**International Journal of Pure and Applied Mathematics**
**Volume 70 No. 3 2011, 379-388**

# A SOLUTION METHOD FOR ENGINEERING MATHEMATICS MODELING

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**Abstract:** In this paper, we consider the vertical nonlinear complementarity problem in engineering equilibrium mathematics modeling (VNCP). To solve the problem, we first establish a global error estimation for VNCP, and then given a solution method to solve the VNCP based on the error bound estimation. The global convergence is also established.

**AMS Subject Classification:** 65H10, 90C33, 90C30
**Key Words:** VNCP, engineering equilibrium mathematics modeling, error estimation, solution method, convergence

## 1. Introduction

Let mappings $F, G : R^n \to R^m$, the vertical nonlinear complementarity problem, abbreviated as VNCP, is to find vector $x^* \in R^n$ such that

$$F(x^*) \geq 0, \quad G(x^*) \geq 0, \quad F(x^*)^\top G(x^*) = 0, \tag{1}$$

where $F$ and $G$ are polynomial functions from $R^n$ to $R^m$, respectively. We denote the solution set of the VNCP by $X^*$ and assume that it is nonempty throughout this paper.

The VNCP is a direct generalization of the classical linear complementarity problem (LCP) which finds applications in engineering, economics, finance, and robust optimization operations research (Refs.[1]). The VNCP plays a significant role in contact mechanics problems, structural mechanics problems, obstacle problems mathematical physics, Elastohydrodynamic lubrication prob-

Received: February 14, 2011 

lems, traffic equilibrium problems(such as a path-based formulation problem, a multicommodity formulation problem, network design problems),etc, and has been received much attention of researchers ([1, 2, 3, 4, 5]), where many effective methods have been proposed for solving VNCP. The basic idea of these methods is to reformulate the problem as an unconstrained or simply constrained optimization problem ([6, 7, 8]). Different from the algorithms listed above, we propose a new type of solution method to solve the VNCP based on the error bound estimation.

Among all the useful tools for theoretical and numerical treatment to variational inequalities, nonlinear complementarity problems and other related optimization problems, the error bound estimation is an important one([9]). To our knowledge, Mangasarian and Shiau ([10]) are the first one who gave the error bound analysis to linear complementarity problems. Latter, Mathias and Pang ([11]) established the global error bound estimation for the LCP with a P-matrix in terms of the natural residual function, and Mangasarian and Ren gave the same error bound of the LCP with an $R_0$-matrix in [12]. Obviously, the VNCP is an extension of the LCP, and this motivates us to consider the error bound estimation for the VNCP. So, in this paper, we are concentrated on establishing a global error bound for the VNCP via the same type of residual function under mild conditions which can be taken as an extension of that for the linear complementarity problem, and discussing its applications on the convergence analysis of method which is given for solving the VNCP, Compared with the existing solution methods in [7, 6], the conditions guaranteed for convergence are weaker in this paper since the assumption on the existence of a nondegenerate solution is removed here.

Some notations used in this paper are in order. Vectors considered in this paper are all taken in Euclidean space equipped with the standard inner product. The norm $\|\cdot\|$ denote the Euclidean 2-norm . We use $x \geq 0$ to denote a nonnegative vector $x \in R^n$ if there is no confusion.

## 2. Error Bound for the VNCP

In this section, we present the error bound for VNCP , First, we need the following definition and a relate property.

**Definition 1.** For the mapping $F, G$ involved in the VNCP, $F$ is said to be an $R_0$-function with respect to $G$ if, for any sequence $\{x^k\} \subseteq R^n$ satisfying $\|x^k\| \to \infty$,

$$\liminf_{k\to\infty} \min_{1\leq i\leq m} \{F_i(x^k)\}/\|x^k\| \geq 0, \tag{2}$$

$$\liminf_{k\to\infty} \min_{1\le i\le m} \{G_i(x^k)\}/\|x^k\| \ge 0, \tag{3}$$

then there exists an index $j$ such that

$$\liminf_{k\to\infty} F_j(x^k)\}/\|x^k\| > 0, \liminf_{k\to\infty}\{G_j(x^k)\}/\|x^k\| > 0.$$

**Lemma 1.** *Given a constant* $\gamma > 0$, *then* $x \in X^*$ *if and only if*

$$r(x) = \|\min\{\gamma F(x), G(x)\}\| = 0.$$

We also need give the following assumption conditions.

**Assumption 1.** For $F(x), G(x)$ defined in (1):

(C1) $F$ be an $R_0$-function with respect to $G$;

(C2) For any the sequence $\{x^k\}$, and $\|x^k\| \to \infty$, then there exists the subsequence $\{x^{k_i}\}$ such that

$$\liminf_{i\to\infty} r(x^{k_i})\}/\|x^{k_i}\| > 0;$$

(C3) Let $X_\varepsilon \overset{\triangle}{=} \{x \in R^n | r(x) \le \varepsilon, \varepsilon > 0\}$, then $X_\varepsilon$ is bounded.

**Theorem 1.** $(C1) \Rightarrow (C2) \Rightarrow (C3)$.

*Proof.* First, we prove that $(C1) \Rightarrow (C2)$. For any the sequence $\{x^k\}$, and $\|x^k\| \to \infty$. If the sequence $\{x^k\}$ satisfy (2),(3), By Definition 1, then there exists an index $j$ such that

$$\liminf_{k\to\infty} F_j(x^k)\}/\|x^k\| > 0, \liminf_{k\to\infty}\{G_j(x^k)\}/\|x^k\| > 0.$$

Combining this, we have that (C2) holds.

On the other hand, if the sequence $\{x^k\}$ does not satisfy (2),(3), then there exist a subsequence $K$ and indices $i$ and $j$ such that

$$\text{either} \liminf_{k\to\infty}\{F_i(x^k)/\|x^k\|\} < 0 \text{or} \quad \liminf_{k\to\infty}\{G_j(x^k)/\|x^k\|\} < 0, k \in K.$$

Combining this, we also have that (C2) holds.

Secondly, we show that $(C2) \Rightarrow (C3)$. Suppose that the level set $X_\varepsilon$ is unbounded for some $\varepsilon > 0$, then there exists a sequence $\{x^k\} \in X_\varepsilon$ such that $\|x^k\| \to \infty$ as $k \to \infty$. By (C2), then there exists the subsequence $\{x^{k_i}\}$ such that $\liminf_{i\to\infty} r(x^{k_i})\}/\|x^{k_i}\| > 0$, combining this, we have $r(x^{k_i}) \to \infty$, This leads to a contradiction. $\square$

In this following, we establish a global error bound for the VNCP under mild condition. First, we can give the needed result.

**Lemma 2.** *Suppose that Assumption (C2) hold,then there exists a constant* $\tau > 0$ *such that*

$$dist(x, X^*) \leq \eta_1 r(x)^{\frac{1}{m}}, \quad \forall x \in X_\varepsilon, \tag{4}$$

*where* $m = \max\{m_1, m_2\}$, $F(x)$, $G(x)$ *are polynomial functions with powers* $m_1$ *and* $m_2$, *respectively.*

*Proof.* Assume that the lemma is false. Then there exist $\varepsilon_0 > 0$, $X_0 = \{x \in X | r(x) \leq \varepsilon_0\}$, for any integer $k$, there exists $x^k \in X_0$, such that $dist(x^k, X^*) > kr(x^k)^{\frac{1}{m}} \geq 0$, i.e.,

$$\frac{r(x^k)^{\frac{1}{m}}}{dist(x^k, X^*)} \to 0, k \to \infty. \tag{5}$$

By Theorem 1, we have that $X_0$ is bounded set, and $r(x)$ is continuous, combining (5), we have $r(x^k) \to 0 (k \to \infty)$, and there exists a subsequence $\{x^{k_i}\}$of $\{x^k\}$ such that $\lim\limits_{k_i \to \infty} x^{k_i} = \overline{x} \in X_0$, where $\overline{x} \in X^*$. Since $F(x)$, $G(x)$ are polynomial functions with powers $m_1$ and $m_2$, respectively, for all sufficiently large $k_i$, we have

$$\frac{r(x^{k_i})^{\frac{1}{m}}}{\|x^{k_i} - \overline{x}\|} = \beta > 0. \tag{6}$$

On the other hand, by (5), we obtain

$$\lim_{k_i \to \infty} \frac{r(x^{k_i})^{\frac{1}{m}}}{\|x^{k_i} - \overline{x}\|} \leq \lim_{k_i \to \infty} \frac{r(x^{k_i})^{\frac{1}{m}}}{dist(x^{k_i}, X^*)} = 0,$$

this contradicts (6), thus, we have that (4) holds. □

Based on Lemma 2, we have the following conclusion.

**Theorem 2.** *Suppose that Assumption (C2) holds, then there exist a constant* $\eta_2 > 0$, *for any* $\forall x \in R^n$, *we have* $dist(x, X^*) \leq \eta_2(r(x) + r(x)^{\frac{1}{m}})$.

*Proof.* Assume that the theorem is false. Then for any integer $k$, there exist $x^k \in R^n$ and $\overline{x} \in X^*$, such that

$$\|x^k - \overline{x}\| > k(r(x^k) + r(x^k)^{\frac{1}{m}}). \tag{7}$$

It follows that there exist $k_0 > 0, \varepsilon_0 > 0, \quad \forall k > k_0$, we have

$$r(x^k) + r(x^k)^{\frac{1}{m}} > \varepsilon_0. \tag{8}$$

In fact, otherwise, for $\forall k > 0$, $\forall 0 < \varepsilon < 1$,there exists $\bar{k} > k$, such that $r(x^{\bar{k}}) + r(x^{\bar{k}})^{\frac{1}{m}} \leq \varepsilon$, we let $\Theta_\varepsilon = \{x \in R^n | r(x^{\bar{k}}) + r(x^{\bar{k}})^{\frac{1}{m}} \leq \varepsilon\}$. Using theorem 1, we know that $X_\varepsilon$ is bounded set, for any $x \in \Theta_\varepsilon$, we have

$$r(x) \leq r(x) + r(x)^{\frac{1}{m}} \leq \varepsilon,$$

i.e., $\Theta_\varepsilon \subseteq X_\varepsilon$, combining this with Theorem 1, we know that $\Theta_\varepsilon$ is bounded. Combining this with Lemma 2 again, for $x^{\bar{k}} \in \Theta_\varepsilon$, there exist $\overline{x}(x^{\bar{k}}) \in X^*$ and constant $\eta_1 > 0$, such that $\|x^{\bar{k}} - \overline{x}(x^{\bar{k}})\| \leq \eta_1 r(x^{\bar{k}})^{\frac{1}{m}}$, combining this with $\|x^{\bar{k}} - \overline{x}(x^{\bar{k}})\| < 1$, we have

$$r(x^{\bar{k}}) + r(x^{\bar{k}})^{\frac{1}{m}} \geq \frac{1}{\eta_1}\|x^{\bar{k}} - \overline{x}(x^{\bar{k}})\| + \frac{1}{\eta_1^m}\|x^{\bar{k}} - \overline{x}(x^{\bar{k}})\|^m \geq (\frac{1}{\eta_1} + \frac{1}{\eta_1^m})\|x^{\bar{k}} - \overline{x}(x^{\bar{k}})\|,$$

combining this with (7), for $x^{\bar{k}}$, and $\overline{x}(x^{\bar{k}}) \in X^*$, we have

$$\frac{\eta_1^m}{\bar{k}(\eta_1^{m-1} + 1)}\|x^{\bar{k}} - \overline{x}(x^{\bar{k}})\| > \frac{\eta_1^m}{(\eta_1^{m-1} + 1)}(r(x^{\bar{k}}) + r(x^{\bar{k}})^{\frac{1}{m}}) \geq \|x^{\bar{k}} - \overline{x}(x^{\bar{k}})\|,$$

i.e.$\frac{\eta_1}{\bar{k}} \geq \frac{\eta_1^m}{\bar{k}(\eta_1^{m-1}+1)} > 1$. Let $\bar{k} \to \infty$, then we have $\frac{\eta_1}{\bar{k}} < 1$, this is contradiction, we have that (8) holds.

By (7) and (8), we have $\|x^k\| > k\varepsilon_0 - \|\overline{x}\|$, i.e., $\|x^k\| \to \infty (k \to \infty)$.

Let $y^k = \frac{x^k}{\|x^k\|}$, then there exist a subsequence $y^{k_i}$ of $\{y^k\}$, such that $y^{k_i} \to \overline{y}(k_i \to \infty)$, note that $\|\overline{y}\| = 1$. Divide both sides of (7) by $\|x^{k_i}\|$, and let $k_i$ go to infinity, we obtain

$$1 = \lim_{i\to\infty} \frac{\|x_{k_i} - \overline{x}\|}{\|x^{k_i}\|} > \lim_{i\to\infty} \frac{k_i(r(x^{k_i}) + r(x^{k_i})^{\frac{1}{m}})}{\|x^{k_i}\|} \to \infty,$$

this is contradiction, then the desired result is followed. □

**Remark 1.** The error bound in the above Theorem 2 is extensions of Theorem 2.1 in [12], Lemma 1 in [11] for linear complementarity problem, respectively.

## 3. Algorithm and Convergence

In this section, we give a new-type method to solve NCP based on the error bound in Section 2, and present the proof for its global convergence. First, we give the following definition.

**Definition 2.** The mapping $F : R^n \to R^m$ is said to be strongly monotone with respect to $G$ if there is constants $\mu > 0$ such that

$$\langle F(x) - F(y), G(x) - G(y)\rangle \geq \mu ||G(x) - G(y)||^2, \quad \forall x, y \in R^n. \tag{9}$$

**Definition 3.** $G$ is said to be nonsingular if there is constant $L > 0$ such that

$$\|G(x) - G(y)\| \geq L\|x - y\|, \quad \forall x, y \in R^n. \tag{10}$$

Now, we formally state our algorithm.

**Algorithm 1.**

*Step 1.* Take $\varepsilon > 0$, parameters $0 < \gamma < 2\mu$, and initial point $x^0 \in R^n_+$. Set $k \triangleq 0$;

*Step 2.* Compute

$$G(x^{k+1}) = P_{R^n_+}(G(x^k) - \gamma F(x^k)); \tag{11}$$

*Step 3.* If $\|G(x^{k+1}) - G(x^k)\| \leq \varepsilon$ stop, otherwise, go to Step 2 with $k \triangleq k+1$.

By the definition of projection operator, we can easily get that problem (11) can be equivalently reformulated as the following constrained optimization problem

$$\begin{array}{ll} \min & (G(x) - G(x^k))^\top (G(x) - G(x^k)) + 2\gamma(G(x) - G(x^k))^\top F(x^k) \\ \text{s.t.} & G(x) \geq 0. \end{array} \tag{12}$$

where $\gamma > 0$is a constant.

**Theorem 3.** *Assume that Assumption (C2) and (9) (10) hold, then the sequence $\{x^k\}$ converges to a solution of the VNCP.*

*Proof.* Obviously, if $G(x^{k+1}) = G(x^k)$, combining Theorem 2 with (9), and then $x^k$ is a solution of VNCP. In the following theoretical analysis, we always assume that Algorithm 1 generates an infinite sequence.

Suppose that $G(x^{k+1}) \neq G(x^k)$ holds, and the objective function of (12) is denoted by $\Phi(x)$ which $G(x^k) = G(x^*)(x^* \in X^*)$. In this following, we would prove that the sequence $\{\Phi(x^k)\}$ is monotone. To this end, we set

$$
\begin{aligned}
&\Theta(k, k+1) = \Phi(x^k) - \Phi(x^{k+1}) \\
=&(G(x^k) - G(x^*))^\top (G(x^k) - G(x^*)) + 2\gamma\langle F(x^*), G(x^k) - x^*\rangle \\
&- (G(x^{k+1}) - G(x^*))^\top (G(x^{k+1}) - G(x^*)) - 2\gamma\langle F(x^*), G(x^{k+1}) - x^*\rangle \\
=&(G(x^k))^\top G(x^k) - (G(x^*))^\top G(x^*) - 2\langle G(x^*), G(x^k) - G(x^*)\rangle \\
&- (G(x^{k+1}))^\top G(x^{k+1}) + (G(x^*))^\top G(x^*) + 2\langle G(x^*), G(x^{k+1}) - G(x^*)\rangle \\
&+ 2\gamma\langle F(x^*), G(x^k) - G(x^{k+1})\rangle \\
=&(G(x^k))^\top G(x^k) - (G(x^{k+1}))^\top G(x^{k+1}) \\
&+ 2\langle G(x^*), G(x^{k+1}) - G(x^k)\rangle + 2\gamma\langle F(x^*), G(x^k) - G(x^{k+1})\rangle \\
=&(G(x^k))^\top G(x^k) - (G(x^{k+1}))^\top G(x^{k+1}) - 2\langle G(x^{k+1}), G(x^k) - G(x^{k+1})\rangle \\
&+ 2\langle G(x^{k+1}) - G(x^*), G(x^k) - G(x^{k+1})\rangle + 2\gamma\langle F(x^*), G(x^k) - G(x^{k+1})\rangle \\
=&(G(x^k) - G(x^{k+1}))^\top (G(x^k) - G(x^{k+1})) \\
&+ 2\langle G(x^{k+1}) - G(x^k), G(x^*) - G(x^{k+1})\rangle + 2\gamma\langle F(x^*), G(x^k) - G(x^{k+1})\rangle \\
\geq&(G(x^k) - G(x^{k+1}))^\top (G(x^k) - G(x^{k+1})) \\
&- 2\gamma\langle F(x^k), G(x^*) - G(x^{k+1})\rangle + 2\gamma\langle F(x^*), G(x^k) - G(x^{k+1})\rangle \\
=&(G(x^k) - G(x^{k+1}))^\top (G(x^k) - G(x^{k+1})) + 2\gamma\langle F(x^k), G(x^k) - G(x^*)\rangle \\
&- 2\gamma\langle F(x^k), G(x^k) - G(x^{k+1})\rangle + 2\gamma\langle F(x^*), G(x^k) - x^{k+1}\rangle \\
\geq&(G(x^k) - G(x^{k+1}))^\top (G(x^k) - G(x^{k+1})) + 2\gamma\mu\|G(x^k) - G(x^*)\|^2 \\
&- 2\gamma\langle F(x^k) - F(x^*), G(x^k) - G(x^{k+1})\rangle \\
\geq&\|G(x^k) - G(x^{k+1})\|^2 + 2\gamma\mu\|G(x^k) - G(x^*)\|^2 \\
&- 2\gamma\langle F(x^k) - F(x^*), G(x^k) - G(x^{k+1})\rangle \\
\geq&\|G(x^k) - G(x^{k+1})\|^2 + 2\gamma\mu\|G(x^k) - G(x^*)\|^2 \\
&- 2\gamma\mu\|F(x^k) - F(x^*)\|^2 - \frac{\gamma}{2\mu}\|G(x^k) - G(x^{k+1})\|^2
\end{aligned}
$$

$$\geq \|G(x^k) - G(x^{k+1})\|^2 - \frac{\gamma}{2\mu}\|G(x^k) - x^{k+1}\|^2.$$

Since (12) can be equivalently reformulated as the following variational inequalities

$$\langle 2(G(x^{k+1}) - G(x^k)), G(x) - G(x^{k+1})\rangle + 2\gamma\langle F(x^k), G(x) - G(x^{k+1})\rangle \geq 0, \quad \forall G(x) \geq 0, \quad (13)$$

let $G(x) = G(x^*)$ in (13), we have that the first inequality holds. Let $F(y) = F(x^*)(x^* \in X^*)$ in (9), scine $G(x^k) \geq 0, F(x^*) \geq 0$, we have $G(x^k)^\top F(x^*) \geq 0$, combining this with Definition 2, we get

$$\begin{aligned} \mu\|G(x^k) - G(x^*)\|^2 &\leq \langle F(x^k) - F(x^*), G(x^k) - G(x^*)\rangle \\ &= \langle F(x^k), G(x^k) - G(x^*)\rangle - G(x^k)^\top F(x^*), \end{aligned} \quad (14)$$

by (14), we have

$$\langle F(x^k), G(x^k) - G(x^*)\rangle \geq \mu\|G(x^k) - G(x^*)\|^2, \quad (15)$$

by (15), we have that the second inequality holds. The fourth inequality is based on Cauchy-Schwarz inequality. By $0 < \gamma < 2\mu$, we have $\Theta(k, k+1) > 0$, the nonnegative sequence $\{\Psi(x^k)\}$ is strictly decreasing.

Combining the definition of $F(x)$, we have

$$\Psi(x^k) = \|G(x^k) - G(x^*)\|^2 + G(x^k)^\top F(x^*) \geq \|G(x^k) - G(x^*)\|^2 \geq 0. \quad (16)$$

So $\{\Psi(x^k)\}$ converges, and we get $\Theta(k, k+1) \to 0$ as $k \to \infty$, and

$$\lim_{k\to\infty} \|G(x^k) - G(x^{k+1})\| = \lim_{k\to\infty} r(x^k) = 0. \quad (17)$$

Moreover, $\{\Psi(x^k)\}$ is bounded since it is convergent, and so is $\{G(x^k)\}$ according to (16). Combining Theorem 2 with (17), there exists a constant $\eta > 0$ such that

$$dist(x^k, X^*) \leq \eta(r(x^k) + r(x^k)^{\frac{1}{m}}) \to 0 \ \ (k \to \infty),$$

where $dist(x^k, X^*) = \|x^k - \bar{x}\|, \bar{x}$ denote the closest solution to $x^k$.

Since $\{G(x^k)\}$ is bounded, let $\{G(x^{k_i})\}$ be a subsequence of $\{G(x^k)\}$ and converges to $G(\bar{x})$, for $\{G(x^{k_i})\}$, we also have

$$dist(x^{k_i}, X^*) \leq \eta(r(x^{k_i}) + r(x^{k_i})^{\frac{1}{m}}) \to 0 \ \ (i \to \infty),$$

by Theorem 2, we get $\bar{x}$ is a solution of VNCP.

Since $\{\Psi(x^k)\}$ converges, substituting $G(x^*)$ in (16) by $G(\bar{x})$ leads to that $\Psi(x^{k_i}) \to 0(i \to \infty)$. Thus, $\Psi(x^k) \to 0(k \to \infty)$. Using (16) again, we know that the sequence $\{G(x^k)\}$ converges to $G(\bar{x})$. By (10), we have

$$\|x^k - \bar{x}\| \leq L\|G(x^k) - G(\bar{x})\| \to 0(k \to \infty),$$

then the desired result is followed. □

In Theorem 3, we have showed that method has a global convergence, and we needn't the condition which there exist nondegenerate solution, it is a new result for NCP.

## Acknowledgments

The author wish to give their sincere thanks to the editor and the anonymous referees for their valuable suggestions and helpful comments which improved the presentation of the paper.

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