. However, the assumption of independence of component lifetimes in the system is a very restrictive assumption, and the components in a system may be dependent (in many ways) rather than being independent. The components in the same system may share the same environment or share the same load or may be subject to same set of stresses. This will cause the lifetime random variables to be related to each other, i.e., dependent. In this context two main types of dependence, positive and negative, can be referred to. If the failure of one component causes the failure of another component to become more (less) likely, their nature of dependence is positive (negative). For details on different kinds of dependence Lehmann [3], Barlow and Proschan [1], Roussas [7] may be referred to. The systems become more complex under the dependence set-up, and the dependence of components is difficult to described even for simple systems consisting of more than two components. We might underestimate or overestimate the system performance if we ignore the effect of dependence.

With this background we think that it is very essential to study how the system performance is affected if the component lifetimes are dependent, and how far it is necessary to incorporate the aspect of dependence in the model for better reliability design and its analysis, and how it depends on the lifetime distributions of the components.

Some studies have been found by relaxing the restriction of independent component lifetimes. Shaked [10] considered the positive dependence by mixture and obtained reliability bounds for k-out-of-n system when the components are identical. Garren and Richards [2] discussed the problem of comparing the reliability of a multi-component m-out-of-n system operating inside a laboratory (i.e., operating under a common environment) with the reliability of the same system operating outside the laboratory. Ma [4] considered a system with n non-renewable dependent components sharing a common random environment and discussed the environmental effect on the performance of the system. It has been considered there that the lifetimes random variables are conditionally independent when the 'common environment' under which the components are being operated and which is a certain random variable with parameter, say γ, is given. Navarro et al [5] discussed the problem of comparison among the coherent systems with components having absolutely continuous exchangeable joint lifetime distribution. Later Navarro and Rychlik [6] considered the system with exchangeable components and derived some reliability bounds. The assumption of exchangeability implies that the components of the system have identical distributions, but the functioning (or failing) of one component influence the functioning (or failing) of the other components. For example, the components connected in parallel and that performs the same task, the failing of one of them may lead to overload for the others and shortens the life span of the functioning components. Roychowdhury and Bhattacharya [9] discussed a unified way of comparing the systems of same or different orders and with dependent component lives assuming a special type of dependence of the component lives.

In this paper we consider different multi-component coherent systems, and make an attempt to explore the effect of dependence on the system performance for various component life distributions.

The organization of this paper is as follows. Section 1 introduces the work, while Section 2 describes some basic concepts needed for the present study. Section 3 includes some results to show the influence of dependence on system reliability. Section 4 considers a special type of dependence model under which the effect of dependence on several frequently discussed coherent systems is observed. A comparison of system reliability has been made under dependence and independence set-ups, to study the effect of dependence on the system reliability at different time points, for different component life distributions.

2. Definitions and Notation

Component lives can be dependent on each other in various ways. Here we discuss different forms of dependence. By positive dependence we mean that the larger (smaller) values of one variable tend to accompany with the larger (smaller) values of the other. For positively dependent component lifetimes, the failure of one component causes the failure of another component to become more likely.

The random variables X_1, X_2, ... ,X_n are said to be positively upper orthant dependent (PUOD) if

$$P(X_1 > x_1, X_2 > x_2, ..., X_n > x_n) \geq \prod_{i=1}^{n} P(X_i > x_i),$$

for all values of $(x_1, x_2, \ldots, x_n)$,

and they are called positively lower orthant dependent (PLOD) if

$$P(X_1 \leq x_1, X_2 \leq x_2, \ ..., X_n \leq x_n) \geq \prod_{i=1}^{n} P(X_i \leq x_i),$$

for all values of $(x_1,x_2 \ , \ldots, \ x_n)$.

Let us define a coherent system of n components with component reliability at time t as p_i, $i=$ 1, 2, ... , n. An n-component system is said to be coherent if every component is relevant, i.e., every component has some contribution towards the system performance, and if the system is monotone, i.e., the performance of the system improves with the improvement of any component or a subset of components.

If X_1,X_2 , ..., X_n are the lives of components 1, 2, ..., n of an n-component series system, then the system reliability (at time t) will be

$$P(X_1 > t, X_2 > t, \ldots, X_n > t),$$

and the reliability (at time t) of an n-component parallel system will be

$$P(X_1 > t \text{ or } X_2 > t \text{ or } \ldots \text{ or } X_n > t)$$

or, equivalently,

$$1 - P(X_1 \leq t, X_2 \leq t, \ldots, X_n \leq t).$$

A coherent system can be considered to be a collection of minimal cut sets. A minimal cut set is a minimal set of those components whose failure guarantees system failure. The idea will be elaborated in Section 4.

3. Some Results

Here we discuss some important results which show the effect of dependence on series and parallel systems. The results reveal that if independence of components is assumed while actually the components are dependent, then the system

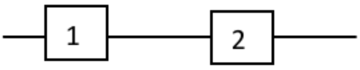

Figure 1: System 1 (series system)

reliability will be either underestimated or overestimated.

Theorem 1. *The reliability of a series system* (*figured as System*1) *with dependent components is higher than a similar system with two independent components, when the component lifetimes are positively dependent* (*positively lower orthant dependent* (*PLOD*) *or positively upper orthant dependent* (*PUOD*)).

Proof. Let X_1 and X_2 be the lives of components 1 and 2, respectively.

Let us denote the event $\{X_i \leq t\}$ by E_i, i =1, 2. For a two-component system with positively lower orthant dependent (PLOD) components,

$$P(X_1 \leq t, X_2 \leq t) \geq P(X_1 \leq t)\ P(X_2 \leq t), \quad \text{for all values of } t,$$

which can equivalently be written as

$$P(E_1 \cap E_2) \geq P(E_1)P(E_2). \tag{1}$$

The reliability of a two-component series system is $P(E_1^c \cap E_2^c)$. Here

$$P(E_1^c \cap E_2^c) = 1 - P(E_1 \cup E_2) = 1 - P(E_1) - P(E_2) + P(E_1 \cap E_2). \tag{2}$$

If the component lives are independent, then the system reliability is

$$P(E_1^c \cap E_2^c) = 1 - P(E_1) - P(E_2) + P(E_1)P(E_2). \tag{3}$$

Because of (1), the expression given in (2) is larger than the one given in (3). Hence the result holds for PLOD components.

Next, let us consider the same system with positively upper orthant dependent (PUOD) components, for which

$$P(X_1 > t, X_2 > t) \geq P(X_1 > t)P(X_2 > t), \quad \text{for all values of } t,$$

or, equivalently,

$$P(E_1^c \cap E_2^c) \geq P(E_1^c)P(E_2^c), \tag{4}$$

Here the right hand side of the above inequality is the reliability of the given system with independent components. Thus the system reliability is shown to be larger in case of PUOD components too. □

Theorem 2. *The reliability of a parallel system (figured as System 2) with dependent components is smaller than the similar system with two independent components, when the component lifetimes are positively dependent*

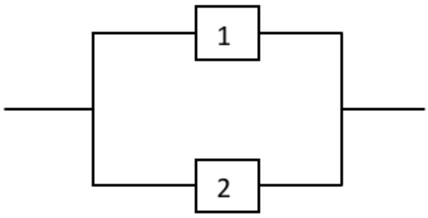

Figure 2: System 2 (parallel system)

(positively lower orthant dependent (PLOD) or positively upper orthant dependent (PUOD)).

Proof. Let X_1 and X_2 be the lives of components 1 and 2, respectively.

Let us denote the event $\{X_i \leq t\}$ by E_i, $i = 1, 2$. For a two-component system with positively lower orthant dependent (PLOD) components

$$P(X_1 \leq t, X_2 \leq t) \geq P(X_1 \leq t)\ P(X_2 \leq t), \text{ for all values of } t,$$

which can equivalently be written as

$$P(E_1 \cap E_2) \geq P(E_1)P(E_2)\ . \tag{5}$$

The reliability of a two-component parallel system is $P(E_1^c \cup E_2^c)$.

Note that

$$P(E_1^c \cup E_2^c) = 1 - P(E_1 \cap E_2). \tag{6}$$

If the component lives are independent, then the system reliability is

$$P(E_1^c \cup E_2^c) = 1 - P(E_1)P(E_2). \tag{7}$$

Thus because of (5), the expression given in (6) is smaller than the one given in (7).

Hence the result holds for PLOD components.

Next, let us consider the same system with positively upper orthant dependent (PUOD) components, for which

$$P(X_1 \geq t, X_2 \geq t) \geq P(X_1 \geq t)P(X_2 \geq t), \text{ for all values of } t,$$

or, equivalently,

$$P(E_1^c \cap E_2^c) \geq P(E_1^c)P(E_2^c)\ . \tag{8}$$

Then we write the reliability of this system as

$$\begin{aligned} P(E_1^c \cup E_2^c) &= P(E_1^c) + P(E_2^c) - P(E_1^c \cap E_2^c) \\ &\leq P(E_1^c) + P(E_2^c) - P(E_1^c)P(E_2^c), \text{ by (8),} \end{aligned}$$

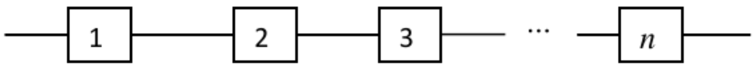

Figure 3: System 3 (n-component series system)

which is the system reliability with independent components.

Thus the result is proved for PUOD components too. □

The above results can be extended to a higher order series or parallel system too.

Theorem 3. *The reliability of a series system (figured as System 3) with n dependent components is higher than its corresponding series system with n independent components, when the component lifetimes are positively upper orthant dependent (PUOD).*

Proof. Let X_i be the life of i-th component, $i = 1, 2, \ldots, n$. For a system with positively upper orthant dependent (PUOD) components,

$$P(X_1 \geq t, X_2 \geq t, \ldots, X_n \geq t) \geq P(X_1 \geq t)P(X_2 \geq t) \ldots P(X_n \geq t), \quad (9)$$

for all values of t.

Let us denote the event $\{X_i \leq t\}$ by E_i, i =1, 2, ..., n. Then (9) can be written as

$$P(E_1^c \cap E_2^c \cap \ldots \cap E_n^c) \geq P(E_1^c)P(E_2^c) \ldots P(E_n^c). \quad (10)$$

The reliability of an $n-$component series system (at time t) is given by

$$P(\bigcap_{i=1}^{n} E_i^c).$$

For an $n-$component series system with PUOD components

$$P(\bigcap_{i=1}^{n} E_i^c) \geq \prod_{i=1}^{n} P(E_i^c),$$

by (10).

The reliability of an $n-$component series system with independent components,

$$P(\bigcap_{i=1}^{n} E_i^c) = \prod_{i=1}^{n} P(E_i^c).$$

Thus the reliability of an $n-$component series system with PUOD components is greater than that of a corresponding series system with independent compo-

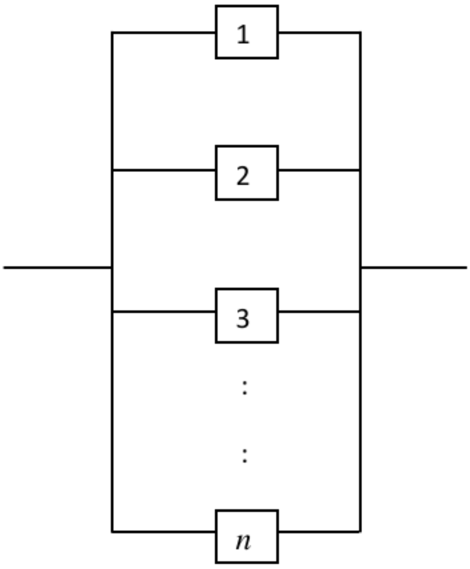

Figure 4: System 4 (n-component parallel system)

nents. □

Theorem 4. *The reliability of a parallel system (figured as System 4) with n dependent components is smaller than its corresponding parallel system with n independent components, when the component lifetimes are positively lower orthant dependent (PLOD).*

Proof. Let X_i be the life of i-th component, $i = 1, 2, \ldots, n$. For a system with positively lower orthant dependent (PLOD) components,

$$P(X_1 \leq t, X_2 \leq t, \ldots, X_n \leq t) \geq P(X_1 \leq t)\ P(X_2 \leq t)\ \ldots\ P(X_n \leq t), \tag{11}$$

for all values of t.

As earlier, let us denote the event $\{X_i \leq t\}$ by E_i, i =1, 2, $\ldots$, n. Then (11) can be written as

$$P(E_1 \cap E_2 \cap \ldots \cap E_n) \geq P(E_1)P(E_2)\ \ldots\ P(E_n). \tag{12}$$

The reliability of an $n-$component parallel system (at time t) is given by

$$P(\bigcup_{i=1}^{n} E_i^c),$$

which is equivalent to $P(\bigcap_{i=1}^{n} E_i)^c = 1 - P(\bigcap_{i=1}^{n} E_i)$.

For an $n-$component parallel system with PLOD components,

$$P(\bigcap_{i=1}^{n} E_i) \geq \prod_{i=1}^{n} P(E_i),$$

by (12).

The reliability of an $n-$component parallel system with independent components,

$$P(\bigcup_{i=1}^{n} E_i^c) = P(\bigcap_{i=1}^{n} E_i)^c = 1 - P(\bigcap_{i=1}^{n} E_i) = 1 - \prod_{i=1}^{n} P(E_i).$$

Thus the reliability of an $n-$component parallel system with PLOD components,

$$P(\bigcup_{i=1}^{n} E_i^c) = P(\bigcap_{i=1}^{n} E_i)^c = 1 - P(\bigcap_{i=1}^{n} E_i) \leq \ 1 - \prod_{i=1}^{n} P(E_i),$$

which is the reliability of an $n-$component parallel system with independent components.

Hence the result holds true. □

4. Determination of System Reliability Using Most Likely Minimal Cut Set

Here we make an attempt to determine the system life as well as the failure probability of a system using the idea of cut set representation of a coherent system. A stochastic ordering criterion has been used to determine the most likely value of system life, based on the said representation.

A system life can be determined from its component lives. For this, we need to represent the system as a combination of a number of sets of components, known as minimal cut sets. A set of components in the system whose failure causes the system to fail is called a cut set of the system. A cut set is said to be minimal if the set cannot be reduced without losing its status as a cut set. A minimal cut set (MCS) of a system is a minimal set of components whose failing causes the system to fail.

The number of different basic components in an MCS is the size or cardinality of the MCS.

Let X_1,X_2 , $\ldots$, X_n be the random lives of the components of an n-

component system which has m minimal cut sets, K_1,K_2 , ..., K_m of sizes N_1,N_2 , ..., N_m, respectively. $\sum_{i=1}^{m} N_i \geq n$. The system life is, then, given by

$$T = \min_{1 \leq i \leq n} \max_{j \in K_i} X_j = \min_{1 \leq i \leq n} Y_i, \tag{13}$$

where $Y_i = \max_{j \in K_i} X_j$ is the maximum of the component lives of the i-th MCS. Let us call it the life of the i-th MCS, since the component having maximum life among the lives of all components in the i-th MCS, by failing, will cause the system to fail. The system life, as given in (13), will be the minimum of the lives of all minimal cut sets.

A minimal cut set will be called the most likely MCS of the system if the chance of its life to be the same as the system life is higher than that of any other minimal cut sets (Roychowdhury [8]). In other words, the maximum component life, Y_i of the i-th MCS corresponds to the most likely MCS if its chance of being least among all such Y_i-values is maximum, i.e., if

$$P(Y_i \leq c) \geq P(Y_h \leq c), \text{ for all } c \text{ and all } h \neq i,\ i,\ h = 1,\ 2,...,m. \tag{14}$$

From (14) we can say that the i-th minimal cut set will be the most likely minimal cut set of the system if its life is stochastically smaller than the life of the any other minimal cut set of the system. Thus the failure probability, i.e., the probability that the system fails before time c is same as $P(Y_i \leq c)$, if Y_i, the maximum component life of the i-th MCS, is the most likely MCS. Hence the system reliability at time c is obtained as $P(Y_i > c)$.

5. Effect of Dependence on System Life and System Reliability

Here we have considered the following positive dependence model. Suppose X_1,X_2, ..., X_n are the random lifetimes of the components of an n-component coherent system. Whenever a component fails, the remaining lives of the other components reduce by a random amount u, where u follows a uniform distribution with parameters (α, 1), $0 < \alpha < 1$.

Under this positive dependence model we observe how the system life and the system reliability change from what they would have been in case of independent component lifetimes. Let us consider the following systems:

We wish to observe the effect of such dependence on the system reliabilities for the systems pictured in Figures 5, 6, and 7.

The minimal cut sets of Systems 5, 6, 7 are as follows:

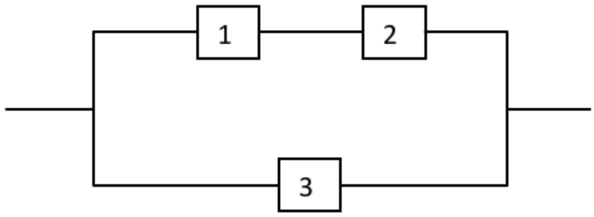

Figure 5: System 5 (parallel-series system)

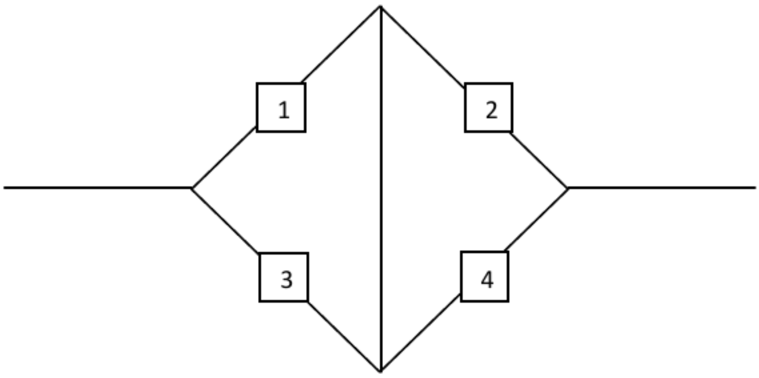

Figure 6: System 6

System 5: $K_1 = \{1, 3\}$, $K_2 = \{2, 3\}$.

System 6: $K_1 = \{1, 3\}$, $K_2 = \{2, 4\}$.

System 7: $K_1 = \{1, 3\}$, $K_2 = \{1, 4\}$, $K_3 = \{2, 3\}$, $K_4 = \{2, 4\}$.

By (13), their system lives are obtained as:

System 5: $T_1 = \min(Y_1, Y_2)$, where $Y_1 = \max(X_1, X_3)$, $Y_2 = \max(X_2, X_3)$.

System 6: $T_2 = \min(Y_1, Y_2)$, where $Y_1 = \max(X_1, X_3)$, $Y_2 = \max(X_2, X_4)$.

System 7: $T_3 = \min(Y_1, Y_2, Y_3, Y_4)$, where $Y_1 = \max(X_1, X_3)$, $Y_2 = \max(X_1, X_4)$, $Y_3 = \max(X_2, X_3)$, $Y_4 = \max(X_2, X_4)$.

5.1. Numerical Examples

Let us assume that the component lifetimes follow exponential distribution with parameter $\lambda = 2$. Whenever one component fails, the remaining life of the other components reduces by a random variable u, which follows a uniform distribution with parameters ($\alpha = 0.85$, 1). The following example shows how the system life changes for Systems 5, 6, 7 because of such dependence of the component lifetimes.

System 5: Let (1.3469492, 0.3734129, 1.0273471) be a random sample of size n= 3 from an exp($\lambda = 2$) distribution for the component lives of System 5. Thus $X_1 = 1.3469492$, $X_2 = 0.3734129$,$X_3 = 1.0273471$.

If the component lifetimes are independent, then by (13), the system life will be $T = \min(Y_1, Y_2) = 1.0273471$, where $Y_1 = \max(X_1, X_3) = 1.3469492$, $Y_2 = \max(X_2, X_3)$ =1.0273471, since the minimal cut sets of System 5 are {1, 3}, {2, 3}.

Now let us see what happens if the components are dependent.

Let (0.8592245, 0.9193445, 0.8971375) be a random sample drawn from a uniform (0.85, 1)-distribution. These are the amounts by which the remaining life of the three components reduces, if one component fails. Naturally there will be no effect of this value on the remaining life of the component that fails.

Here at first component 2 fails at time t= 0.3734129. Then the remaining life of the components will be

$l_1 = 0.8592245 \times (1.3469492 - 0.3734129) = 0.836486$,

$l_2 = 0.9193445 \times (0.3734129 - 0.3734129) = 0$,

$l_3 = 0.8971375 \times (1.0273471 - 0.3734129) = 0.586669$.

Thus the component lives are

new $X_1 = 0.3734129 + 0.8592245 \times (1.3469492 - 0.3734129) = 0.3734129 + 0.836486 = 1.209899$,

new $X_2 = 0.3734129 + 0.9193445 \times (0.3734129 - 0.3734129) = 0.3734129 + 0 = 0.3734129$,

new $X_3 = 0.3734129 + 0.8971375 \times (1.0273471 - 0.3734129) = 0.3734129 + 0.586669 = 0.960082$.

Component 3 fails next, at time $t = 0.3734129 + 0.586669 = 0.960082$, and with the failure of this component the system will also fail.

Thus, in dependent case, finally we have the component lives as $X_1 = 1.209899$, $X_2 = 0.3734129$, $X_3 = 0.960082$. Hence the system life will be $T_1 = 0.960082$, which is less than $T = 1.0273471$, the system life when the component lives are independent.

Note that new Y_1 = max(new X_1, new X_3) = 1.209899, new Y_2 = max(new

Component life distribution	System	Dependent component lives				Inependent component lives				% decrease in system life
		Mean T	s.d.(T)	95% confidence limit for system life		Mean T'	s.d.(T')	95% confidence limit for system life		
				Lower limit	Upper limit			Lower limit	Upper limit	
Exponential (λ=2)	System 5	0.7128	0.5773	0.5997	0.8260	0.7737	0.6249	0.6512	0.8962	7.87
	System 6	0.4786	0.2532	0.4290	0.5282	0.5180	0.2919	0.4608	0.5752	7.61
	System 7	0.2293	0.0913	0.2114	0.2472	0.2445	0.0988	0.2251	0.2368	6.22
Lognormal (μ = 2, σ =1)	System 5	14.4029	20.6163	10.3621	18.4437	15.3158	22.3465	10.9359	19.6957	5.96
	System 6	10.8752	7.6100	9.3837	12.3668	11.8559	8.7071	10.1493	13.5625	8.27
	System 7	7.9142	4.6602	7.0008	8.8276	8.3260	4.9096	7.3637	9.2883	4.97
Weibull (shape =5, scale = 3)	System 5	2.6255	0.4848	2.5304	2.7205	2.7470	0.6061	2.6282	2.8658	4.42
	System 6	2.8123	0.3642	2.7409	2.8836	2.8883	0.3954	2.8108	2.9658	2.63
	System 7	2.6383	0.3055	2.5785	2.6982	2.6910	0.3342	2.6255	2.7565	1.96
Gamma (shape = 5, scale = 3)	System 5	1.7302	0.3739	1.6570	1.8035	1.7815	0.3909	1.7049	1.8582	2.88
	System 6	1.4588	0.3921	1.3820	1.5357	1.5165	0.4517	1.4279	1.6050	3.80
	System 7	1.4116	0.4968	1.3139	1.5094	1.4645	0.5509	1.3565	1.5725	3.61
Beta (shape1= 0.2, shape2= 0.03)	System 5	0.8661	0.2302	0.8210	0.9112	0.8962	0.2199	0.8531	0.9393	3.36
	System 6	0.8959	0.1743	0.8617	0.9301	0.9488	0.1595	0.9175	0.9800	5.58
	System 7	0.8265	0.2895	0.7698	0.8832	0.8921	0.3086	0.8316	0.9525	7.35

Table 1: Simulation result showing the change in system life due to dependence

X_2, new X_3) $= 0.960082$. Hence, by (13), the system life for dependent case is $T_1 = \min$ (new Y_1, new Y_2) $= 0.960082$.

Hence the system life decreases by 6.547456 %, because of the dependence of the components.

System 6: Let (0.02705882, 0.52411065, 0.52855343, 0.93733562) be a random sample of size $n = 4$ from an $\exp(\lambda = 2)$ distribution for the component

c		*System* 5	*System* 6	*System* 7
0.001	r_1	0.9994	0.9989	0.9989
	r_2	0.9995	0.9989	0.9989
	loss in reliability (%)	0.01	0	0
0.1	r_1	0.9351	0.9315	0.8809
	r_2	0.9406	0.9381	0.8889
	loss in reliability (%)	0.58	0.70	0.90
0.5	r_1	0.4178	0.3155	0.2142
	r_2	0.4462	0.3656	0.2468
	loss in reliability (%)	6.36	13.70	13.21
0.7	r_1	0.2607	0.1439	0.0935
	r_2	0.2932	0.1822	0.1203
	loss in reliability (%)	11.08	21.02	22.28
1.0	r_1	0.1255	0.0436	0.0246
	r_2	0.1507	0.062	0.0357
	loss in reliability (%)	16.72	29.68	31.09
1.5	r_1	0.0338	0.0042	0.0025
	r_2	0.0494	0.0086	0.0044
	loss in reliability (%)	31.58	51.16	43.18

Table 2: Comparison of reliabilities of systems with dependent and independent exponential component lives

lives of System 6. We can write $X_1 = 0.02705882$, $X_2 = 0.52411065$, $X_3 = 0.52855343$, $X_4 = 0.93733562$.

If the components are independent, then, by (13), the system life will be $T = \min(Y_1, Y_2) = 0.52855343$, where $Y_1 = \max(X_1, X_3) = 0.52855343$, $Y_2 = \max(X_2, X_4) =$0.93733562, since the minimal cut sets of System 6 are {1, 3} and {2, 4}.

Now let us suppose the component lives to be dependent. Here component 1 will fail, at first, at the time $t = 0.5285534$.

We draw a random sample of size 4 from a uniform (0.85, 1)-distribution. The sample is (0.9331286, 0.8754590, 0.9392228, 0.8583949). Because of the failure of one component, due to dependence, the component lives will reduce. As earlier, the remaining life of the components are obtained as $l_1 = 0$, $l_2 = 0.435148$, $l_3 = 0.471015$, $l_4 = 0.781377$. Then the components lives will be $new\ X_1 = 0.02705882$, $new\ X_2 = 0.462207$, $new\ X_3 = 0.498074$, $new\ X_4 =$

c		*System* 5	*System* 6	*System* 7
0.7	r_1	0.9994	0.9999	0.9998
	r_2	0.9994	0.9999	0.9998
	loss in reliability (%)	0	0	0
1.0	r_1	0.9985	0.9992	0.9985
	r_2	0.9987	0.9992	0.9986
	loss in reliability (%)	0.02	0	0.01
2.0	r_1	0.9818	0.9792	0.9645
	r_2	0.9938	0.9817	0.9666
	loss in reliability (%)	0.20	0.25	0.22
3.0	r_1	0.9341	0.9290	0.8750
	r_2	0.9392	0.9356	0.8850
	loss in reliability (%)	0.54	0.71	1.13
4.0	r_1	0.8587	0.8506	0.7610
	r_2	0.8687	0.8660	0.7777
	loss in reliability (%)	1.15	1.78	2.15
7.0	r_1	0.6293	0.5556	0.4331
	r_2	0.6515	0.5926	0.4623
	loss in reliability (%)	3.41	6.24	6.32
9.0	r_1	0.5002	0.3981	0.2970
	r_2	0.5273	0.4374	0.3252
	loss in reliability (%)	5.14	8.98	8.67
12.0	r_1	0.3535	0.2336	0.1626
	r_2	0.3821	0.2749	0.1872
	loss in reliability (%)	7.84	15.02	13.14

Table 3: Comparison of reliabilities of systems with dependent and independent lognormal component lives

0.808436.

Next fails component 2, since its life is minimum among the remaining ones, i.e., components 2, 3 and 4. It fails at time $t = 0.462207$.

Because of the failure of component 2, the lives of the remaining components will reduce further, due to the dependence. Each of the remaining lives of the components 2, 3, 4 will reduce by a random amount. Let (0.9183803, 0.9470116, 0.9456199) be a random sample from a uniform (0.85, 1)-distribution. Actually the first observation does not have any effect of the remaining life of the com-

c		*System* 5	*System* 6	*System* 7
1.0	r_1	0.9998	0.9999	0.9999
	r_2	0.9998	0.9999	0.9999
	loss in reliability (%)	0	0	0
1.5	r_1	0.9968	0.9989	0.9969
	r_2	0.9971	0.9999	0.9973
	loss in reliability (%)	0.03	0.02	0.04
2.0	r_1	0.9668	0.9655	0.9428
	r_2	0.9712	0.9719	0.9489
	loss in reliability (%)	0.45	0.66	0.64
2.5	r_1	0.7945	0.7591	0.6599
	r_2	0.8183	0.7966	0.6951
	loss in reliability (%)	2.91	4.71	5.06
2.7	r_1	0.6529	0.5784	0.4736
	r_2	0.6913	0.6407	0.5247
	loss in reliability (%)	5.55	9.72	9.73
2.8	r_1	0.5764	0.4755	0.3645
	r_2	0.6154	0.5462	0.4211
	loss in reliability (%)	6.34	12.94	13.44
2.9	r_1	0.4900	0.3774	0.2769
	r_2	0.5376	0.4536	0.3341
	loss in reliability (%)	8.85	16.80	17.12
3.0	r_1	0.3971	0.2902	0.2061
	r_2	0.4517	0.3711	0.2573
	loss in reliability (%)	12.09	21.80	19.90

Table 4: Comparison of reliabilities of systems with dependent and independent Weibull component lives

ponent 2, as it has already failed. The remaining lives of the components are $l_2 = 0$, $l_3 = 0.033966$, $l_4 = 0.327401$. Hence the lives of all components become *new* $X_1 = 0.02705882$, *new* $X_2 = 0.462207$, *new* $X_3 = 0.496173$, *new* $X_4 = 0.789608$. Note that the first two values will not change further, as they have already failed.

Now, between the lives of components 3 and 4, life of component 3 is smaller. Therefore this component will fail next, at the time t= 0.496173, and with the failure of component 3, the system will also fail. Hence the system life will be

T_2 =0.496173, which is less than T =0.52855343, the system life that would have been if the components were independent. Note that here, by (13) also, the system life is found to be $T_2 = \min(Y_1, Y_2) = 0.496173$, where $Y_1 = \max(new\ X_1, new\ X_3) = 0.496173$, $Y_2 = \max(new\ X_2, new\ X_4) = 0.789608$.

Here the system life will decrease by 6.126236 %, because of the dependence of the component.

System 7: Let (0.02769449, 0.17492380, 1.58672295, 0.12025870) be a random sample of size n= 4 from an exp($\lambda = 2$) distribution for the component lives of System 7. Let us write $X_1 = 0.02769449$, $X_2 = 0.17492380$, $X_3 = 1.58672295$, $X_4 = 0.12025870$.

The minimal cut sets of System 7 are {1, 3}, {1, 4}, {2, 3} and {2, 4}.

If the component lives are independent, the system life, by (13), will be $T = \min(Y_1, Y_2, Y_3, Y_4) = 0.12025870$, where $Y_1 = \max(X_1, X_3) = 1.58672295$, $Y_2 = \max(X_1, X_4) = 0.12025870$, $Y_3 = \max(X_2, X_3) = 1.58672295$, $Y_4 = \max(X_2, X_4) = 0.17492380$.

Now suppose the components are dependent, and if one fails, the remaining lives of the other will also reduce.

Here component 1 fails first. It fails at time t= 0.02769449. Then the remaining life of the other components will reduce by a random quantity which follows a uniform (0.85, 1)-distribution.

Let (0.9992983, 0.8580538, 0.9106982, 0.8868552) be a random sample from uniform (0.85, 1).

The remaining lives of the components are $l_1 = 0$, $l_2 = 0.126331$, $l_3 = 1.419804$, $l_4 = 0.082091$.

Then the components lives will be $new\ X_1 = 0.02769449$, $new\ X_2 = 0.154025$, $new\ X_3 = 1.447498$, $new\ X_4 = 0.10979$.

Next fails component 4, since it is minimum among the lives of components 2, 3 and 4. Component 1 has already failed. With the failure of component 4 the system will also fail. The system life is then $T_3 = \min(Y_1, Y_2, Y_3, Y_4) = 0.10979$, which is less than $T = 0.12025870$, where $Y_1 = \max(new\ X_1, new\ X_3) = 1.447498$, $Y_2 = \max(new\ X_1, new\ X_4) = 0.10979$, $Y_3 = \max(new\ X_2, new\ X_3) = 1.447498$, $Y_4 = \max(new\ X_2, new\ X_4) = 0.154025$. Here the system life decreases by 8.70515% due to dependence.

Next we carry out a simulation to observe the change in system reliability due to dependence of component lifetimes. We study the effect of different component life distributions on the change in system life and system reliability

c		*System* 5	*System* 6	*System* 7
0.4	r_1	0.9999	0.9997	0.9996
	r_2	0.9999	0.9997	0.9996
	loss in reliability (%)	0	0	0
0.5	r_1	0.9998	0.9991	0.9986
	r_2	0.9999	0.9992	0.9983
	loss in reliability (%)	0.01	0.01	0.03
1.0	r_1	0.9294	0.9257	0.8757
	r_2	0.9361	0.9367	0.8893
	loss in reliability (%)	0.72	1.17	1.53
1.5	r_1	0.6363	0.5556	0.4396
	r_2	0.6682	0.6063	0.4766
	loss in reliability (%)	4.77	8.36	7.76
1.8	r_1	0.4198	0.3120	0.2213
	r_2	0.4572	0.3733	0.2596
	loss in reliability (%)	8.18	16.42	14.75
2.0	r_1	0.2985	0.1835	0.1202
	r_2	0.3423	0.2380	0.1546
	loss in reliability (%)	12.80	22.90	22.25
3.0	r_1	0.0397	0.0042	0.0030
	r_2	0.0577	0.0108	0.0063
	loss in reliability (%)	31.20	61.11	52.38

Table 5: Comparison of reliabilities of systems with dependent and independent gamma component lives

as well, due to dependence of component lives.

5.2. Simulation Results Showing the Effect of Dependence on System Life and System Reliability

In this section we carry out a simulation to see the effect of dependence on the system life and system reliability for different component life distributions. Depending upon the nature of failure rate curve, the appropriate life time distribution can be chosen. Failure rate is constant in time for exponential life distribution, while for the lognormal life distribution the failure rate curve starts at zero, then go up to its peak and finally moves toward zero, asymptotically.

c		*System* 5	*System* 6	*System* 7
0.001	r_1	0.9979	0.9976	0.9962
	r_2	0.9979	0.9976	0.9963
	loss in reliability (%)	0	0	0.01
0.01	r_1	0.9954	0.9939	0.9896
	r_2	0.9958	0.9939	0.9897
	loss in reliability (%)	0.04	0	0.01
0.1	r_1	0.9861	0.9859	0.9739
	r_2	0.9865	0.9962	0.9749
	loss in reliability (%)	0.04	0.03	0.10
0.2	r_1	0.9809	0.9768	0.9640
	r_2	0.9812	0.9773	0.9646
	loss in reliability (%)	0.03	0.05	0.06
0.5	r_1	0.9640	0.9649	0.9400
	r_2	0.9664	0.9674	0.9428
	loss in reliability (%)	0.25	0.26	0.30
0.7	r_1	0.9594	0.9529	0.9190
	r_2	0.9623	0.9578	0.9240
	loss in reliability (%)	0.30	0.51	0.54

Table 6: Comparison of reliabilities of systems with dependent and independent beta component lives

It is commonly seen in semiconductor failure mechanism. For a Weibull life distribution, the failure rate curve is monotonically increasing (decreasing) for its shape parameter value greater (smaller) than unity. At the early life (wear-out) phase, when failure rate decreases (increases) with time, a Weibull distribution with shape parameter value less (more) than unity will be the most appropriate distribution. At steady state phase, a Weibull distribution with shape parameter unity will be appropriate. Actually a Weibull distribution can have different shapes based on the value of its shape parameter. We can get an exponential, a beta or a gamma distribution from a Weibull distribution for its different parameter values. Here we observe how these life distributions influence the variation in change in system life and system reliability for different system designs.

Let us first carry out 10^4 simulation runs to see the effect of dependence on the system life for different component life distributions. Table 1 displays the

c		*System* 5	*System* 6	*System* 7
0.8	r_1	0.9457	0.9299	0.8993
	r_2	0.9527	0.9453	0.9134
	loss in reliability (%)	0.73	1.63	1.54
0.9	r_1	0.8407	0.7792	0.7322
	r_2	0.9442	0.9367	0.8986
	loss in reliability (%)	10.96	16.81	18.52
0.95	r_1	0.7148	0.6114	0.5762
	r_2	0.9270	0.9237	0.8694
	loss in reliability (%)	22.89	33.81	33.72
0.99	r_1	0.5721	0.4587	0.4447
	r_2	0.9013	0.8982	0.8298
	loss in reliability (%)	36.52	48.93	46.41
0.999	r_1	0.4555	0.3461	0.3396
	r_2	0.8572	0.8442	0.7532
	loss in reliability (%)	46.86	59.00	54.91
0.9999	r_1	0.3716	0.2571	0.2510
	r_2	0.8169	0.7854	0.6911
	loss in reliability (%)	54.51	67.27	63.68

Table 7: Continuation

result. In Table 1 we can see that the change in system life is not same for all distributions. It is negligible for some distributions, and significant for some. The extent of this change depends on the system design too.

Next we perform a simulation with 10^4 runs to observe how the system reliability changes with time and with the component life distributions. Here we estimate the system reliability for independent component lives and for dependent component lives, and determine the change (decrease) in reliability due to dependence of the components. Here we consider three system designs. The simulation has been performed for different component life distributions. We observe how the system designs or the component life distributions affect the change. We compare the system reliability in the phase of dependence and independence of component lives supposing r_1 and r_2 to be the system reliability, when the components are dependent and independent, respectively.

Now we carry out the simulations assuming component lives to follow different distributions, viz., exponential, lognormal, Weibull, gamma and beta. In each case we perform a simulation with 10^4 runs to observe how dependence of components reduces the system reliability and how the percent loss varies with the variation in time (c) for different system designs. In the course of our study, we have considered an exponential distribution with parameter $\lambda = 2$, i.e., mean $= 0.5$, a lognormal distribution with parameters μ and σ, where μ is the mean of logarithm of the component life and σ is the standard deviation of logarithm of component life, a Weibull distribution with shape parameter $=$ 5 and scale $= 3$, a gamma distribution with shape parameter $= 5$ and scale $=$ 3 and a beta distribution with two shape parameters, shape1$= 0.2$ and shape2 $= 0.03$. The results we have obtained are displayed in Tables 2-6.

In all of the cases we can see that the percentage of loss due to dependence of component lives increases with time. Initially the loss is not that significant. But as the time passes, the percentage of loss rises. Its extent depends on the system design, as well as the distribution of the component lives.

6. Conclusion and Discussion

This paper studied the effect of dependence of components on system life and system reliabilities for different system designs and different component life distributions. If the component lifetimes are assumed to be independent, while actually they are dependent, then system reliability will be under- or over-estimated, depending on the system designs. The extent of the change in system reliability or system life depends on the component life distribution and the system design as well. It has been seen that the difference in system reliabilities in dependence and independence set-ups is not much initially, when the system starts functioning. But as the time passes, the difference in system reliabilities in the two set-ups gets larger.

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