International Journal of Pure and Applied Mathematics
Volume 70 No. 3 2011, 389-399

GALERKIN-FINITE ELEMENT METHOD FOR THE NUMERICAL SOLUTION OF ADVECTION-DIFFUSION EQUATION

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Abstract: In this article, Galerkin-finite element method is proposed to find the numerical solutions of advection-diffusion equation. The equation is generally used to describe mass, heat, energy, velocity, vorticity etc. As test problem, three different solutions of advection-diffusion equation are chosen. Maximum errors norm L_∞ are calculated and found that the errors are small and negligible.

AMS Subject Classification: 65Mxx, 65Nxx
Key Words: finite element method, finite difference method, advection-diffusion equation

1. Introduction

Consider the one-dimensional advection-diffusion equation

$$\frac{\partial U}{\partial t} - \lambda \frac{\partial^2 U}{\partial x^2} + \alpha \frac{\partial U}{\partial x}, \quad 0 \leq x \leq L,\ t > 0, \tag{1}$$

with the initial condition

$$U(x,0) = \phi(x), \quad 0 \leq x \leq L, \tag{2}$$

and the boundary conditions

Received: March 15, 2011

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$$U(0,t) = f(t), \quad U(L,t) = h(t), \quad t > 0, \tag{3}$$

where U represents concentration of the pollutant at point x, the advection coefficient α is the velocity of water flow and λ is the diffusion coefficient.

The advection-diffusion equation is generally used to describe mass, heat, energy, velocity, and vorticity [13]. The equation has been used as a model equation in many chemistry and engineering problems such as, thermal pollution in river systems [3], flow in porous media [9], dispersion of tracers in porous media [4], the dispersion of dissolved material in estuaries and coastal seas [6], the intrusion of salt water into fresh water aquifers, the spread of pollutants in rivers and streams [2], forced cooling by fluids of solid material such as windings in turbo generators [5], the spread of solute in a liquid flowing through a tube, long-range transport of pollutants in the atmosphere [17], contaminant dispersion in shallow lakes [15], model water transport in soils [14], the absorption of chemicals into beds [7], etc. Isenberg and Gutfinger used the advection-diffusion equation to describe heat transfer in a draining film [7]. Mortan has used the advection-diffusion equation to model some economics and financial forecasting [10]. Besides this, the equation has great importance in civil engineers and hydrologists. Thus, the advection-diffusion equation is very interesting linear partial differential equation from numerical study point of view.

Many researchers have developed numerical techniques to study the numerical solutions of advection-diffusion equation. Celia et al [1] have proposed an Eulerian-langrangian localized adjoint method for the advection-diffusion equation. Spalding [16]proposed the hybrid scheme which is a combination of three straight lines to correlate the exact curve. Kakuda and Tosaka [8]used time splitting or fractional steps method for advection-diffusion and viscous fluid flows problems. Nico et al [12] proposed a finite volume upwind scheme for the solution of the linear advection-diffusion equation with sharp gradients in multiple dimensions for the solution of advection-diffusion equation while Neubauer and Bastian [11] used a monotonicity preserving Eulerian-Lagrangian localized adjoint method for advection-diffusion equations.

2. Semidiscrete Finite Element Models

The semi discrete formulation involves approximation of the spatial variation of the dependent variable. The first step involves the construction of the weak form of the given problem over a typical element. In second step, we develop the finite element model by seeking approximation of the solution.

2.1. Weak Formulation of the Problem

The weak formulation of the given problem (1) over a typical linear element (x_a, x_b) is given by

$$\int_{x_a}^{x_b} w(x)\left\{\frac{\partial U}{\partial t} - \lambda\frac{\partial^2 U}{\partial x^2} + \alpha\frac{\partial U}{\partial x}\right\} dx = 0, \tag{4}$$

where $w(x)$ are arbitrary test functions and may be viewed as the variation in $U(x)$. After reducing the order of integration, we arrive at the following system of equations

$$\int_{x_a}^{x_b} w(x)\left\{\frac{\partial U}{\partial t} + \lambda\frac{\partial w}{\partial x}\frac{\partial U}{\partial x} + \alpha w(x)\frac{\partial U}{\partial x}\right\} dx = 0. \tag{5}$$

2.2. Finite Element Formulation of the Problem

The finite-element model may be obtained from equations (5) by substituting finite element approximations in the decoupled form

$$U(x,t) = \sum_{j=1}^{N} U_j^e(t_s)\psi_j^e(x), \quad s = 1, 2.... \tag{6}$$

Substituting $w = \psi_i(x)$ and (6) in equation (5) to obtain the ith equation of the system, we have

$$\int_{x_a}^{x_b}\left[\psi_i\left(\sum_{j=1}^{N}\frac{\dot{U}_j}{dt}\psi_j\right) + \alpha\psi_i\left(\sum_{j=1}^{N}U_j\frac{d\psi_j}{dx}\right) + \lambda\frac{d\psi_i}{dx}\left(\sum_{j=1}^{N}U_j\frac{d\psi_j}{dx}\right)\right] dx = 0, \tag{7}$$

$$\sum_{j=1}^{N}\left[\left(\int_{x_a}^{x_b}\psi_i\psi_j dx\right)\frac{dU_j}{dt} + \alpha\left(\int_{x_a}^{x_b}\psi_i\frac{d\psi_j}{dx}dx\right)U_j + \lambda\left(\int_{x_a}^{x_b}\frac{d\psi_i}{dx}\frac{d\psi_j}{dx}dx\right)U_j\right] = 0 \tag{8}$$

The equation (8) can be written in the matrix form

$$[M]\left\{\dot{U}\right\} + [K^1]\{U\} + [K^2]\{U\} = 0, \tag{9}$$

where $M_{ij} = \int_{x_a}^{x_b} \psi_i \psi_j dx$, $K_{ij}^1 = \int_{x_a}^{x_b} \psi_i \frac{d\psi_j}{dx} dx$,

$$K_{ij}^2 = \int_{x_a}^{x_b} \frac{d\psi_i}{dx} \frac{d\psi_j}{dx} dx. \tag{10}$$

The system (9) can be written as

$$[M] \left\{ \dot{U} \right\} + [K] \{U\} = 0, \tag{11}$$

where$[K] = [K^1] + [K^2]$.

We use the linear piecewise approximation in the space variable and the Galerkin method to obtain the semi discrete approximation to equation (1)

$$U^e (x, t) = \psi_{i-1} (x) U_{i-1} (t) + \psi_i (x) U_i (t), \tag{12}$$

where

$$\psi_{i-1} (x) = \frac{x - x_{i-1}}{x_i - x_{i-1}}, \quad \psi_i (x) = \frac{x_i - x}{x_i - x_{i-1}}. \tag{13}$$

We have used the linear piecewise approximation (12) and (13) to find out the integral in the equation (10). Then, the system (11) become

$$[M] \left\{ \dot{U} \right\} + [K] \{U\} = 0, \tag{14}$$

where:

$$[M] = \frac{h}{6} \left[\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix} \right], \quad [K] = \frac{\alpha}{2} \left[\begin{smallmatrix} -1 & 1 \\ -1 & 1 \end{smallmatrix} \right] + \frac{\lambda}{h} \left[\begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix} \right],$$
$$\left\{ \dot{U} \right\} = \left[\begin{smallmatrix} \dot{U_{i-1}} \\ \dot{U_i} \end{smallmatrix} \right], \quad \{U\} = \left[\begin{smallmatrix} U_{i-1} \\ U_i \end{smallmatrix} \right].$$

3. Fully Discretized Finite Element Equations

We have the system of ordinary differential equations as follows

$$[M] \left\{ \dot{U} \right\} + [K] \{U\} = 0, \tag{15}$$

subject to the initial condition

$$\{U\}_0 = \phi(x) = \{U\}_0, \tag{16}$$

where $\{U\}_0$ denotes the vector of nodal values of U at time$t = 0$ whereas $\{U\}_0$ denotes the column of nodal values U_{j0}.

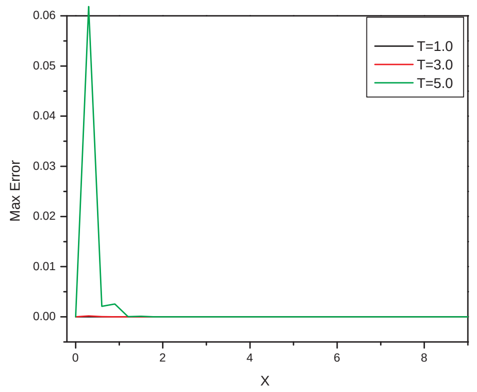

Figure 1: Maximum errors in Example 1 for $\lambda = 0$ (purely advection equation) at different time

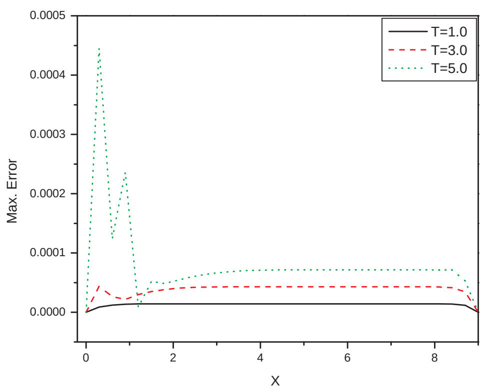

Figure 2: Maximum errors in Example 1 for $\lambda = 0.1$ (purely advection equation) at different time

As applied to a vector of time derivatives of the nodal values the weighted average of approximation on the equation (15), we have

$$[M]\left(\frac{\{U\}_{s+1} - \{U\}_s}{\Delta t}\right) + \theta[K]\{U\}_{s+1} + (1-\theta)[K]\{U\}_s = 0. \tag{17}$$

The equation (17) can be written in simple form as

$$([M] + \Delta t\theta[K])\{U\}_{s+1} = [M]\{U\}_s - \Delta t\,(1-\theta)\,[K]\{U\}_s\,. \tag{18}$$

The algebraic system (18) is solved by Gauss elimination method by taking Crank-Nicolson Scheme i.e $\theta = \frac{1}{2}$ in equation (18).

4. Numerical Experiment and Discussion

In this section, we have studied three test examples to check the accuracy of the proposed numerical scheme.

Example 1. In the first test example, the advection-diffusion equation (1) is considered with domain $[0, 9]$ and the analytical solution

$$U(x,t) = 10\exp\left(\frac{-(x - x_0 - \alpha t)^2}{2\rho_2}\right). \tag{19}$$

T	x	Num Sol	Exact Sol	Max Error
0.5	1.5	9.99991	9.99993	
	4.5	9.99970	9.99971	1.43479
	7.5	9.99819	9.99821	
1.0	1.5	9.99968	9.99971	
	4.5	9.99989	9.99993	4.30570
	7.5	9.99881	9.99885	
3.0	1.5	9.99930	9.99935	
	4.5	9.99993	10.0000	4.45166
	7.5	9.99928	9.99935	

Table 1: Comparison of numerical and analytic solutions of Example 1 for $\alpha = 0.5, \lambda = 0.1$ with maximum absolute error

The initial and boundary conditions are taken from the analytical solution (19). We have considered $\rho = 264$m, $\alpha = 0.5$m/s and $x_0 = 2$. In this example, we have considered two cases. In the first case, we study the purely advection equation by taking $\lambda = 0$.In second case, we take advection-diffusion equation. Figure 1 depicts the absolute errors at different time for first case. Table 1 shows the comparison of numerical and analytic solutions at different time and x with maximum absolute error for the second case. Figure 2 shows the absolute errors for the second case. The Table 1 and Figures 1-2 show that the proposed method has good accuracy.

Example 2. In the second test, the analytical solution of equation (1) is given by

$$U(x,t) = 100\left(\frac{e^{\frac{Px}{L}-1}}{e^P - 1} + \frac{4\pi e^{\frac{Px}{L}}\sinh(P/2)}{e^P - 1}\sum_{m=1}^{\infty} A_m + 2\pi e^{\frac{Px}{2L}}\sum_{m=1}^{\infty} B_m\right), \quad (20)$$

where:

$$A_m = (-1)^m \frac{m}{\beta_m}\sin\left(\frac{m\pi x}{L}\right)e^{-\lambda_m t},$$

$$B_m = \left[(-1)^m \frac{m}{\beta_m}\left(1 + \frac{P}{\beta_m}\right)e^{\frac{-P}{2}} + \frac{mP}{\beta_m^2}\right]\sin\left(\frac{m\pi x}{L}\right)e^{-\lambda_m t},$$

with

$$\beta_m = \frac{p^2}{4} + (m\pi)^2 \quad \text{and} \quad \lambda_m = \frac{\alpha^2}{4\lambda} + \frac{m^2\pi^2\lambda}{L^2},$$

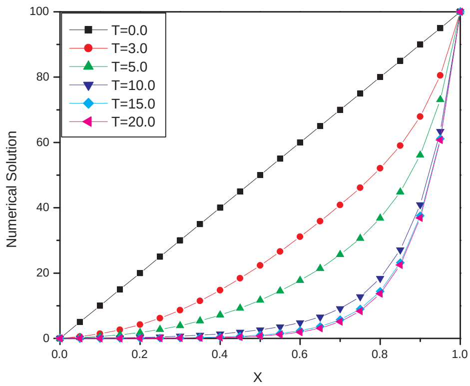

Figure 3: Numerical solution of Example 2 for $\alpha = 0.1$, $\lambda = 0.01$ at different time

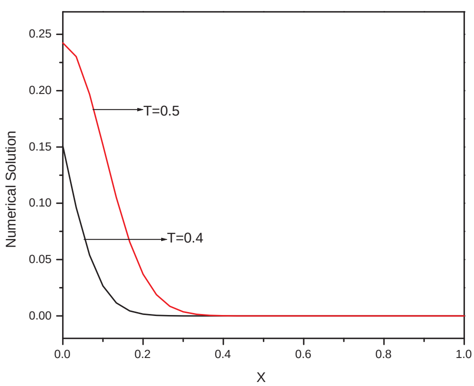

Figure 4: Numerical solution of Example 3 for $\alpha = 1.0$, $\lambda = 0.01$ at different time

T	x	Num Sol	Exact Sol	Max Error
3.0	0.25	6.19441	6.19505	
	0.50	22.34930	22.3500	6.4125
	0.75	46.15160	46.15206	
5.0	0.25	2.72582	2.72601	
	0.50	11.69460	11.69100	3.6142
	0.75	30.63180	30.63030	
10.0	0.25	0.46021	0.46009	
	0.50	2.60798	2.60830	3.2890
	0.75	12.78790	12.7904	

Table 2: Comparison of numerical and analytic solutions of Example 1 for $\alpha = 0.1, \lambda = 0.01$ with maximum absolute error

where $P = \frac{\alpha L}{\lambda}$ is the Peclet number.

In the numerical experiment, we have considered the initial and boundary conditions $U(x,0) = \frac{100x}{L}$ and $U(0,t) = 0$, $U(L,t) = 100$ with $L = 1.0$m, $\alpha = 0.1$m/s, $\lambda = 0.01m^2/s$.

In Table 2, a comparison is made between the analytical solutions and the numerical solution with maximum absolute error. The Figure 3 shows the behavior of the numerical solutions at different time and from figure it is clear

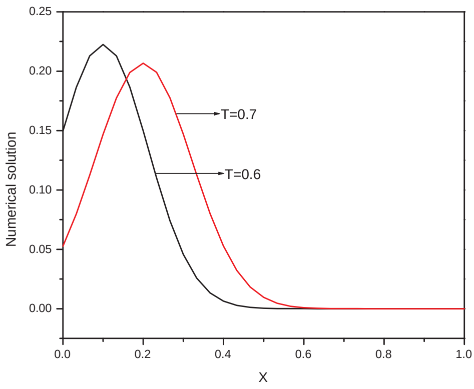

Figure 5: Numerical solution of Example 3 for $\alpha = 1.0$, $\lambda = 0.01$ at different time

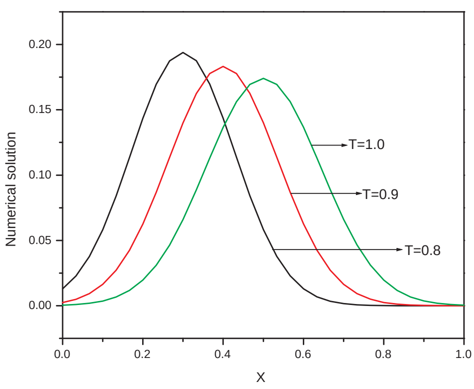

Figure 6: Numerical solution of Example 3 for $\alpha = 1.0$, $\lambda = 0.01$ at different time

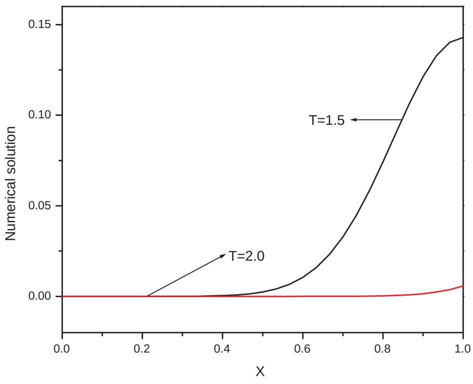

Figure 7: Numerical solution of Example 3 for $\alpha = 1.0$, $\lambda = 0.01$ at different time

that as the time increase the profile behavior of the wave decreases.

Example 3. The analytical solution of the equation (1) in the region bounded by $0 \leq x \leq 1$ is given by

$$U(x,t) = \frac{0.025}{\sqrt{0.000625 + 0.02t}} \exp\left(\frac{-(x-0.5-t)^2}{(0.00125 + 0.04t)}\right). \tag{21}$$

The initial and boundary conditions are taken from the analytical solution. The values of advection and diffusion coefficients are chosen by $\alpha = 1.0, \lambda = 0.01$.The Figures 4-7 show the behavior of numerical solutions at different times.

5. Conclusion

In this article, Galerkin-finite element method is proposed to find the numerical solutions of advection-diffusion equation. In the solution procedure, the first step is to make weak formulation and then develop finite element formulation. Lastly, weighted average is used for fully discretization. As test problem, three different solutions of advection-diffusion equation are chosen. Maximum errors norm L_∞ are calculated and found that the errors are small. Also, a comparison of numerical and analytical solutions is made and found that the proposed scheme has good accuracy.

6. Acknowledgment

One of the authors Dinkar Sharma thankfully acknowledges the financial assistance provided by MHRD India.

References

[1] M.A. Celia, T.F. Russell, I. Herrera, R.E. Ewing, An Eulerian-langrangian localized adjoint method for the advection-diffusion equation, *Advances in Water Resources*, **13** (1990), 187-206.

[2] P.C. Chatwin, C.M. Allen, Mathematical models of dispersion in rivers and estuaries, *Ann. Rev. Fluid Mech.*, **17** (1985), 119-149.

[3] M.H. Chaudhry, D.E. Cass, J.E. Edinger, Modelling of unsteady-flow water temperatures, *J. Hydraul. Eng.*, **109**, No. 5 (1983),657-669.

[4] Q.N. Fattah, J.A. Hoopes, Dispersion in anisotropic homogeneous porous media, *J. Hydraul. Eng.*, **111** (1985), 810-827.

[5] C.R. Gane, P.L. Stephenson, An explicit numerical method for solving transient combined heat conduction and convection problems, *Int. J. Numer. Meth. Eng.*, **14** (1979), 1141-1163.

[6] F.M. Holly, J.M. Usseglio-Polatera, Dispersion simulation in two-dimensional tidal flow, *J. Hydraul Eng.*, **111** (1984), 905-926.

[7] J. Isenberg, C. Gutfinger, Heat transfer to a draining film, *Int. J. Heat Transf.*, **16** (1972), 505-512.

[8] K. Kakauda, N. Tosaka, Boundary element analysis of viscous flow problems using the time splitting method, *Advances in Water Resources* (1990) 87-93.

[9] N. Kumar, Unsteady flow against dispersion in finite porous media, *J. Hydrol.*, **63** (1988), 345-358.

[10] K.W. Morton, *Numerical Solution of Convection-Advection Equation*, Chapman and Hall, London (1996).

[11] T. Neubauer, P. Bastian, On a monotonicity preserving Eulerian-Lagrangian localized adjoint method for advection-diffusion equations, *Adv. Wat. Res.*, **28**, No. 12 (2005), 1292-1309.

[12] F.B. Nico, F. Brissaud, V. Guinot, A finite volume upwind scheme for the solution of the linear advection-diffusion equation with sharp gradients in multiple dimension, *Advances in Water Resources*, **30**, No. 9 (2007), 2002-2025.

[13] B.J. Noye, *Numerical Solution of Partial Differential Equations*, Lecture Notes (1990).

[14] J.Y. Parlarge, Water transport in soils, *Ann. Rev. Fluids Mech.*, **2** (1980), 77-102.

[15] J.R. Salmon, J.A. Liggett, R.H. Gallager, Dispersion analysis in homogenous lakes, *Int. J. Numer. Meth. Eng.*, **15** (1980), 1627-1642.

[16] D.B. Spalding, A novel finite difference formulation for differential expressions involving in both first and second derivatives, *International Journal for Numerical Methods in Engineering*, **4** (1972), 551-559.

[17] Z. Zlatev, R. Berkowicz, L.P. Prahm, Implementation of a variable stepsize variable formula in the time-integration part of a code for treatment of long-range transport of air pollutants, *J. Comput. Phys.*, **55** (1984), 278-301.

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