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PICONE-TYPE IDENTITIES AND INEQUALITIES FOR GENERAL QUASILINEAR ELLIPTIC EQUATIONS PART I: OSCILLATION AND UNIQUENESS RESULTS

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Abstract: We investigate some sufficient conditions for the solutions of the equations below, extended to the whole space to be oscillatory, using only some Picone-type formulas. Some uniqueness results are given. The important fact of the use of the Picone-type formulas displayed here is that it leads to some interesting results with some easy hypotheses. In Part II we extend the results to some known equations, using some comparison results.

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1. Introduction

We consider the following equations in any regular (having a C^1-boundary) bounded domains $G \subset \mathbb{R}^n$; $n \geq 3$:

$$Pu := \nabla\Big\{a(x)\Phi(\nabla u)\Big\} + a(x)B(x).\Phi(\nabla u) + c(x)\phi(u) = 0, \tag{1.1}$$

$$P_0 v := \nabla\Big\{a(x)\Phi(\nabla v)\Big\} + c(x)\phi(v) = 0, \quad \text{and} \tag{1.2}$$

$$Qw := \nabla\Big\{a(x)\Phi(\nabla w)\Big\} + a(x)B(x).\Phi(\nabla w) + c(x)\phi(w) + f(x,w) = 0 \tag{1.3}$$

in G,

where for some $\eta, \alpha > 0$, $\forall \zeta \in \mathbb{R}^n$, $t \in \mathbb{R}$

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$$\Phi(\zeta) := |\zeta|^{\alpha-1}\zeta \quad \text{and} \quad \phi(t) := |t|^{\alpha-1}t; \tag{1.4}$$

$f \in C(\mathbb{R}^n \times \mathbb{R}; \mathbb{R})$; $a \in C^1(\overline{G}; (0, \infty))$; $B \in C(\overline{G}; \mathbb{R}^n)$ and $c \in C^{0,\eta}(\overline{G}; \mathbb{R})$.

For boundary conditions, we set when necessary

$$w|_{\partial G} = 0 \quad \text{and} \quad w \neq 0 \ \text{ in } \ G. \tag{1.5}$$

The dot between two vectorfields denotes the usual scalar product in $\mathbb{R}^n$.

We recall that $\forall t, t_i \in \mathbb{R},\ \zeta \in \mathbb{R}^n$

$$\phi(t)\Phi(\zeta) = \Phi(t\zeta), \quad \phi(t_1)\phi(t_2) = \phi(t_1t_2) \quad \text{and} \quad t\phi'(t) = \alpha\phi(t).$$

A classical solution of (1.1), (1.2) or (1.3) (assumed non trivial) in G will denote solutions which belong to $D(G)$ where

$$D(G) := \{w \in C^1(\overline{G}; \mathbb{R}) \text{ such that } a(x)|\nabla w|^{\alpha-1}\nabla w \in C^1(G; \mathbb{R}^n) \bigcap C(\overline{G}; \mathbb{R}^n)\}.$$

The principal part of those equations, namely the $p-$Laplacian $\triangle_p v := \nabla.\{|\nabla v|^{p-2}\nabla v\}$ (here $\alpha := p-1 > 0$) arises in various physical phenomena; e.g. non-Newtonian fluids, reaction-diffusion problems, glaciology, flows through porous media (see [7, 1] and references therein).

The main goal of this work is to establish some sufficient conditions on a, c, B, f such that any (non trivial) classical solution of (1.1) and (1.3) (extended) in $\mathbb{R}^n$ is strongly oscillatory. We will also look for uniqueness conditions. We recall that a function u is said to be strongly oscillatory in $\mathbb{R}^n$ if $\forall M > 0$, there is a bounded domain $G \subset \Omega^M := \{x \in \mathbb{R}^n \ ; \ |x| > M\}$ such that $u \neq 0$ in G and $u|_{\partial G} = 0$.

Such a G is called a "nodal" set for u.The regularity condition on c here garanties that any oscillatory solution of (1.1) is a strongly oscillatory one (see [3]). The weakly oscillatory u would just require for u to have a zero in any Ω^M.

In the sequel, G will always denote a bounded and C^1-domain in $\mathbb{R}^n$.

Define for $r,\ r_0 > 0$

$$A(r) := \max_{|x|=r} a(x)\ ; \quad \gamma(r) := \min_{|x|=r} c(x); \quad q(r) := r^{n-1}\gamma(r)\ ; \tag{1.6}$$

$$p(r) := r^{n-1}A(r)\ ; \quad P(t) := \int_{r_0}^t p(s)^{-1/\alpha}\, ds \quad \text{and} \quad \pi(t) := \int_t^\infty p(s)^{-1/\alpha}\, ds.$$

We recall the following well known result on oscillatory solutions (see [2]):

The sufficient conditions for non trivial solutions (extended to the whole $\mathbb{R}^n$)

$$\text{of (1.2), (1.5) to be oscillatory in } \mathbb{R}^n \text{ are} \tag{1.7}$$

(Cia) $\lim_{r\nearrow\infty} P(r) = \infty$ and $\int_{r_0}^{\infty} q(r)dr = \infty$, or

(Cib) $\displaystyle\int_{r_0}^{\infty} q(r)dr < \infty$ and $\lim_{r\nearrow\infty} \inf \left\{ P(r)^\alpha \int_r^\infty q(s)ds \right\} > \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}$,

or

(Ciia) $\lim_{r\nearrow\infty} P(r) < \infty$ and $\int_{r_0}^{\infty} \pi(r)^{\alpha+1}\, q(r)dr = \infty$ or

(Ciib) $\displaystyle\int_{r_0}^{\infty} \pi(r)^{\alpha+1} q(r)dr < \infty$ and

$$\lim_{r\nearrow\infty} \inf \left\{ \frac{1}{\pi(r)} \int_r^\infty \pi(t)^{\alpha+1} q(t)dt \right\} > \left[\frac{\alpha}{\alpha+1} \right]^{\alpha+1}.$$

We define here some identities and notations which will be used later on: For any $w_1, w_2 \in C^1(\mathbb{R}^n)$ the two-form

$$Z(w_1, w_2) = \left\{ |\nabla w_1|^{\alpha+1} - (\alpha+1) \left| \Phi\left(\frac{w_1}{w_2} \nabla w_2 \right) \right|^{\alpha-1} \left(\frac{w_1}{w_2} \nabla w_2 \right) . \nabla w_1 + \alpha \left| \frac{w_1}{w_2} \nabla w_2 \right|^{\alpha+1} \right\} \tag{1.8}$$

is non negative and is null only if $\exists k \in \mathbb{R};\ w_1 = kw_2$ (see, e.g. [2, 4]). For ease writing, we define for any two differentiable functions u and v

$$\Gamma(u, v) := \{u\Phi(\nabla u) - u\Phi(\frac{u}{v}\ \nabla v)\}. \tag{1.9}$$

2. Picone Type Formulas

Let u and v be two elements of $C^2(G)$. If

$$\text{There is } b \in C^1(\mathbb{R}^n) \text{ such that } \forall x \in \mathbb{R}^n \quad \nabla b(x) = B(x), \tag{2.1}$$

then if u satisfies (1.1) and v (1.2)

$$\nabla\{ub(x)a(x)\Phi(\nabla u)\} = a(x)b(x)|\nabla u|^{\alpha+1} + ua(x)B(x).\Phi(\nabla u)$$

$$-ua(x)b(x)B(x).\Phi(\nabla u) - b(x)c(x)|u|^{\alpha+1} \tag{2.2}$$

$$\text{and} \quad \nabla.\Big\{v\phi(\frac{v}{u})a(x)b(x)\Phi(\nabla u)\Big\} = a(x)vB(x).\Phi(\frac{v}{u}\nabla u) \tag{2.3}$$

$$+b(x)\Big\{a(x)|\nabla v|^{\alpha+1} - a(x)Z(v,u) - a(x)vB(x).\Phi(\frac{v}{u}\nabla u) - c(x)|v|^{\alpha+1}\Big\}.$$

We then have the following

Lemma 2.1. *Assume that there are $u, v \in D(G)$ which satisfy respectively (1.1) and (1.2). Then if (2.1) holds and $v \neq 0$ in G*

$$\nabla.\Big\{ua(x)\Phi(\nabla u) - u\phi\Big(\frac{u}{v}\Big)a(x)\Phi(\nabla v) + ub(x)a(x)\Phi(\nabla u)\Big\} \tag{2.4}$$

$$= a(x)Z(u,v) + b(x)\Big[a(x)|\nabla u|^{\alpha+1} - ua(x)B(x).\Phi(\nabla u) - c(x)|u|^{\alpha+1}\Big].$$

Also if $u \neq 0$ in G then

$$\nabla.\Big\{va(x)\Phi(\nabla v) - v\phi\Big(\frac{v}{u}\Big)a(x)\Phi(\nabla u) - v\phi\Big(\frac{v}{u}\Big)a(x)b(x)\Phi(\nabla u)\Big\}$$

$$= a(x)Z(v,u) + b(x)\Big\{a(x)Z(v,u) + c(x)|v|^{\alpha+1}$$

$$+b(x)\Big\{a(x)Z(v,u) + c(x)|v|^{\alpha+1} + a(x)vB(x).\Phi\Big(\frac{v}{u}\nabla u\Big) - a(x)|\nabla v|^{\alpha+1}\Big\}. \tag{2.5}$$

Proof. Assume that u satisfies (1.1).

The application of the formulas (2.1) through (2.3) leads to (2.4). For (2.5) we just interchange the functions u and u_1 in those formulas taking in account the equations (1.1) and (1.2). □

We have the following Picone-type formulas for two solutions u and u_1 of (1.1) such that $u_1 \neq 0$ inside G. If (2.1) is satisfied then

$$\left.\begin{array}{l}
(i) \quad \nabla.\Big\{a(x)ub(x)\phi(\nabla u)\Big\} = a(x)uB(x).\Phi(\nabla u) \\
+\, b(x)\Big\{a(x)|\nabla u|^{\alpha+1} - a(x)uB(x).\Phi(\nabla u) - c(x)|u|^{\alpha+1}\Big\}; \\
(ii) \quad \nabla.\Big\{u\phi(\frac{u}{u_1})\, a(x)b(x)\Phi(\nabla u_1)\Big\} = a(x)uB(x).\Phi(\frac{u}{u_1}\nabla u_1) \\
+\, b(x)\Big[a(x)|\nabla u|^{\alpha+1} - a(x)Z(u,u_1) - a(x)uB(x).\Phi(\frac{u}{u_1}\nabla u_1) - c(x)|u|^{\alpha+1}\Big].
\end{array}\right\} \tag{2.6}$$

So, finally, from (2.6) i) and ii), and (1.9)

$$\nabla.\Big\{a(x)ub(x)\Phi(\nabla u) - u\phi(\frac{u}{u_1})a(x)b(x)\Phi(\nabla u_1)\Big\} = a(x)uB(x).\Gamma(u,u_1)$$
$$+b(x)\Big\{a(x)Z(u,u_1) - a(x)uB(x).\Gamma(u,u_1)\Big\}. \tag{2.7}$$

We then have the following

Lemma 2.2. *Assume that (2.1) holds and u is a solution of (1.1), (1.5), non zero inside G. Then if u_1 is another solution of (1.1) in G,*
(1) $u_1 > 0$ in $\overline{G}$ cannot hold;
(2) if $u_1 > 0$ inside G and $u_1|_{\partial G} = 0$ then $\exists k \in \mathbb{R}\setminus\{0\}$ such that $u = ku_1$;
(3) otherwise u_1 has a zero inside G.

Proof. If we replace $b(x)$ in (2.1) by $(b(x)+\mu)$; any $\mu \in \mathbb{R}$, then (2.7) reads after integrating over G, $\forall \mu \in \mathbb{R}$,

$$0 = \int_G a(x)uB(x).\Gamma(u,u_1)dx$$
$$+ \int_G (b(x)+\mu)\Big\{a(x)Z(u,u_1) - a(x)uB(x).\Gamma(u,u_1)\Big\}dx. \tag{2.8}$$

This can hold only if

$$\begin{aligned} &(i) \qquad \int_G \Big\{a(x)Z(u,u_1) - a(x)uB(x).\Gamma(u,u_1)\Big\}dx = 0 \\ &\qquad (ii) \qquad \text{and} \qquad \int_G a(x)uB(x).\Gamma(u,u_1)dx = 0. \end{aligned} \tag{2.9}$$

The equations (2.9) imply that $\int_G a(x)Z(u,u_1)dx = 0$ with $a(x) > 0$ and $Z(u,u_1) \geq 0$. The only condition for (2.9) to hold is to have $Z(u,u_1) = 0$ leading to the conclusions of the lemma. ☐

Let $w \in D(G)$ be a solution of (1.3), v a solution of (1.2), (1.5) and b the function in (2.1). Then proceeding as before we have

$$\nabla.\Big\{va(x)\Phi(\nabla v) - v\phi(\frac{v}{w})a(x)\Phi(\nabla w) - v\phi(\frac{v}{w})a(x)b(x)\Phi(\nabla w)\Big\}$$
$$= a(x)Z(v,w) + |v|^{\alpha+1}\frac{f(x,w)}{\phi(w)} + +b(x)\Big\{a(x)Z(v,w) - a(x)|\nabla v|^{\alpha+1} \tag{2.10}$$
$$+ va(x)B(x).\Phi(\frac{v}{w}\nabla w) + |v|^{\alpha+1}\Big(c(x) + \frac{f(x,w)}{\phi(w)}\Big)\Big\};$$

for $V(x) := v(x) + k$; $k \in \mathbb{R}$,

$$\begin{aligned}\nabla.\Big\{ & wa(x)b(x)\Phi(\nabla w) - w\phi(\frac{w}{V})a(x)b(x)\Phi(\nabla V)\Big\} \\ & = a(x)wB(x).\Gamma(w, v+k) \\ & \quad + b(x)\Big[a(x)Z(w, v+k) - a(x)wB(x).\Phi(\nabla w) - wf(x,w)\Big].\end{aligned} \tag{2.11}$$

Lemma 2.3. *Let $v, w \in D(G)$ be respectively non trivial solutions of (1.2) and (1.3). If*

$$\forall x \in G \text{ and } t \in \mathbb{R} \setminus \{0\} \qquad tf(x,t) > 0; \tag{2.12}$$

then if $v|_{\partial G} = 0$, w cannot be non zero throughout $\overline{G}$ and inversely if $w|_{\partial G} = 0$, v cannot be non zero throughout $\overline{G}$.

Moreover since

$$\int_G |\frac{|v|^{\alpha+1}}{\phi(w)}| dx < \infty \tag{2.13}$$

when the two solutions satisfies (1.5), these results hold also with G replacing $\overline{G}$.

Proof. 1) Assume that $v|_{\partial G} = 0$ and $w \neq 0$ throughout $\overline{G}$.

If we replace in (2.10) $b(x)$ by any $b(x) + \mu$, $\mu \in \mathbb{R}$ and integrate the resulting equation over G, we get

$$\begin{aligned}\forall \mu \in \mathbb{R}, \quad \int_G & a(x)Z(v,w) + |v|^{\alpha+1}\frac{f(x,w)}{\phi(w)} \\ & + [b(x) + \mu]\Big\{a(x)Z(v,w) - a(x)|\nabla v|^{\alpha+1} \\ & + va(x)B(x).\Phi(\frac{v}{w}\nabla w) + |v|^{\alpha+1}\Big(c(x) + \frac{f(x,w)}{\phi(w)}\Big)\Big\}\, dx = 0\end{aligned}$$

implying among the other that $\int_G [a(x)Z(v,w) + |v|^{\alpha+1}\frac{f(x,w)}{\phi(w)}]dx = 0$. But this cannot hold as the two integrand-members are non negative and none of them can be zero unless $v = 0$ or $w = 0$ throughout $\overline{G}$.

2) Assume that $w|_{\partial G} = 0$ and $v \neq 0$ throughout $\overline{G}$.

Using $V(x) := v(x) + k$; $k > 0$ and taking $b(x) + \mu$ as before, (2.11) leads to

$$\int_G \Big[a(x)Z(w, v+k) - a(x)wB(x).\Phi(\nabla w) - wf(x,w)\Big] dx = 0 \quad \forall k > 0.$$

But from the definition of Z, this equation, when $k \nearrow \infty$, tends to

$$\int_G \Big[a(x)|\nabla w|^{\alpha+1} - a(x)wB(x).\Phi(\nabla w) - wf(x,w) \Big] dx = 0,$$

which cannot hold as from the equation of

$$w \int_G \Big[a(x)|\nabla w|^{\alpha+1} - a(x)wB(x).\Phi(\nabla w) - wf(x,w) \Big] dx = \int_G c(x)|w|^{\alpha+1} > 0,$$

unless $w \equiv 0$ in G. □

Let $w,\ w_o \in D(G)$ be non zero in G and satisfy (1.3). Then b being the function defined earlier, simple calculations show that

$$(a) \quad \nabla.\Big\{ wa(x)b(x)\Phi(\nabla w) - w\phi(\frac{w}{w_o})a(x)b(x)\Phi(\nabla w_o) \Big\}$$
$$= a(x)wB(x).\Gamma(w,w_o) + b(x)\Big\{ a(x)Z(w,w_o) - a(x)wB(x).\Gamma(w,w_o)$$
$$+|w|^{\alpha+1}\left(\frac{f(x,w_o)}{\phi(w_o)} - \frac{f(x,w)}{\phi(w)} \right) \Big\}.$$

Similarly interchanging w and w_o we have

$$(b) \quad \nabla.\Big\{ w_oa(x)b(x)\Phi(\nabla w_o) - w_o\phi(\frac{w_o}{w})a(x)b(x)\Phi(\nabla w) \Big\}$$
$$= a(x)w_oB(x).\Gamma(w_o,w) + b(x)\Big\{ a(x)Z(w_o,w) - a(x)w_oB(x).\Gamma(w_o,w)$$
$$+|w_o|^{\alpha+1}\left(\frac{f(x,w)}{\phi(w)} - \frac{f(x,w_o)}{\phi(w_o)} \right) \Big\}.$$

After adding (a) and (b) we get

$$\nabla.\Big\{ a(x)b(x)\Phi(\nabla w) - w\phi(\frac{w}{w_o})a(x)b(x)\Phi(\nabla w_o)$$
$$+w_oa(x)b(x)\Phi(\nabla w_o) - w_o\phi(\frac{w_o}{w})a(x)b(x)\Phi(\nabla w) \Big\}$$
$$= a(x)B(x).\Big[w\Gamma(w,w_o) + w_o\Gamma(w_o,w) \Big]$$
$$+b(x)\Big\{ a(x)\{\ Z(w,w_o) + Z(w_o,w)\ \} - a(x)B(x).\Big[w\Gamma(w,w_o) + w_o\Gamma(w_o,w) \Big]$$
$$+\left(\frac{f(x,w_o)}{\phi(w_o)} - \frac{f(x,w)}{\phi(w)} \right)(\ |w|^{\alpha+1} - |w_o|^{\alpha+1}\ ) \Big\}. \tag{2.14}$$

3. Main Results

3.1. Comparison Results

Lemma 3.1. *Let G be a regular bounded domain in $\mathbb{R}^n$ and $u, v \in D(G)$ be respectiely classical (non trivial) solutions of (1.1) and (1.2).*

1) If u satisfies (1.5) then v has a zero inside G. Inversely if v satisfies then u has a zero inside G.

2) Let v and w be respectively non trivial solutions of (1.2) and (1.3).Assume that (2.1), (2.12) and (2.13) hold. If v satisfies (1.5), then w has a zero inside G and reversely if w satisfies (1.5) then v has a zero inside G.

Proof. 1) Assume that u satisfies (1.5). Then (2.4) of Lemma 2.1 applies. In fact as in the proof of Lemma 2.3, if we assume that $v \neq 0$ in G then (2.4) implies that $\int_G Z(u, v)dx = 0$ whence $\exists k \in \mathbb{R} \setminus \{0\}$ such that $u = kv$ and this cannot hold since the two equations cannot have the same non trivial solutions.

If v satisfies (1.5), by means of (2.5) we reach the same conclusion.

2) If (2.12) and (2.13) hold then Lemma 2.3 applies. □

3.2. Oscillation Result

Theorem 3.2. *With the functions A, γ, q, p, P and π as defined in (1.6), assume that (Ci) or (Cii) of (1.7) and (2.1) hold. Then extended to the whole $\mathbb{R}^n$*

A) the non trivial solutions of (1.1) are strongly oscillatory;

B) if in addition (2.12) and (2.13) hold, the non trivial solutions of (1.3) are strongly oscillatory.

Proof. Under the respective hypotheses displayed above, let u and w be respectively non trivial solutions of (1.1), (1.5) and (1.3), (1.5), each one extended in the whole space. From the hypotheses, (1.2) has a strong oscillatory solution v, say, in $\mathbb{R}^n$.

A) From Lemma 3.1, between any two zeros of v lies one zero of u and viseversa.

B) This holds also between v and w if (2.12) and (2.13) hold. □

3.3. Uniqueness Result

Theorem 3.3. *Assume that (2.1) holds and*

$$\forall x \in G \qquad t \mapsto \frac{f(x,t)}{\phi(t)} \quad \textit{is decreasing} \; ; \tag{3.1}$$

then (1.3), (1.5) has at most one classical solution.

Proof. If w and w_o are two solutions, then, repralacing $b(x)$ by $b(x)+\mu$, (2.14) implies that $\forall \mu \in \mathbb{R}$

$$\begin{aligned} 0 = &\int_G a(x)B(x).\Big[w\Gamma(w,w_o) + w_o\Gamma(w_o,w)\Big]dx \\ &+ \int_G (b(x)+\mu)\Big\{a(x)\{\ Z(w,w_o)+Z(w_o,w)\ \} \\ &- a(x)B(x).\Big[w\Gamma(w,w_o)+w_o\Gamma(w_o,w)\Big] \\ &+ \left(\frac{f(x,w_o)}{\phi(w_o)} - \frac{f(x,w)}{\phi(w)}\right)(\ |w|^{\alpha+1} - |w_o|^{\alpha+1}\ )\Big\}dx \end{aligned}$$

implying that each integral is zero. Thus the first integral being zero, the second becaumes

$$\begin{aligned} 0 = \int_G \Big\{a(x)\{\ Z(w,w_o)+Z(w_o,w)\ \} + \Bigg(&\frac{f(x,w_o)}{\phi(w_o)} \\ &- \frac{f(x,w)}{\phi(w)}\Bigg)(\ |w|^{\alpha+1} - |w_o|^{\alpha+1}\ )\Big\}dx, \end{aligned}$$

which holds only if $w = w_o$. □

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