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COMPUTING SIMPLE ROOTS BY A SIXTH-ORDER ITERATIVE METHOD

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Abstract: This paper studies a novel without memory sixth-order method for computing simple roots of nonlinear scalar equations. Using the well-known technique of un-determined coefficients, we derive an iterative scheme which includes two evaluations of the function and two evaluations of the first derivative per full cycle. Numerical comparisons are made to reveal the efficiency of the developed method.

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1. Preamble

For more than three centuries, the quadratically Newton's method $x_{n+1} = x_n - f(x_n)/f'(x_n)$ has remained the most common iterative technique for finding approximate solutions to the nonlinear equations of the general form $f(x) = 0$. Third-order iterative schemes, like Halley's method (see [4], [10]), despite its cubic convergence, are considered less practical from a computational point of view; because of the costly second derivatives. However, during the last ten

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years, many cubically two-point iterative methods free from second derivatives have been derived and studied; see [6] and the references therein. They can be obtained from quadrature formulae or approximation to the second derivatives. These methods require the function or its first derivative evaluated at two different points, usually using Newton's method as the first step. After that many methods of higher orders have been discussed in literature, too (see [7]). And consequently, many iterative methods for the solution of nonlinear systems have also been presented to literature, such as [1], [2]. For further reading on this topic, we refer the readers to [3], [5] and the references therein.

This research unfolds the material in what follows. In Section 2, the main result of this paper is given theoretically where it is followed by Section 3 in which after presenting the known literatures, we make the comparison to show that the new method is accurate and efficient. In the end, the concluding remarks are furnished in Section 4.

2. The Developed Technique

Consider that the scalar function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ has a simple zero in the open domain D. We assume the following three-step cycle

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \dfrac{2f(x_n) + (y_n - x_n)[f'(y_n) + f'(x_n)]}{2f'(y_n)}, \\ x_{n+1} = z_n - \dfrac{f(z_n)}{f'(z_n)}, \end{cases} \tag{1}$$

in which there are five evaluations per iteration to proceed. The first and second steps of (1) are the two-step two-point method of Soleymani and Sharifi [6] of third-order of convergence. To obtain a novel modification of the Newton's method and also Soleymani and Sharifi's scheme, with better order of convergence and efficiency index, we have added a new step by the Newton's method. It is crystal clear that (1) is of sixth-order of convergence and has 1.430 as its index of efficiency. The main challenge is to approximate $f'(z_n)$ as the order does not decrease and the index of efficiency goes up. We use the method of un-determined coefficients to do this efficiently. Toward this end, we assume

$$f'(z_n) \approx f'(x_n)\frac{\alpha f'(x_n) + \beta f'(y_n) + Af(x_n) + Bf(z_n)}{\theta f'(x_n) + (\alpha + \beta - \theta)f'(y_n) + Cf(x_n) + Df(z_n)}, \tag{2}$$

where A, B, C and D are unknowns and α, β and θ are three parameters. Expand the terms $f'(z_n)$, $f'(y_n)$ and $f(y_n)$ around the point x_n up to the second derivatives and collect the obtained terms. Next, by comparing the coefficient of the derivatives of f at x_n, we attain a system of linear equations for the unknowns A, B, C and D as comes next

$$\begin{cases} \delta D = \delta B, \\ \theta\delta + (\alpha+\beta-\theta)\epsilon + (\alpha+\beta-\theta)\delta + \delta^2 D = \beta\epsilon, \\ A + B = C + D, \\ \delta C + \delta D = 0, \\ (\alpha+\beta-\theta)\delta\epsilon = 0, \end{cases} \tag{3}$$

wherein $\delta = z_n - x_n$ and $\epsilon = y_n - x_n$. This system has the solution $A = C = (\theta\delta - \beta\epsilon)/\delta^2$, $B = D = -(\theta\delta - \beta\epsilon)/\delta^2$. That is, $A = C = -B = -D$. Note that we should choose $\alpha = 0$ form now on to obtain the sixth-order convergence; because in other case by $\alpha \neq 0$ a scheme of order five will be obtained. Thus, the following accurate three-step method can be attained

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \dfrac{2f(x_n) + (y_n - x_n)[f'(y_n) + f'(x_n)]}{2f'(y_n)}, \\ x_{n+1} = z_n - \dfrac{\delta^2 f'(x_n) + (\delta - \epsilon)[f(x_n) - f(z_n)]}{\delta^2 f'(y_n) + (\delta - \epsilon)[f(x_n) - f(z_n)]} \dfrac{f(z_n)}{f'(x_n)}, \end{cases} \tag{4}$$

where the parameters β and θ automatically vanished. The proof of this method is given in Theorem 1.

Theorem 1. *If an initial guess x_0 is sufficiently close to the root α of the function f, then the convergence order of the three-step without memory scheme (4) is six.*

Proof. We demonstrate the order of (4) by providing its Taylor expansion in the last step. Hence we start by writing the Taylor expansion of $f(x_n)$ and $f'(x_n)$ about the simple root. Let us consider $e_n = x_n - \alpha$ and $c_j = \frac{1}{j!}\frac{f^{(j)}(\alpha)}{f'(\alpha)}$, $j > 2$. Thus, we have

$$f(x_n) = f'(\alpha)[e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + c_6 e_n^6 + O(e_n^7)].$$

Furthermore, for the first derivative, we get

$$f'(x_n) = f'(\alpha)[1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4 + 6c_6 e_n^5 + O(e_n^6)].$$

Therefore, $y_n - \alpha = c_2 e_n^2 + (-2c_2^2 + 2c_3)e_n^3 + (4c_2^3 - 7c_2c_3 + 3c_4)e_n^4 + (-8c_2^4 + 20c_2^2c_3 - 6c_3^2 - 10c_2c_4 + 4c_5)e_n^5 + (16c_2^5 - 52c_2^3c_3 + 33c_2c_3^2 + 28c_2^2c_4 - 17c_3c_4 - 13c_2c_5)e_n^6 + O(e_n^7)$. Using the new obtained relation and the second step of (4), we obtain

$$z_n - \alpha = (\frac{c_3}{2})e_n^3 + (c_2^3 - \frac{3c_2c_3}{2} + c_4)e_n^4 + O(e_n^5). \tag{5}$$

By applying (5) and the Taylor expansion, we obtain that

$$\frac{\delta^2 f'(x_n) + (\delta - \epsilon)[f(x_n) - f(z_n)]}{\delta^2 f'(y_n) + (\delta - \epsilon)[f(x_n) - f(z_n)]} = 1 + 2c_2e_n^1 + 3c_3e_n^2$$
$$+ (-2c_2^3 + 2c_2c_3 + 4c_4)e_n^3 + (-9c_2^2c_3 + \frac{9c_3^2}{2} + 2c_2c_4 + 5c_5)e_n^4 + O(e_n^5). \tag{6}$$

Similarly, we attain $\frac{f(z_n)}{f'(x_n)} = (c_3e_n^3)/2 + (c_2^3 - (5c_2c_3)/2 + c_4)e_n^4 + 1/2(-12c_2^4 + 25c_2^2c_3 - 9c_3^2 - 8c_2c_4 + 3c_5)e_n^5 + 1/4(88c_2^5 - 222c_2^3c_3 + 141c_2c_3^2 + 64c_2^2c_4 - 54c_3c_4 - 22c_2c_5)e_n^6 + O(e_n^7)$. Now by applying this and (6) in the last step of (4), we get that

$$e_{n+1} = x_{n+1} - \alpha = (c_2^3c_3 - \frac{5c_2c_3^2}{4})e_n^6 + O(e_n^7) \tag{7}$$

which shows that the technique (4) is of sixth-order convergence. This ends the proof. □

We should remark that, although we have doubled the convergence rate of the two-point method considered in the first two steps of our cycle, the convergence rate 6 with four evaluations per cycle is not optimal. The optimal order with four evaluations is eight according to the still un-proved conjecture of Kung-Traub (1974). Anyhow, there is no optimal 8-th order method with two evaluations of the function and two evaluations of the first derivatives. In fact, the optimal without memory 8-th order methods which are available in literature are three-step methods in which there are four function evaluations or three function and one first derivative evaluations per full iteration.

Remark 1. The method (4) comprises two evaluations of the function and two evaluations of the first derivatives per full cycle. (4) is free from second-derivative and possesses 1.565 as the index of efficiency which is bigger than $2^{1/2} \approx 1.414$ of Newton's, $3^{1/3} \approx 1.442$ of cubically iterative methods, such as Soleymani and Sharifi's [6], $6^{1/5} \approx 1.430$ of (1) and is equal to the methods (carried forward) (8) and (10).

3. Numerical Results

Before comparing the proposed technique with the well-known schemes from the literature in this section, we need to review some of the techniques of the same order here. Soleymani in [9] proposed a sixth-order convergence method as follows

$$\begin{cases} y_n = x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}, \\ z_n = x_n - \frac{3\frac{a_1 - f(x_n)a_2}{(1+a_2(y_n-x_n))^2} + f'(x_n)}{6\frac{a_1 - f(x_n)a_2}{(1+a_2(y_n-x_n))^2} - 2f'(x_n)}\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = z_n - \frac{(1+b_2(z_n-x_n))^2 f(z_n)}{f'(x_n) + b_1(z_n-x_n)(2+b_2(z_n-x_n))}, \end{cases} \tag{8}$$

where the parameters a_1, a_2, b_1 and b_2 are provided in the following way

$$\begin{cases} a_1 = -\frac{f'(x)f(y_n)}{f(x_n)-f(y_n)} + \frac{f(x_n)}{x_n - y_n}, \\ a_2 = \frac{f'(x)}{f(y_n)-f(x_n)} + \frac{1}{x_n - y_n}, \\ b_1 = \frac{f'(x)f[y_n,z_n] - f[x_n,y_n]f[x_n,z_n]}{x_n f[y_n,z_n] + \frac{y_n f(z_n) - z_n f(y_n)}{y_n - z_n} - f(x_n)}, \\ b_2 = \frac{b_1}{f[x_n,y_n]} + \frac{f'(x_n) - f[x_n,y_n]}{(y_n-x_n)f[x_n,y_n]}. \end{cases} \tag{9}$$

[8] presented a sixth-order scheme as comes next

$$\begin{cases} y_n = x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}, \\ z_n = x_n - \frac{3f'(y_n)+f'(x_n)}{6f'(y_n)-2f'(x_n)}\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = z_n - \frac{f(z_n)}{f'(y_n) + 2f[z_n,x_n,x_n](z_n-y_n)}, \end{cases} \tag{10}$$

which is a high-order method with better convergence radius and accuracy in contrast to the optimal 8-th order methods when the starting points are in the vicinity of the root but not so close. This feature of the predictor-corrector method (10) comes from its first step, i.e., high-order Jarratt-type methods possess better predictors only when the starting points are in the neighborhood but some far from the sought zero. So this will be obvious that, if one guesses

Functions	Roots
$f_1(x) = \sin(x) - x/100$	0
$f_2(x) = (1/3)x^4 - x^2 - (1/3)x + 1$	1
$f_3(x) = e^{\sin(x)} - 1 - x/5$	0

Table 1: Test functions and their roots

Methods	$\|x_1 - \alpha\|$	$\|x_2 - \alpha\|$	$\|x_3 - \alpha\|$	$\|x_4 - \alpha\|$
NM	0.9e-2	0.2e-6	0.7e-20	0.1e-60
SSM	0.2e-2	0.1e-8	0.1e-27	0.3e-84
(8)	0.2e-5	0.2e-30	0.3e-155	0.3e-780
(10)	0.2e-5	0.1e-41	0.2e-295	0.1e-2070
(4)	0.2e-5	0.1e-40	0.1e-288	0.1e-2022

Table 2: Results of comparisons for f_1 with $x_0 = 0.3$

the starting points very close to α then optimal 8-th order methods perform better than (10) or the Jarratt-type method in [7].

The proposed method (4) is compared with the second-order Newton's method (NM), the cubically iterative method of Soleymani and Sharifi (SSM), the sixth-order method (8) and the sixth-order scheme (10). The test functions and their roots are listed in Table 1.

All numerical tests were done on a personal computer using *Matlab 7.6* with *Intel (R) Pentium 4* while the operating system was *Windows XP (Vista)*. We have used the following stopping criteria for computer programs (i) $|f(x_n)| < 1.E-2500$, and (ii) $|x_{n+1} - x_n| < 1.E-2500$. The results of comparisons for the initial guesses are given in Tables 2-4. As can be observed, (4) is competitive with the other famous methods in literature except from (10).

From the numerical results displayed in Tables 2-4, it can be concluded that the convergence of the tested multi-point method is good. Although three full iterative steps are quite satisfactory in solving most practical problems when the starting guess is reasonably good, we have displayed the numerical reports of the fourth iteration to demonstrate remarkably fast convergence of the derived method. Since the approximations of great accuracy are obtained using only a few function evaluations per iteration. Here, we should remark that the convergence behavior of the considered multi-point methods strongly depends on the structure of the nonlinear tested functions and the accuracy of starting points.

Methods	$\lvert x_1-\alpha\rvert$	$\lvert x_2-\alpha\rvert$	$\lvert x_3-\alpha\rvert$	$\lvert x_4-\alpha\rvert$
NM	0.1e-1	0.2e-3	0.6e-7	0.4e-14
SSM	0.1e-2	0.1e-8	0.2e-26	0.1e-79
(8)	0.3e-6	0.4e-39	0.2e-236	0.2e-1420
(10)	0.6e-5	0.1e-30	0.1e-183	0.5e-1103
(4)	0.1e-4	0.2e-28	0.1e-170	0.9e-1025

Table 3: Results of comparisons for f_2 with $x_0 = 1.1$

Methods	$\lvert x_1-\alpha\rvert$	$\lvert x_2-\alpha\rvert$	$\lvert x_3-\alpha\rvert$	$\lvert x_4-\alpha\rvert$
NM	0.9e-1	0.4e-2	0.1e-4	0.1e-9
SSM	0.3e-1	0.1e-6	0.2e-28	0.2e-115
(8)	0.5e-2	0.8e-16	0.1e-98	0.3e-595
(10)	0.1e-2	0.1e-20	0.7e-148	0.9e-1038
(4)	0.1e-2	0.2e-20	0.5e-145	0.4e-1018

Table 4: Results of comparisons for f_3 with $x_0 = 0.7$

4. Concluding Remarks

Solving single variable nonlinear equations is one of the old and classical problems in applied mathematics which has a lot of applications in scientific fields. In this article, we have presented a new sixth-order technique consisting of three steps and three points. The error equation of the main theorem was furnished analytically and a comparison with the celebrated techniques of various orders was provided in Section 3. The method reaches the 1.565 as its efficiency index. Eventually, some illustrative examples were provided to support the theory given in this contribution.

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