A NOTE ON UPPER BOUND FOR $d$-COVERED TRIANGULATION OF CLOSED SURFACES

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Abstract: A triangulation of a closed surface is said to be $d$-covered if at least one vertex in each edge has degree $d$. In [7] the authors have shown that: (a) if a closed surface of non-positive Euler characteristic $\chi$ admits a $d$-covered triangulation then $d \leq 2 \left\lfloor \frac{5 + \sqrt{49 - 24\chi}}{2} \right\rfloor$, and (b) the upper bound for $d$ is attained when: (i) $\chi = 0$ and for (ii) $d = 2(n - 1)$, whenever $n = \frac{7 + \sqrt{49 - 24\chi}}{2}$ is a positive integer. In this note we show that the upper bound is attained for $\chi \in S := \{-2, \ldots, -8\} \cup \{-13, -14, -15\}$.

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1. Introduction and Definitions

A closed surface $F$, is a connected, compact two dimensional manifold without boundary. Let $G$ denote a finite graph which has no loops or multiple edges and each vertex of $G$ has valence $\geq 3$ (see [1]). An embedding of $G$ in $F$ such that closure of each component of $F \setminus G$ is a closed 2-cell is called a map on $F$. When each of these closed 2-cells is a triangle such that a non-empty intersection of any two of these triangles is either a common edge or a common vertex, the map is said to be a triangulation of $F$. A triangulation of $F$ in which exactly $q$ triangles meet at each vertex is called a $q$-equivelar triangulation. The number of edges incident at a vertex is called the degree of the vertex. The surface of Icosahedron...
is a 5-equivelar triangulation of the 2-sphere. The \(\{3,6\}\) tessellation of the Euclidean plane is an example of 6–equivelar triangulation. In [2], 6–equivelar triangulations of the torus and Klein bottle on few vertices are presented. To a triangulation of \(F\), we associate an integer called its Euler characteristic as follows: let \(n\) denote the number of vertices, \(e\) the number of edges and \(f\) the number of triangles in the triangulation. Then \(\chi(F) = n - e + f\) is an integer, known as Euler characteristic of the triangulation (see [6] for details).

In an equivelar triangulation of a closed surface, consider the set of all triangles containing a given vertex \(v\). Let \(e\) denote those edges in these triangles which are not incident with \(v\) and denote this set by \(\text{lk}(v)\). Let \(|\text{lk}(v)|\) denote the cycle obtained by joining common vertices of the edges in \(\text{lk}(v)\). It is easy to see that \(|\text{lk}(v)|\) is a polygon called the link of \(v\). Similarly, let \(\text{st}(v)\) denote the set of all triangles containing \(v\) and \(|\text{st}(v)|\) denote the 2-cell formed by joining triangles in \(\text{st}(v)\) along their common edges. Then \(|\text{st}(v)|\) is called the star of \(v\). It is easy to see that \(|\text{st}(v)|\) is a closed topological 2-disc with polygonal boundary \(|\text{lk}(v)|\). If the triangulation is \(q\)-equivelar then \(|\text{lk}(v)|\) is a \(q\)-gon for each vertex \(v\).

Given an integer \(\chi\), consider the set \(S_\chi = \{(n,d) | \chi = n - \frac{nd}{2} + \frac{nd}{3}, d \leq (n-1)\}\). A pair \((n,d) \in S_\chi\) is called an admissible pair for \(\chi\). If \(d = 7\) and \(n = -6\chi\) then \(S_\chi \neq \emptyset\) for all integers \(\chi < -1\). Let \(F\) be a closed surface such that \(S_{\chi(F)} \neq \emptyset\) and \((n,d) \in S_{\chi(F)}\) be an admissible pair for \(\chi(F)\). It is easy to see that \((12,7) \in S_{-2}\), \((9,8) \in S_{-3}\), \((12,8) \in S_{-4}\), \((10,9) \in S_{-5}\), \((12,9) \in S_{-6}\), \((14,9) \in S_{-7}\), \((12,10) \in S_{-8}\), \((13,12) \in S_{-13}\), \((14,12) \in S_{-14}\) and \((15,12) \in S_{-15}\). Various results about equivelar triangulations of surfaces presented in [4] and [9] lead to the following:

**Proposition 1.** For each \(\chi \in \{-2, \ldots, -8, -13, -14, -15\}\) there exist equivelar triangulations of closed surfaces of Euler characteristic \(\chi\). These triangulations correspond to the pairs \((n,d) = (12,7), (9,8), (12,8), (10,9), (12,9), (14,9), (12,10), (13,12), (14,12)\) and \((15,12)\) respectively.

The authors have, in fact, listed the number of equivelar triangulations of surfaces with up to 12 vertices (in [9], pp. 12) and vertex transitive triangulations of closed surfaces with \(\leq 15\) vertices (in [4], pp. 45-46). These and much more are also available in tabular form from [11]. In [3], 7-equivelar triangulations of double torus on 12 vertices have been constructed.

A triangulation of a closed surface \(F\) is said to be \(d\)-covered (see [7]), if at least one vertex in each edge has degree \(d\). In other words, we say that edges are covered by vertices of degree \(d\).

For \(\chi \leq 0\), in [7], the authors have shown following:
Proposition 2. If a closed surface $F$ admits a $d$-covered triangulation then $d \leq 2 \left\lfloor \frac{5+\sqrt{49-24\chi(F)}}{2} \right\rfloor$.

The authors have also shown that the upper bound in Proposition 2 is attained when (i) $\chi = 0$ and when (ii) $n = \frac{7+\sqrt{49-24\chi(F)}}{2}$ is a positive integer and $F$ is not the Klein bottle. The second case is established by considering embeddings of the complete graph $K_n$ in $F$, see [8]. Here we show:

Theorem 3. Let $F$ be a closed surface with Euler characteristic $\chi(F) \in S = \{-2, \ldots, -8, -13, -14, -15\}$. Then there exists $d = 2 \left\lfloor \frac{5+\sqrt{49-24\chi(F)}}{2} \right\rfloor$-covered triangulation of $F$.

2. Results

Lemma 4. Let $S$ be as in Theorem 3 and $F$ be a closed surface with $\chi(F) \in S$ and $q = \left\lfloor \frac{5+\sqrt{49-24\chi(F)}}{2} \right\rfloor$. Then, there exists $n = \frac{6\chi(F)}{6-\frac{5+\sqrt{49-24\chi(F)}}{2}}$ vertex $q$-equivelar triangulation of $F$.

Proof. The proof of lemma follows by observing that the Euler characteristic relation $\chi(F) = n - \frac{3nq}{2} + \frac{q}{3}$ for $F$ can be re-written as $\chi(F) = n \left(\frac{6-q}{6}\right)$. Then, for the given values of $\chi$, i.e., for $\chi \in S$, we have $(n,q) \in \{(12,7), (9,8), (12,8), (10,9), (12,9), (14,9), (12,10), (13,12), (14,12), (15,12)\}$. The existence of triangulation now follows by Proposition 1.

Proof. of Theorem 3. Given $\chi \in S$, consider a $q = \left\lfloor \frac{5+\sqrt{49-24\chi}}{2} \right\rfloor$-equivelar triangulation of a closed surface of Euler characteristic $\chi$. Following [7], subdivide each triangle of this triangulation into three by introducing a vertex of degree 3 (as shown in example below). Note that the value of $\chi$ does not change in this subdivision. Then, the resulting triangulation is $d = 2q = 2 \left\lfloor \frac{5+\sqrt{49-24\chi}}{2} \right\rfloor$ covered.

3. Example

Example. In [3], classification of equivelar triangulations of double torus is presented. In this case $S_{-2} = \{(12,7)\}$, i.e. $(n,d) = (12,7)$ and $\chi = -2$. The figure below shows $|st(v)|$, star of a vertex $v$ in these triangulations before and after subdivision see Figure 1.
Remark 5. The authors in [7] have also shown that $d$ covered triangulations exists for all $\chi > 0$ and that the upper bound is attained for both the values $\chi = 1$ and 2.

Remark 6. The case of $\chi = -10$ has been covered in [7]. After some simple calculations one observes that for $\chi = -1, -9, -11$ and $-12$ the values of $q = 6, 10, 11$ and 11 respectively. For these values of $q$, $n = \frac{6\chi(F)}{6 - \left[5 + \sqrt{25 - 24\chi(F)}\right]}$ is not an integer. So, the observations and method used in this article do not work for these cases. On the other hand, for $\chi \leq -16$ the values of $d \geq 12$. In [5] existence of such triangulations is shown for $-16 \leq \chi \leq -127$ and whenever $n = \frac{6\chi}{(6-d)}$ is a positive integer. In [10] 12-covered triangulations on surface of Euler characteristic $-1$ has been constructed.

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References

A NOTE ON UPPER BOUND FOR $d$-COVERED... 5


6