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THE LERAY-LIONS EQUATION AND ITS DOUBLE OBSTACLE PROBLEM WITH VARIABLE EXPONENT

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Abstract: The relation between the solutions to the Leray-Lions equation and the solutions to its double obstacle problem with variable exponent has been studied. The Caccioppoli estimate of the solution to double obstacle problem has been obtain, which is important in studying the integrablity of the solutions.

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1. Introduction

Partial differential equations with nonlinearities involving nonconstant exponents have attracted an increasing amount of attention in recent years. The development, mainly by [1], of a theory modeling the behavior of electrorheological fluids, an important class of non-Newtonian fluids, seems to have boosted a still far from completed effort to study and understand nonlinear PDEs involving variable exponents. This, in turn, gave rise to a revival of the interest in Lebesgue and Sobolev spaces with variable exponent.

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There has been new interesting development in the study of the Leray-Lions equation, largely pertaining to applications in quasiformal analysis and nonlinear elasticity, that is

$$divA\,(x, \nabla u) = 0 \tag{1.1}$$

where the operator $A : \Omega \times \mathbf{R}^n \to \mathbf{R}^n$ satisfies the conditions

$$\begin{gathered} |A\,(x,\xi)\,| \leq \alpha|\xi|^{p-1}, \quad \langle A\,(x,\xi)\,,\xi\rangle \geq \beta|\xi|^p, \\ \langle A(x,\xi) - A(x,\zeta), \xi - \zeta\rangle \geq 0. \end{gathered} \tag{1.2}$$

for almost every $x \in \Omega$ and all $\xi, \zeta \in \mathbf{R}^n$. Here $\alpha, \beta > 0$ are constants and $1 < p < \infty$ is a fixed exponent associated with (1.1).

In this paper, we consider the Leray-Lions equation with potential applications to the modeling of combustion, thermal explosions, nonlinear heat generation, gravitational equilibrium of polytropic stars, glaciology, non-Newtonian fluids, and the flow through porous media. Many of properties of the Leray-Lions equation have already been analyzed for constant exponents of nonlinearity (cf. [2]-[5], and the references therein) but it seems to be more realistic to assume the exponent to be variable.

The aim of the present paper is to study the Leray-Lions equation of type (1.1) with the more general growth conditions than (1.2), that is, we assume the operator $A : \Omega \times \mathbf{R}^n \to \mathbf{R}^n$ satisfies the following assumptions

$$\langle A(x,\xi), \xi\rangle \geq \alpha|\xi|^{p(x)}; \tag{1.3}$$

$$|A(x,\xi)| \leq \beta|\xi|^{p(x)-1}; \tag{1.4}$$

for almost every $x \in \Omega$ and for every $\xi \in \mathbf{R}^n$, where α, β is a positive constant;

$$\langle A(x,\xi) - A(x,\zeta), \xi - \zeta\rangle > 0; \tag{1.5}$$

for almost every $x \in \Omega$ and for every $\xi, \zeta \in \mathbf{R}^n$, with $\xi \neq \zeta$.

Concerning the exponent $p(\cdot)$ appearing in (1.3) and (1.4) , it is a measurable function $p(\cdot) : \Omega \to \mathbf{R}^n$ such that

$$p(\cdot) \in W^{1,\infty}(\Omega) \qquad \text{and} \qquad 1 < \inf_{x\in\Omega} p(x) \leq \sup_{x\in\Omega} p(x) \leq n. \tag{1.6}$$

The following is a brief overview of the state of the art concerning Lebesgue spaces with variable exponent, and Sobolev spaces modeled upon them. Given a measurable function $p(\cdot) : \Omega \to [1, +\infty)$, we will use the following notation throughout the paper:

$$p_- = \inf_{x\in\Omega} p(x) \quad \text{and} \quad p_+ = \sup_{x\in\Omega} p(x). \tag{1.7}$$

Let

$$L^{p(x)}(\Omega) = \left\{ u \middle| u \text{ is a measurable real valued function on } \Omega, \int_\Omega |u|^{p(x)} dx < \infty \right\},$$

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) \middle| |\nabla u| \in L^{p(x)}(\Omega) \right\}.$$

We can introduce norms on $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$, respectively, as

$$\|u\|_{p(x)} = \inf \left\{ \lambda > 0 \middle| \int_\Omega \left| \frac{u(x)}{\lambda} \right| dx \leq 1 \right\}, u \in L^{p(x)}(\Omega)$$

and

$$\|u\|_{1,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}$$

then $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ and $(W^{1,p(x)}(\Omega), |\cdot|_{p(x)})$ are both separable and reflexive Banach spaces.

Denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$; we know that $|\nabla u|_{p(x)}$ is an equivalent norm on $W_0^{1,p(x)}(\Omega)$.

In this work we do not study variable exponent spaces themselves, but rather related differential equations. For our purposes, the most important facts about the variable exponent Lebesgue spaces are the following. If E is a measurable set with a finite measure, p and q are variable exponents satisfying $q(x) \leq p(x)$ for almost every $x \in E$, then $L^{p(\cdot)}(E)$ embeds continuously into $L^{q(\cdot)}(E)$ and the norm of the embedding cannot exceed $1 + |E|$. We use also a variable exponent version of Hölders inequality

$$\int_\Omega fg dx \leq C \|f\|_{p(\cdot)} \|g\|_{p'(\cdot)} \tag{1.8}$$

where the constant C depends only on p^- and p^+. Here $1/p(x) + 1/p'(x) = 1$ for every x. For all these facts we refer the reader to [9].

2. The Relation between the Solutions to (1.1) and the Solutions to its Double Obstacle Problem

By the similar definitions as the solution, supersolution (or subsolution) to quasilinear elliptic equation, we can give the definitions of the solution, supersolution (or subsolution) to (1.1).

Definition 2.1. If a function $u \in W^{1,p(x)}_{loc}(\Omega)$ satisfies

$$\int_\Omega \langle A(x, \nabla u), \nabla \varphi \rangle dx = 0, \tag{2.1}$$

for any $\varphi \in W_0^{1,p(x)}(\Omega)$, then we say that u is a solution to (1.1). If for any $0 \leq \varphi \in W_0^{1,p(x)}(\Omega)$, we have

$$\int_\Omega \langle A(x, \nabla u), \nabla \varphi \rangle dx \geq 0 (\leq 0), \tag{2.2}$$

then we say that u is a supersolution(subsolution) to (1.1).

We can see that if u is a subsolution to (1.1), then for $0 \geq \varphi \in W_0^{1,p(x)}(\Omega)$, we have

$$\int_\Omega \langle A(x, \nabla u), \nabla \varphi \rangle dx \geq 0. \tag{2.3}$$

A function u is a subsolution in Ω if $-u$ is a supersolution in Ω, and a solution in Ω if it is both a supersolution and a subsolution in Ω.

According to the above definitions, we can get the following theorem.

Theorem 2.1. *A function $u \in W^{1,p(x)}_{loc}(\Omega)$ is a solution to (1.1) if and only if u is both supersolution and subsolution to(1.1).*

Proof. The sufficiency is obvious, we only prove the necessity. For any $\varphi \in W_0^{1,p(x)}(\Omega)$, let

$$\varphi_1 = \varphi^+ \geq 0, \qquad \varphi_2 = \varphi^- \leq 0.$$

By definition 2.1, for u is a supersolution to (1.1), it holds that

$$\int_\Omega \langle A(x, \nabla u), \nabla \varphi_1 \rangle dx \geq 0, \tag{2.4}$$

and for u is a subsolution to (1.1), we have

$$\int_\Omega \langle A(x, \nabla u), \nabla \varphi_2 \rangle dx \geq 0 \tag{2.5}$$

So

$$\begin{aligned} 0 &\leq \int_\Omega \langle A(x, \nabla u), \nabla \varphi_1 \rangle dx + \int_\Omega \langle A(x, \nabla u), \nabla \varphi_2 \rangle dx \\ &= \int_\Omega \langle A(x, \nabla u), \nabla \varphi_1 + \nabla \varphi_2 \rangle dx \\ &= \int_\Omega \langle A(x, \nabla u), \nabla \varphi \rangle dx. \end{aligned} \tag{2.6}$$

Using $-\varphi$ in place of φ, we also can get

$$\int_\Omega \langle A(x, \nabla u), \nabla \varphi \rangle dx \leq 0. \tag{2.7}$$

Thus

$$\int_\Omega \langle A(x, \nabla u), \nabla \varphi \rangle dx = 0. \tag{2.8}$$

Therefore u is a solution to (1.1).

The next we will introduce the double obstacle problem to (1.1), which definition is according to the similar definition as the obstacle problem of quasilinear elliptic equation. We refer the reader to [7] for some details of onstacle problem of quasilinear elliptic equation.

Supposed that Ω is a bounded domain. ψ_1, ψ_2 are any functions in Ω with values in the extended reals $[-\infty, \infty]$, and $\theta \in W^{1,p(x)}(\Omega)$ is a function which gives the boundary values. Let

$$K^{\theta,p(x)}_{\psi_1,\psi_2}(\Omega) = \left\{ v \in W^{1,p(x)}(\Omega) : \psi_1 \leq v \leq \psi_2 \ a.e., v - \theta \in W^{1,p(x)}_0(\Omega) \right\}.$$

The problem is to find a function in $K^{\theta,p(x)}_{\psi_1,\psi_2}(\Omega)$ such that for any $v \in K^{\theta,p(x)}_{\psi_1,\psi_2}(\Omega)$ we have

$$\int_\Omega \langle A(x, \nabla u), \nabla (v - u) \rangle dx \geq 0. \tag{2.9}$$

Definition 2.2. *A function $u \in K^{\theta,p(x)}_{\psi_1,\psi_2}(\Omega)$ is called a solution to the double obstacle problem of Leray-Lions equation (1.1) with obstacles ψ_1, ψ_2 and boundary values θ or a solution to the double obstacle problem of Leray-Lions equation (1.1) in $K^{\theta,p(x)}_{\psi_1,\psi_2}(\Omega)$ if u satisfies (2.9) for any $v \in K^{\theta,p(x)}_{\psi_1,\psi_2}(\Omega)$.*

If $u \in K^{\theta,p(x)}_{\psi_1,\psi_2}(\Omega)$ is a solution to the double obstacle problem of (1.1), then for any constant $\varepsilon > 0$, we have $u - \varepsilon \in K^{\theta-\varepsilon,p(x)}_{\psi_1-\varepsilon,\psi_2-\varepsilon}(\Omega)$ is a solution to the double obstacle problem of (1.1).

A function $u \in K^{\theta,p(x)}_{\psi_1,\psi_2}(\Omega)$ is a solution of the obstacle problem if and only if it is a quasiminimizer of the Dirichlet energy integral among the functions in $K^{\theta,p(x)}_{\psi_1,\psi_2}(\Omega)$. This is shown in the usual way. Explicitly, if $u \in K^{\theta,p(x)}_{\psi_1,\psi_2}(\Omega)$ is a solution of the obstacle problem, then

$$\begin{aligned}
\alpha\int_\Omega |\nabla u|^{p(x)}dx &\le \int_\Omega \langle A(x,\nabla u),\nabla u\rangle dx \\
&\le \int_\Omega \langle A(x,\nabla u),\nabla v\rangle dx \\
&\le \int_\Omega |A(x,\nabla u)||\nabla v| dx \\
&\le \beta\int_\Omega |\nabla u|^{p(x)-1}|\nabla v|dx \\
&\le \beta\int_\Omega \left(\frac{\varepsilon(p(x)-1)}{p(x)}|\nabla u|^{p(x)}+\frac{\varepsilon^{1-p(x)}}{p(x)}|\nabla v|^{p(x)}\right)dx \\
&\le \beta\frac{\varepsilon(p^+-1)}{p^-}\int_\Omega |\nabla u|^{p(x)}dx+\beta\frac{\varepsilon^{1-p_+}}{p_-}\int_\Omega |\nabla v|^{p(x)}dx,
\end{aligned} \tag{2.10}$$

for every $v\in K^{\theta,p(x)}_{\psi_1,\psi_2}(\Omega)$, where we have used Young's inequalirt. Moving all the terms with u to the left hand side, and let ε satisfies that $\beta\frac{\varepsilon(p^+-1)}{p^-}=\frac{\alpha}{2}$, we see that

$$\int_\Omega |\nabla u|^{p(x)}dx \le C(\alpha,\beta,p^-,p^+)\int_\Omega |\nabla v|^{p(x)}dx \tag{2.11}$$

So it is a quasiminimizer of the Dirichlet energy integral among the functions in $K^{\theta,p(x)}_{\psi_1,\psi_2}(\Omega)$.

We have some relations between the solution to quasilinear elliptic equation and the solution to obstacle problem in PDE. As to the Leray-Lions equation and its double obstacle problem with variable exponent, we also have some relations between the solution to nonhomogeneous Leray-Lions equation and the solution to obstacle problem of it. We have the following two theorems.

Theorem 2.2. *If u is a supersolution to (1.1), then u is a solution to the obstacle problem of (1.1) in $K^{u,p(x)}_{u,\psi_2}(\Omega)$.*

Proof. If u is a supersolution to (1.1) in Ω then for any $v\in K^{u,p(x)}_{u,\psi_2}(\Omega)$, we have $v-u\ge 0$, $v-u\in W_0^{1,p(x)}(\Omega)$. Thus let

$$\varphi = v-u, \tag{2.15}$$

we have

$$0\le \int_\Omega \langle A(x,\nabla u),\nabla\varphi\rangle dx=\int_\Omega \langle A(x,\nabla u),\nabla v-\nabla u\rangle dx. \tag{2.16}$$

So u is a solution to the obstacle problem of (1.1) in $K^{u,p(x)}_{u,\psi_2}(\Omega)$.

Theorem 2.3. *A function $u \in W^{1,p(x)}(\Omega)$ is a solution to (1.1) if only if u is a solution to the obstacle problem of (1.1) in $K^{\theta,p(x)}_{-\infty,+\infty}(\Omega)$ with θ satisfying $u - \theta \in W^{1,p(x)}_0(\Omega)$.*

Proof. If u is a solution to the obstacle problem of (1.1) in $K^{\theta,p(x)}_{-\infty,+\infty}(\Omega)$, then for any $\theta \in W^{1,p(x)}_0(\Omega)$, let $v = u + \varphi$, then

$$v - \theta = u - \theta + \varphi \in W^{1,p(x)}_0(\Omega).$$

Then we have $v \in K^{\theta,p(x)}_{-\infty,+\infty}(\Omega)$. So we can obtain

$$\int_\Omega \langle A(x, \nabla u), \nabla\varphi\rangle dx = \int_\Omega \langle A(x, \nabla u), \nabla v - \nabla u\rangle dx \geq 0. \tag{2.17}$$

By using $-\varphi$ in place of φ, we also can get

$$\int_\Omega \langle A(x, \nabla u), \nabla(-\varphi)\rangle dx = \int_\Omega \langle A(x, \nabla u), \nabla v - \nabla u\rangle dx \geq 0. \tag{2.18}$$

So

$$\int_\Omega \langle A(x, \nabla u), \nabla\varphi\rangle dx = 0. \tag{2.19}$$

Thus u is a solution to (1.1) in Ω.

Conversely, if u is a solution to (1.1) in Ω, then for any $v \in K^{\theta,p(x)}_{-\infty,+\infty}(\Omega)$, we have $v - u \in W^{1,p(x)}_0(\Omega)$. Now let $\varphi = v - u$, then we have

$$0 = \int_\Omega \langle A(x, \nabla u), \nabla\varphi\rangle dx = \int_\Omega \langle A(x, \nabla u), \nabla v - \nabla u\rangle dx. \tag{2.20}$$

Thus

$$0 \leq \int_\Omega \langle A(x, \nabla u), \nabla v - \nabla u\rangle dx. \tag{2.21}$$

So the theorem 2.3 is proved.

3. Caccioppoli Type Estimate of the Solutions to Obstacle Problem

Next we consider a Caccioppoli type estimate of the solutions to obstacle problem taking into account its importance in studying the integrablity of the solutions.

Theorem 3.1. *Assume u is a solution to the obstacle problem $K^{\theta,p(x)}_{\psi_1,\psi_2}(\Omega)$ with positive obstacle ψ_1, non-positive obstacle ψ_2. Let $B \subset\subset \Omega$, $R > 0$ and $\eta \in C_0^\infty(B)$ with $0 \le \eta \le 1$. Then*

$$\int_\Omega |\nabla u_+|^{p(x)} \eta^{p_B^+} \le C \int_\Omega u_+^{p(x)} |\nabla \eta|^{p(x)} dx. \tag{3.1}$$

Here the constant C depends only on p^- and p^+.

Proof. Let $v = u - sgn(u)\varepsilon_0\phi$, where $\phi = u_+\eta^{p_B^+}$,

$$sgn(u) = \begin{cases} -1, & if \;\; u(x) < 0 \\ 1, & if \;\; u(x) \ge 0 \end{cases}$$

$\varepsilon_0 > 0$ is a constant which is small enough to ensure that v is between the positive obstacle ψ_1 and non-positive obstacle ψ_2. It is clearly that v has the same boundary values as u. Since

$$|\nabla \phi| \le \left| \eta^{p_B^+} \nabla u_+ + u_+ p_B^+ \eta^{p_B^+ - 1} \nabla \eta \right| \le |\nabla u| + C|u_+| \tag{3.2}$$

we observe that $|\nabla \phi| \in L^{p(\cdot)}(\Omega)$, and $|\phi| \in L^{p(\cdot)}(\Omega)$, thus, it is enough to show that $\phi \in W^{1,p(\cdot)}(\Omega)$.

Since $v = u - sgn(u)\varepsilon_0\phi \in K^{\theta,p(x)}_{\psi_1,\psi_2}(\Omega)$, we find

$$\begin{aligned} 0 &\le \int_\Omega \langle A(x,\nabla u), \nabla(-sgn(u)\varepsilon_0\phi)\rangle dx \\ &= -\int_\Omega \langle A(x,\nabla u), sgn(u)\varepsilon_0\eta^{p_B^+}\nabla u_+ + sgn(u)\varepsilon_0 u_+ p_B^+ \eta^{p_B^+ - 1}\nabla\eta\rangle dx \\ &\le -\alpha\varepsilon_0 \int_\Omega |\nabla u_+|^{p(x)}\eta^{p_B^+} + \beta\varepsilon_0 \int_\Omega |\nabla u_+|^{p(x)-1} u_+ p_B^+ \eta^{p_B^+ - 1} |\nabla \eta| dx \end{aligned} \tag{3.3}$$

This implies that

$$\int_\Omega |\nabla u_+|^{p(x)} \eta^{p_B^+} \le \frac{\beta p_B^+}{\alpha} \int_\Omega |\nabla u_+|^{p(x)-1} u_+ \eta^{p_B^+ - 1} |\nabla \eta| dx \tag{3.4}$$

We denote the right hand side of (3.4) by I. By Younger's inequality we obtain

$$\begin{aligned} I &= \frac{\beta p_B^+}{\alpha}\int_\Omega u_+|\nabla\eta|\eta^{p_B^+-1-\frac{p_B^+}{p'(x)}}|\nabla u_+|^{p(x)-1}\eta^{\frac{p_B^+}{p'(x)}}dx \\ &\le \frac{\beta p_B^+}{\alpha}\left(\int_\Omega \frac{\varepsilon^{1-p(x)}}{p(x)}u_+^{p(x)}|\nabla\eta|^{p(x)}\eta^{p_B^+-p(x)} + \frac{\varepsilon(p(x)-1)}{p(x)}|\nabla u_+|^{p(x)}\eta^{p_B^+}dx\right) \\ &\le \frac{\beta p_B^+}{\alpha}\left(\int_\Omega \frac{\varepsilon^{1-p_+}}{p_-}u_+^{p(x)}|\nabla\eta|^{p(x)}\eta^{p_B^+-p(x)} + \frac{\varepsilon(p^+-1)}{p_-}|\nabla u_+|^{p(x)}\eta^{p_B^+}dx\right) \\ &\le \frac{\beta p_B^+}{\alpha}\left(\varepsilon^{1-p_+}\int_\Omega u_+^{p(x)}|\nabla\eta|^{p(x)}\eta^{p_B^+-p(x)}dx + \varepsilon\int_\Omega |\nabla u_+|^{p(x)}\eta^{p_B^+}dx\right), \end{aligned} \tag{3.5}$$

where ε satisfties $\frac{\beta p_B^+}{\alpha}\varepsilon < 1$. For this and the face that $\int_\Omega |\nabla u_+|^{p(x)}\eta^{p_B^+} < \infty$, which combining with the inequality (3.4), we obtain

$$\int_\Omega |\nabla u_+|^{p(x)}\eta^{p_B^+} \le C\int_\Omega u_+^{p(x)}|\nabla\eta|^{p(x)}dx. \tag{3.6}$$

The proof completed.

We can also proof a Caccioppoli type estimate for the negative parts of solutions to obstacle problem.

Theorem 3.2. *Assume u is a solution to the obstacle problem $K^{\theta,p(x)}_{\psi_1,\psi_2}(\Omega)$ with positive obstacle ψ_1, non-positive obstacle ψ_2. Let $B \subset\subset \Omega$, $R > 0$ and $\eta \in C_0^\infty(B)$ with $0 \le \eta \le 1$. Then*

$$\int_B |\nabla u_-|^{p(x)}\eta^{p_B^+} \le C\int_B |u_-|^{p(x)}|\nabla\eta|^{p(x)}dx. \tag{3.7}$$

Here the constant C depends only on p_B^- and p_B^+.

Proof. Let $v = u + sgn(u)\varepsilon_0 u_-\eta^{p_B^+}$. we find that

$$\begin{aligned} 0 &\le \int_\Omega \langle A(x,\nabla u), \nabla(sgn(u)\varepsilon_0 u_-\eta^{p_B^+})\rangle dx \\ &= \int_\Omega \langle A(x,\nabla u), sgn(u)\varepsilon_0\eta^{p_B^+}\nabla u_- + sgn(u)u_- p_B^+\eta^{p_B^+-1}\nabla\eta\rangle dx \\ &\le -\alpha\varepsilon_0\int_\Omega |\nabla u_-|^{p(x)}\eta^{p_B^+} - \beta\varepsilon_0\int_\Omega |\nabla u_-|^{p(x)-1}u_- p_B^+\eta^{p_B^+-1}|\nabla\eta| dx \end{aligned} \tag{3.8}$$

This implies that

$$
\begin{aligned}
&\int_\Omega |\nabla u_-|^{p(x)}\eta^{p_B^+} \\
\leq\ & \frac{\beta p_B^+}{\alpha}\int_\Omega -|\nabla u_-|^{p(x)-1}u_-\eta^{p_B^+-1}|\nabla\eta|dx \\
=\ & \frac{\beta p_B^+}{\alpha}\int_\Omega -u_-|\nabla\eta|\eta^{p_B^+-1-\frac{p_B^+}{p'(x)}}|\nabla u_-|^{p(x)-1}\eta^{\frac{p_B^+}{p'(x)}}dx
\end{aligned}
\tag{3.9}
$$

Notice that $\int_\Omega |\nabla u_-|^{p(x)}\eta^{p_B^+} < \infty$, then by the same methods as the proof in Theorem 3.1, we obtain

$$\int_\Omega |\nabla u_-|^{p(x)}\eta^{p_B^+} \leq C\int_\Omega |u_-|^{p(x)}|\nabla\eta|^{p(x)}dx \tag{3.10}$$

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References

[1] M. Ružička, *Electrorheological Fluids: Modeling and Mathematical Theory*, Lecture Notes in Mathematics, **1748**, Berlin, Springer-Verlag (2000).

[2] Gao Hongya, Tian Huiying, Local regularity result for solutions of obstacle problem, *Acta Math. Sci.*, **24B**, No. 1 (2004), 71-74.

[3] Gao Hongya. Regularity for very weak solutions of A-harmonic equation, *Acta Math. Sci.*, **44**, No. 4 (2001), 605-610.

[4] Ph. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, J.L. Vázquez, An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations, *Ann.Sc. Norm. Super. Pisa, Cl. Sci.*, **22** (1995), 241-273.

[5] A. Alvino, L. Boccardo, V. Ferone, L. Orsina, G. Trombetti, Existence results for nonlinear elliptic equations with degenerate coercivity, *Ann. Mat. Pura Appl.*, **182** (2003), 53-79.

[6] Y. Fu, The existence of solutions for elliptic systems with nonuniform growth, *Studia Math.*, **151** (2002), 227-246.

[7] J. Heinonen, T. Kilpeläinen, O. Martio, *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, NY, USA (1993).

[8] Yu.A. Alkhutov, *The Harnack inequality and the Hölder property of solutions of nonlinear elliptic equations with nonstandard growth condition*, *Differ. Uravn.*, **33**, No. 12 (1997), 1651-1660; *Translation in Differ. Eq.*, **33**, No. 12 (1997), 1653-1663.

[9] O. Kováčik, J. Rákosník, On spaces $L^{p(x)}$ and $W^{1,p(x)}$, *Czechoslovak Math. J.*, **41**, No. 116 (1991), 592-618.

[10] P. Harjulehto, P. Hästö, et al, An obstacle problem and superharmonic functions with nonstandard growth, *Nonlinear Analysis*, **67** (2007), 3424-3440.