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FUZZY GREEDOIDS

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Abstract: Greedoid theory has several intresting applications in system analysis, operations research and economics. Since most of the time the aspects of greedoid problems are uncertain, it is nice to deal with these aspects via the methods of fuzzy logic. In this paper, we introduce the notions of fuzzy feasible sets and fuzzy greedoids providing several examples. We show that the levels of the fuzzy greedoids introduced are indeed crisp greedoids. Moreover, we study some fuzzy greedoid preserving operations.

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1. Introduction

Matroid theory and greedoid theory have several interesting applications in system analysis, operations research and economics. Since most of the time the aspects of matroid and greedoid problems are uncertain, it is nice to deal with these aspects via the methods of fuzzy logic. The notion of fuzzy matroids was first introduced by Geotschel and Voxman in their landmark paper [4] using the notion of fuzzy independent set. The notion of fuzzy independent set was also explored in [12, 13]. Some constructions, fuzzy spanning sets, fuzzy rank and fuzzy closure axioms were also studied in [5, 6, 7, 16]. Several other fuzzifications of matroids were also discussed in [11, 14]. Since the notion of feasible set in traditional greedoids is one of the most significant notions that plays a very important rule in characterizing strong maps (see for example [1, 15]), in this paper we introduce the notions of fuzzy feasibles and fuzzy feeble feasible sets providing several examples. Thus fuzzy greedoids are defined via fuzzy feasible

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axioms. We show that the levels of the fuzzy greedoid introduced are indeed crisp greedoids.

Let E be any be any non-empty set. By $\wp(1)$ we denote the set of all fuzzy sets on E. That is $\wp(1) = [0,1]^E$, which is a completely distributive lattice. Thus let 0^E and 1^E denote its greatest and smallest elements, respectively. That is $0^E(e) = 0$ and $1^E(e) = 1$ for every $e \in E$. A fuzzy set μ_1 is a subset of μ_2, written $\mu_1 \leq \mu_2$, if $\mu_1(e) \leq \mu_2(e)$ for all $e \in E$. If $\mu_1 \leq \mu_2$ and $\mu_1 \neq \mu_2$, then μ_1 is a proper subset of μ_2, written $\mu_1 < \mu_2$. Moreover, $\mu_1 \prec \mu_2$ if $\mu_1 < \mu_2$ and there does not exist μ_3 such that $\mu_1 < \mu_3 < \mu_2$. Finally, $\mu_1 \vee \mu_2 = \sup\{\mu_1, \mu_2\}$ and $\mu_1 \wedge \mu_2 = \inf\{\mu_1, \mu_2\}$. The support of a fuzzy set μ is the set $supp(\mu) = \{x \in E : \mu(x) \neq 0^E\}$ and $m(\mu) := \inf\{\{\mu(x) : x \in E\} - \{0^E\}\}$.

Greedoids were invented in 1981 by Korte and Lovász [10]. Originally, the main motivation for proposing this generalization of the matroid concept came from combinatorial optimization. Korte and Lovász had observed that the optimality of a "greedy" algorithm could in several instances be traced back to an underlying combinatorial structure that was not a matroid—but (as they named it) a greedoid. In 1991, Korte, Lovász and Schrader [9] introduced greedoid as a special kind of antimatroids. In 1992, Björner and Ziegler [17] explained the basic ideas and gave a few glimpses of more specialized topics related to greedoids. In 1992, Broesma and Li [3] extended the "connectivity" concept from matroids to greedoids and in 1997, Gordon [8] extended Crapo's β invariant from matroids to greedoids. In this paper, we study properties of fuzzy greedoid deletion and contraction operations and show that these operations commute. In addition, we study some fuzzy greedoid preserving operations.

2. Fuzzy Greedoids

In this section, axioms for fuzzy greedoids are given. Properties and several examples are also provided.

Definition 1. Let E be a finite set and let $\mathfrak{F}$ be a family of fuzzy sets satisfying the following two conditions:

(F1) For every $\mu \in \mathfrak{F}$ with $\mu \neq 0^E$, there exists $x \in supp(\mu)$ such that $\mu - \{x\} \in \mathfrak{F}$, where $(\mu - \{x\})(y) = \mu(x)$, if $y \neq x$ and 0^E otherwise.

(F2) For any $\mu_1, \mu_2 \in \mathfrak{F}$ with $0^E < |supp(\mu_1)| < |supp(\mu_2)|$, then there exists $\mu \in \mathfrak{F}$ such that:

(i) $\mu_1 < \mu \leq \mu_1 \vee \mu_2$.

(ii) $m(\mu) \geq \min\{m(\mu_1), m(\mu_2)\}$.

Then the system $FG = (E, \mathfrak{F})$ is called *fuzzy greedoid* and the elements of $\mathfrak{F}$ are *fuzzy feasible sets of* FG:

Thus every fuzzy matroid [2] is a fuzzy greedoid and a fuzzy greedoid is a fuzzy matroid if and only if the axiom

(F3) If $\mu \in \mathfrak{F}$ and $\eta \leq \mu$, then $\eta \in \mathfrak{F}$.

is satisfied.

Definition 2. Let $FG = (E, \mathfrak{F})$ be a fuzzy greedoid. A map $r : [0,1]^E \to [0,\infty)$, defined by $r(\mu) = \sup p\{|\eta| : \eta \in \mathfrak{F}, \eta \leq \mu\}$ is called the *fuzzy rank function of* FG.

Thus a fuzzy set μ is feasible if and only if $r(\mu) = |\mu|$ and it is called a *fuzzy basis* if $r(\mu) = |\mu| = |E|$. The collection of all fuzzy basis of FG is denoted by $F\mathfrak{B}(FG)$. Axiom (F2) implies that bases elements have same size r (or $r(FG)$).

Definition 3. Let $FG = (E, \mathfrak{F})$ be a fuzzy greedoid. For $\mu \in \mathfrak{F}$, define

$$\mathfrak{F} \backslash \mu = \{\eta \leq [0,1]^E - \mu : \eta \in \mathfrak{F}\}$$

and, if μ is feasible, define

$$\mathfrak{F}/\mu = \{\eta \leq [0,1]^E - \mu : \eta \vee \mu \in \mathfrak{F}\}.$$

Then it is easy to see that the set systems obtained in both cases are fuzzy greedoids on the ground set $[0,1]^E - \mu$. The greedoid $FG\backslash\mu = ([E - \mu, \mathfrak{F}\backslash\mu)$ is called *FG delete μ* or the *restriction of FG to $E - \mu$* and $FG/\mu = (E - \mu, \mathfrak{F}/\mu)$ is called *FG contract μ*. For all $\eta \leq [0,1]^E - \mu$, it is easy to see that

$$r_{FG\backslash\mu}(\eta) = r(\eta) \text{ and } r_{FG/\mu}(\eta) = r(\eta \vee \mu) - r(\mu).$$

A fuzzy greedoid $FG = (E, \mathfrak{F})$ is called an *interval fuzzy greedoid* if it satisfies the *fuzzy interval property* if $\mu \leq \eta \leq \gamma$, $\mu, \eta, \gamma \in \mathfrak{F}$, $x \in [0,1]^E - \gamma$, $\mu \vee x \in \mathfrak{F}$, and $\gamma \vee x \in \mathfrak{F}$, imply that $\eta \vee x \in \mathfrak{F}$. Thus, fuzzy matroids are interval fuzzy greedoids.

One might ask whether the levels of fuzzy greedoid introduced are crisp greedoids. We shall prove in this section that this is indeed the case. For this purpose we will first recall the definition of the level of fuzzy set.

Definition 4. For $r \in (0,1]$, let $C^r(\mu) = \{e \in E | \mu(e) \geq r\}$ be the r-level of a fuzzy set $\mu \in \mathfrak{F}$, and let $\mathfrak{F}^r = \{C^r(\mu) : \mu \in \mathfrak{F}\}$ be the r-level of the family $\mathfrak{F}$ of fuzzy feasible sets. Then for $r \in (0,1]$, $(E, \mathfrak{F}^r)$ is the *r-level* of the fuzzy set system $(E, \mathfrak{F})$.

Theorem 1. *For every $r \in (0,1]$, $\mathfrak{F}^r = \{C^r(\mu) : \mu \in \mathfrak{F}\}$ the r-levels of a family of fuzzy feasible sets $\mathfrak{F}$ of a fuzzy greedoid $FG = (E, \mathfrak{F})$ is a family of crisp feasible sets.*

Proof. (F1) If $C^r(\mu) \in \mathfrak{F}^r$ such that $C^r(\mu) \neq 0^E$, then there exists $x \in E$ such that $\mu(x) \geq r > 0$. Thus $x \in supp(\mu)$ and $C^r(\mu) - \{x\} \in \mathfrak{F}^r$.

(F2) Let $C^r(\mu_1), C^r(\mu_1) \in \mathfrak{F}^r$ with $0^E < |supp(C^r(\mu_1))| < |supp(C^r(\mu_2))|$. Let $\mu = \mu_1 \vee \mu_2$. Then:

(i) $C^r(\mu_1) < C^r(\mu) \leq C^r(\mu_1 \vee \mu_2)$.

(ii) $m(C^r(\mu)) \geq \min\{m(C^r(\mu_1)), m(C^r(\mu_2))\}$. □

To illustrate the preceding result, we next provide a non-trivial example of fuzzy set system $(E, \mathfrak{F})$, satisfying the axioms (i) and (ii), and to show that the levels of the fuzzy greedoid are crisp greedoids.

Example 1. Let $E = \{a, b\}$ and let $\mu_1, \mu_2, \mu_3, \mu_4$ be a family of fuzzy sets defined as follows:

$$\mu_1 = \begin{cases} 1/2 & , e = a \\ 1/2 & , e = b \end{cases}, \mu_2 = \begin{cases} 1 & , e = a \\ 1/2 & , e = b \end{cases}$$

$$\mu_3 = \begin{cases} 1/2 & , e = a \\ 1 & , e = b \end{cases}, \mu_4 = \begin{cases} 1 & , e = a \\ 1 & , e = b \end{cases}$$

Example 2. Consider $\mathfrak{F} = \{\mu :$ for every $x \in \sup p(\mu)$, $\mu - \mu_i = \{x\}$ for some $i = 1, 2, 3, 4\}$. For $r \in (0, 1/2]$, $C^r(\mu_1) = C^r(\mu_2) = C^r(\mu_3) = C^r(\mu_4) = \phi$. For $r \in (1/2, 1]$, $C^r(\mu_1) = \emptyset$, $C^r(\mu_2) = \{a\}$, $C^r(\mu_3) = \{b\}$ and $C^r(\mu_4) = E$. Therefore the fuzzy set system $(E, \mathfrak{F})$, has two distinct levels: $\mathfrak{F}^r = \{\phi\}$, for $r \in (0, 1/2]$ and $\mathfrak{F}^r = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$, for $r \in (1/2, 1]$.

It is easy to see that $(E, \mathfrak{F}^r)$ are crisp greedoids for $r \in (0, 1]$ (two distinct fuzzy greedoids and therefore two distinct families of crisp feasibles).

Next, we explore alternative ways to define several interesting fuzzy greedoids.

Theorem 2. *Let $M(E, \mathfrak{F}')$ be a matroid and $\mathfrak{F} = \{\chi_\lambda : \lambda \in \mathfrak{F}'\}$. Then $FM = (E, \mathfrak{F}')$ is a fuzzy matroid [2] and thus a fuzzy greedoid.*

We remark that $(E, \mathfrak{F} = \{\chi_\lambda : \lambda \leq E\})$ need not be a fuzzy greedoid and as we have seen in Theorem 2 starting with a greedoid, the set of all characteristic functions with feasible sets as bases is a fuzzy greedoid. Hence any greedoid generates a unique fuzzy greedoid.

Example 3. Let E be any set with n-elements and $\mathfrak{F} = \{\chi_\lambda : \lambda \leq E, |\lambda| = n \text{ or } |\lambda| < m\}$ where m is a positive integer such that $m \leq n$. Then $(E, \mathfrak{F})$ is a fuzzy matroid [2] and hence a fuzzy greedoid.

Definition 5. The fuzzy greedoid described in Example 3 is called the *fuzzy uniform greedoid on n-elements and rank m*, denoted by $F_{m,n}$. $F_{m,m}$ is called the *free fuzzy uniform greedoid on n-elements.*

Next, several constructions of fuzzy greedoids are discussed. We first prove that given a non-zero $X \subseteq E$, every fuzzy greedoid on E generates a fuzzy greedoid on X. Its called the *induced* (or *relative*) fuzzy greedoid on X.

Theorem 3. *Let $FM = (E, \mathfrak{F})$ be a fuzzy matroid and X be a non-empty subset of E. Then $(X, \mathfrak{F}_X)$ is a fuzzy greedoid, where $\mathfrak{F}_X = \{\mu : \forall x \in \mu, \mu - \chi_X \wedge \eta = \{x\}, \eta \in \mathfrak{F}\}$.*

Proof. Follows from the fact that $FM = (E, \mathfrak{F}_X = \{\chi_X \wedge \mu : \mu \in \mathfrak{F}\})$ is a fuzzy matroid, see [2]. □

Let $FG = (E, \mathfrak{F})$ be a fuzzy greedoid, X be a non-empty subset of E and μ be a fuzzy set in X. We may realize μ as a fuzzy set in E by the convention that $\mu(e) = 0$ for all $e \in E - X$. It can be easily shown that $\mathfrak{F}_X = \{\mu|_X : \mu \in \mathfrak{F}\}$, where $\mu|_X$ is the restriction of μ to X.

Given any non-empty set E and a point $e \in E$, we clearly note that $G = (E, \mathfrak{F}')$ is a greedoid where $\mathfrak{F}' = \{\mu \leq E : e \in \mu\}$. This greedoid is called the *point greedoid determined by e*. It heavily depends on the cardinality of E. We say e is a *fuzzy singleton on E* if $e(x) = 1$ for all $x \in E$ except one. Next, we define the corresponding *fuzzy point greedoid.*

Theorem 4. *Let E be a non-empty set and e be a fuzzy singleton on E. Then $(E, \mathfrak{F}_e)$ is a fuzzy greedoid, where $\mathfrak{F}_e = \{\mu : \forall x \in \mu, \mu - \eta = \{x\}, \eta \in \cup\{\mu : e \leq \mu, \mu \in \wp(1)\}\}$.*

Proof. Follows from the fact that $(E, \vee\{\mu : e \leq \mu, \mu \in \wp(1)\})$ is a fuzzy matroid, see [2]. □

The fuzzy greedoid described in the preceding theorem is called the *induced fuzzy singleton e-greedoid on E* or simply *fuzzy e-greedoid.* Another type of fuzzy point greedoids is given next.

Theorem 5. *Let E be a non-empty set and e be a fuzzy singleton on E. Then $(E, \mathfrak{F}^e)$ is a fuzzy greedoid, where $\mathfrak{F}^e = \{\mu : \forall x \in \mu, \mu - \eta = \{x\}, \eta \in \{1^E\} \cup \{\mu : e \leq 1^E - \mu, \mu \in \wp(1)\}\}$.*

Proof. Follows from the fact that $(E, \{1^E\} \cup \{\mu : e \leq 1^E - \mu, \mu \in \wp(1)\})$ is a fuzzy matroid, see [2]. □

We end this section by giving non trivial examples of fuzzy greedoids based on the concept of fuzzy feasible sets.

Example 4. Consider $E = \{a, b, c\}$ and $\mathfrak{F} = \{\mu : \forall x \in \mu, \mu - \eta = \{x\}, \eta \in \{1^E - \mu : \mu(x) = a_x, a_x \in [0,1]\}\}$.That is $\mathfrak{F}$ is the collection of all complements of constant maps. Then $(E, \mathfrak{F})$ is a fuzzy greedoid.

Operations as basic as deletion and contraction are those of *direct sum* and *ordered sum*. Let $FG_1 = (E_1, \mathfrak{F}_1)$ and $FG_2 = (E_2, \mathfrak{F}_2)$ be two fuzzy greedoids on disjoint ground sets. Then their direct sum is the greedoid $FG_1 \oplus FG_2 = (E_1 \cup E_2, \mathfrak{F}_1 \oplus \mathfrak{F}_2)$, where

$$\mathfrak{F}_1 \oplus \mathfrak{F}_2 = \{\mu_1 \vee \mu_2 \, : \, \mu_1 \in \mathfrak{F}_1 \text{ and } \mu_2 \in \mathfrak{F}_2\},$$

and the ordered sum of FG_1 and FG_2 is the greedoid $FG_1 \otimes FG_2 = (E_1 \cup E_2, \mathfrak{F}_1 \otimes \mathfrak{F}_2)$, where

$$\mathfrak{F}_1 \otimes \mathfrak{F}_2 = \mathfrak{F}_1 \cup \{B \vee \mu : \, B \in \mathcal{B}(FG_1), \, \mu \in \mathfrak{F}_2\}.$$

Observe that $0 \in \mathfrak{F}_1 \cap \mathfrak{F}_2$ and $\mathfrak{F}_1 \otimes \mathfrak{F}_2 \subseteq \mathfrak{F}_1 \oplus \mathfrak{F}_2$, thus $FG_1 \otimes FG_2$ is a subgreedoid of $FG_1 \oplus FG_2$.

3. Deletion and Contraction Fuzzy Greedoids

In this section, we study properties of fuzzy greedoid deletion and contraction operations and show that these operations commute. We start by proving the following.

Proposition 1. *If μ_λ is a fuzzy basis for the restriction $FG|\lambda$ of FG to λ, then*

$$\begin{aligned} \mathfrak{F}(FG/\lambda) &= \{\eta \in E - \lambda \, : \, FG|\lambda \text{ has a fuzzy basis } \delta \text{ such that } \eta \vee \delta \in \mathfrak{F}(FG)\} \\ &= \{\eta \in E - \lambda \, : \, \eta \vee \mu_\lambda \in \mathfrak{F}(FG)\}. \end{aligned}$$

Proof. Clearly $\{\eta \in E - \lambda \, : \, FG|\lambda$ has a fuzzy basis δ such that $\eta \vee \delta \in \mathfrak{F}(FG)\}$ contains the set $\{\eta \in E - \lambda \, : \, \eta \vee \mu_\lambda \in \mathfrak{F}(FG)\}$. Suppose $\eta \vee \delta \in \mathfrak{F}(FG)$ for some fuzzy basis δ of $FG|\lambda$. We shall show that $\eta \in \mathfrak{F}(FG/\lambda)$. Clearly $\eta \vee \delta$ is a fuzzy basis of $\eta \cup \lambda$, so $r(\eta \vee \delta) = r(\delta \cup \lambda)$. Therefore,

$$r_{FG/\lambda}(\eta) = r(\delta \cup \lambda) - r(\mu_\lambda) = r(\eta \vee \delta) - r(\delta) = |\eta \vee \delta| - |\delta| = |\eta|,$$

that is, $\eta \in \mathfrak{F}(FG/\lambda)$. Hence,

$$\begin{aligned}&\mathfrak{F}(FG/\lambda)\\ &= \{\eta \in E-\lambda : FG|\lambda \text{ has a fuzzy basis } \delta \text{ such that } \eta\vee\delta \in \mathfrak{F}(FG)\} \subseteq \mathfrak{F}(FG/\lambda).\end{aligned}$$

Finally we show $\{\eta \in E - \lambda : \eta \vee \mu_\lambda \in \mathfrak{F}(FG)\}$ contains $\mathfrak{F}(FG/\lambda)$. If $\eta \in \mathfrak{F}(FG/\lambda)$, then

$$\begin{aligned}|\eta| &= r_{FG/\lambda}(\eta)\\ &= r(\eta \cup \lambda) - r(\lambda)\\ &= r(\eta \vee \mu_\lambda) - |\mu_\lambda|.\end{aligned}$$

Hence $|\eta \vee \mu_\lambda| = r(\eta \vee \mu_\lambda)$, so $\eta \vee \mu_\lambda \in \mathfrak{F}(FG)$. □

Corollary 1. *If μ_λ is a fuzzy basis for $FG|\lambda$, then a fuzzy bases of FG/λ is*

$$\begin{aligned}\mathcal{B}(FG/\lambda) &= \{\eta \in E - \lambda : FG|\lambda \textit{ has a fuzzy basis } \delta \textit{ such that } \eta \vee \delta \in \mathcal{B}(FG)\}\\ &= \{\eta \in E - \lambda : \eta \vee \mu_\lambda \in \mathcal{B}(FG)\}.\end{aligned}$$

Observe that $\mathfrak{F}(FG/\lambda) \subseteq \mathfrak{F}(FG\backslash\lambda)$ for every fuzzy feasible set λ in FG. Next, we give a necessary and sufficient condition for the contraction of a fuzzy feasible set to be the same as the deletion of that set.

Proposition 2. *If λ is a fuzzy feasible set in FG, then*

$$FG/\lambda = FG\backslash\lambda \textit{ if and only if } r(FG\backslash\lambda) = r(FG) - r(\lambda).$$

Proof. Suppose $FG/\lambda = FG\backslash\lambda$ and let μ be a fuzzy basis of $FG\backslash\lambda$. Then μ is a fuzzy basis of FG/λ and hence by Corollary 1, $B\dot\cup\mu_\lambda$ is a fuzzy basis of FG for some fuzzy basis μ_λ of $FG|\lambda$. Thus

$$\begin{aligned}r(FG) &= |\mu\dot\cup\mu_\lambda|\\ &= |\mu| + |\mu_\lambda|\\ &= r(\lambda) + r(FG\backslash\lambda).\end{aligned}$$

Suppose $r(FG\backslash\lambda) = r(FG) - r(\lambda)$. Since $\mathfrak{F}(FG/\lambda) \subseteq \mathfrak{F}(FG\backslash\lambda)$, to show $FG/\lambda = FG\backslash\lambda$, we need only show $\mathfrak{F}(FG\backslash\lambda) \subseteq \mathfrak{F}(FG/\lambda)$. But if $\mu \in \mathfrak{F}(FG\backslash\lambda)$, then $\mu \leq \eta$ for a fuzzy basis η of $FG\backslash\lambda$ and η is contained in a fuzzy basis $\eta\dot\cup\delta$ of FG. Evidently

$$r(FG) = |\delta\dot\cup\eta|$$

$$
\begin{aligned}
&= |\eta| + |\delta| \\
&= r(FG\backslash\lambda) + |\delta|.
\end{aligned}
$$

Since $r(FG\backslash\lambda) = r(FG) - r(\lambda)$, we have $r(\lambda) = |\delta|$, that is, δ is a fuzzy basis of $FG|\lambda$. Hence $\eta \in \mathcal{B}(FG/\lambda)$, so $\mu \in \mathfrak{F}(FG/\lambda)$ and $FG/\lambda = FG\backslash\lambda$. □

Corollary 2. *For all $\lambda \in \mathfrak{F}$, $FG/\lambda = FG\backslash\lambda$ if and only if $r(FG\backslash\lambda) \leq r(FG/\lambda)$.*

Proof. If $FG/\lambda = FG\backslash\lambda$, then clearly $r(FG\backslash\lambda) \leq r(FG/\lambda)$. If $r(FG\backslash\lambda) \leq r(FG/\lambda)$, then as $\mathfrak{F}(FG/\lambda)$ is a fuzzy subset of $\mathfrak{F}(FG\backslash\lambda)$ we must have $r(FG\backslash\lambda) \geq r(FG/\lambda)$. Thus $FG/\lambda = FG\backslash\lambda$. □

In the next proposition, we show that the operations of deletion and contraction commute.

Proposition 3. *Let $FG = (E, \mathfrak{F})$ be a fuzzy greedoid. Then*

$$(FG\backslash\acute{\lambda})/\lambda = (FG/\lambda)\backslash\acute{\lambda} = \{\mu \in E - (\acute{\lambda} \vee \lambda) \,:\, \mu \vee \lambda \in \mathfrak{F}\},$$

for $\lambda \wedge \acute{\lambda} = 0^E$, $\lambda \in \mathfrak{F}$ and $\acute{\lambda} \in E$.

Proof. We need only show $(FG\backslash\acute{\lambda})/\lambda$ and $(FG/\lambda)\backslash\acute{\lambda}$ have the same collections of fuzzy feasible sets. If $\mu \in \mathfrak{F}_{(FG\backslash\acute{\lambda})/\lambda}$, then $\mu \in (E - \acute{\lambda}) - \lambda$ and $\mu \vee \lambda \in \mathfrak{F}$. That is, $\mu \in (E - \lambda) - \acute{\lambda}$ and $\mu \in \mathfrak{F}_{FG/\lambda}$ and hence $\mu \in \mathfrak{F}_{(FG/\lambda)\backslash\acute{\lambda}}$. Conversely, if $\mu \in \mathfrak{F}_{(FG/\lambda)\backslash\acute{\lambda}}$, then $\mu \in (E - \lambda) - \acute{\lambda}$ and $\mu \in \mathfrak{F}_{FG/\lambda}$. That is, $\mu \in (E - \acute{\lambda}) - \lambda$ and $\mu \vee \lambda \in \mathfrak{F}$ and hence $\mu \in \mathfrak{F}_{(FG\backslash\acute{\lambda})/\lambda}$. Therefore, $\mathfrak{F}_{(FG\backslash\acute{\lambda})/\lambda} = \mathfrak{F}_{(FG/\lambda)\backslash\acute{\lambda}}$. □

The straightforward proof of the following proposition is omitted.

Proposition 4. *$\{\mu_1 \vee \mu_2 \,:\, \mu_1 \in \mathcal{B}(FG_1)$ and $\mu_2 \in \mathcal{B}(FG_2)\} = \mathcal{B}(FG_1 \otimes FG_2)$ which is equal to $\mathcal{B}(FG_1 \oplus FG_2)$.*

Corollary 3. *Let $FG_1 = (E_1, \mathfrak{F}_1)$ and $FG_2 = (E_2, \mathfrak{F}_2)$ be fuzzy greedoids on disjoint ground sets. If $\mu \in E_1 \cup E_2$, then*

$$r_{FG_1 \otimes FG_2}(\mu) = r_{FG_1 \oplus FG_2}(\mu) = r_{FG_1}(\mu \wedge E_1) + r_{FG_2}(\mu \wedge E_2).$$

4. On Fuzzy Greedoid Preserving Operations

In this section, we prove the operations of direct sum and ordered sum take interval fuzzy greedoids to interval fuzzy greedoids. In fact, we show that the direct sum and ordered sum of fuzzy greedoids FG_1 and FG_2 is an interval fuzzy greedoid if and only if FG_1 and FG_2 are both interval fuzzy greedoids.

Theorem 6. *Let $FG_1 = (E_1, \mathfrak{F}_1)$ and $FG_2 = (E_2, \mathfrak{F}_2)$ be fuzzy greedoids on disjoint ground sets. Then FG_1 and FG_2 are interval fuzzy greedoids if and only if $FG_1 \oplus FG_2$ is an interval fuzzy greedoid.*

Proof. Suppose FG_1 and FG_2 are interval fuzzy greedoids. If $\lambda \leq \mu \leq \delta$, $\lambda, \mu, \delta \in \mathfrak{F}_1 \oplus \mathfrak{F}_2$, $x \in E_1 \cup E_2 - C$, $\lambda \vee x \in \mathfrak{F}_1 \oplus \mathfrak{F}_2$, and $\delta \vee x \in \mathfrak{F}_1 \oplus \mathfrak{F}_2$, then $\lambda = \lambda_1 \vee \lambda_2$, $\mu = \mu_1 \vee \mu_2$, $\delta = \delta_1 \vee \delta_2$ where $\lambda_i, \mu_i, \delta_i$ are fuzzy feasible sets in FG_i for $i = 1, 2$, $\lambda_i \vee x \in \mathfrak{F}_i$ (as $\lambda_i \vee x = (\lambda_1 \vee \lambda \vee x) \cap E_i$). Similarly, $\delta_i \vee x \in \mathfrak{F}_i$. Moreover, $x \in (E_1 \cup E_2 - \delta_1) \wedge (E_1 \cup E_2 - \delta_2)$. Hence suppose $x \in E_i - \delta_i$ for $i = 1$ or $i = 2$ and as $\lambda_i \leq \mu_i \leq \delta_i$, $\delta_i \vee x \in \mathfrak{F}_i$. But

$$\delta \vee x = \delta_1 \vee \delta_2 \vee x = (\delta_1 \vee x) \vee \delta_2 \in \mathfrak{F}_1 \oplus \mathfrak{F}_2.$$

Therefore, $FG_1 \oplus FG_2$ is an interval greedoid.

Suppose $FG_1 \oplus FG_2$ is an interval greedoid. If $\lambda \leq \mu \leq \delta$, $\lambda, \mu, \delta \in \mathfrak{F}_1$, x an element in $E_1 - \delta$, $\lambda \vee x \in \mathfrak{F}_1$, and $\delta \vee x \in \mathfrak{F}_1$, then as $0^E \in \mathfrak{F}_2$, $\lambda \vee 0^E \leq \mu \vee 0^E \leq \delta \vee 0^E$, $\lambda \vee 0^E, \mu \vee 0^E, \delta \vee 0^E \in \mathfrak{F}_1 \oplus \mathfrak{F}_2$, $x \in E_1 \cup E_2 - \delta$, $(\lambda \vee x) \vee 0^E, (\delta \vee x) \vee 0^E \in \mathfrak{F}_1 \oplus \mathfrak{F}_2$ and as $FG_1 \oplus FG_2$ is an interval greedoid, $\delta \vee x = (\delta \vee 0^E) \vee x \in \mathfrak{F}_1 \oplus \mathfrak{F}_2$. But $\delta \vee x = (\delta \vee x) \cap E_1 \in \mathfrak{F}_1$ and hence FG_1 is an interval fuzzy greedoid. Similarly, FG_2 is an interval fuzzy greedoid. □

Theorem 7. *Let $FG_1 = (E_1, \mathfrak{F}_1)$ and $FG_2 = (E_2, \mathfrak{F}_2)$ be fuzzy greedoids on disjoint ground sets. Then FG_1 and FG_2 are interval fuzzy greedoids if and only if $FG_1 \otimes FG_2$ is an interval fuzzy greedoid.*

Proof. The proof of the necessary condition is similar to that of the direct sum one in the preceding theorem and is left to the reader. Suppose $FG_1 \otimes FG_2$ is an interval greedoid. If $\lambda \leq \mu \leq \delta$, $\lambda, \mu, \delta \in \mathfrak{F}_1$, $x \in E_1 - \delta$, $\lambda \vee x \in \mathfrak{F}_1$, and $\delta \vee x \in \mathfrak{F}_1$, then $\lambda, \mu, \delta \in \mathfrak{F}_1 \otimes \mathfrak{F}_2$, $x \in E_1 \cup E_2 - \delta$, $\lambda \vee x, \delta \vee x \in \mathfrak{F}_1 \otimes \mathfrak{F}_2$ and as $FG_1 \otimes FG_2$ is an interval fuzzy greedoid, $\delta \vee x \in \mathfrak{F}_1 \otimes \mathfrak{F}_2$. But as $\mathfrak{F}_1 \otimes \mathfrak{F}_2 \subseteq \mathfrak{F}_1 \oplus \mathfrak{F}_2$, $\delta \vee x \in \mathfrak{F}_1 \oplus \mathfrak{F}_2$. Thus $\delta \vee x = (\delta \vee x) \cap E_1 \in \mathfrak{F}_1$ and hence FG_1 is an interval fuzzy greedoid. Similarly, FG_2 is an interval fuzzy greedoid. □

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